Constrained and Unconstrained Optimization

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Lecture – 05
Graphical Solution of LLP- II

Let us start with the; from the last lecture. Last lecture we started the graphical solution of LLP consisting of 2 decision variables only in the objective function. We have talked about 2 things one is the search method another one is the Iso-profit method. So, let us see how it works let us take this example; maximize z equals x 1 plus 3 x 2 subject to 3 x 1 plus 6, x 2 less than equals 8 and 5 x 1 plus 2 x 2 less than equals 10 x 1 x 2 greater than equals 0.

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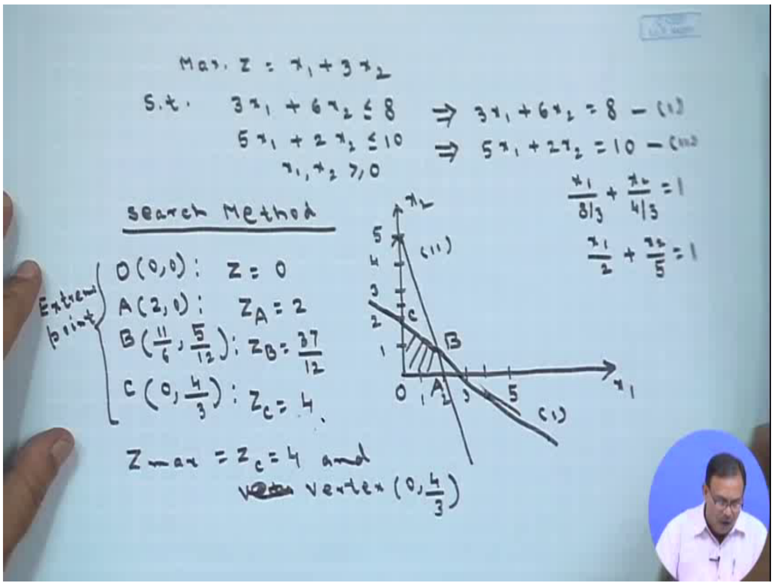

So, first let us do this problem using the search method, then we will see the same problem with the Iso-profit method in the search method first what we have told you convert this in equation into equation, that is this we will write it as 3 x 1 plus 6 x 2. This is equals 8 and the second one will be 5 x 1 plus 2 x 2 equals 10. So, that now there are 2 lines and this 2 lines we can draw very easily, suppose, this is your axis as x 1 and x 2. So, this goes to 0. And one suppose we are assuming this is 1 2 3 4 and 5. Similarly on this side also 1 2 3 4 and this will be 5 say.

So, that first line whatever you see 3 x 1 plus 6 x 2, this is equals x this can be writ10 in the form of x 1 by 8 by 3 plus x 2 by 4 by 3 this is equals 1. So, that the interception x 1 and x 2 are 8 by 3 and 4 by 3 respectively. So, that the drawing becomes much easier. So, one line will go through just below 3 here and the point 2, that is it will be something like this and there will be another line which is going through 5 and 2 that is it is something like this.

So, your 5 and 2 is the line 2 if this is the line 2, if this is the line one. So, this represents the line 2 whereas, this represents the line one. So, first what you are doing you are just writing the equalities 3 x 1 plus 6 x 2 equals 8 and 5 x 1 plus 2 x 2 equals 10 this you can write down in standard intercept format second one, you can write down x 1 by 2 plus x 2 by 5, this is equals 1 also from their you are drawing 2 lines this is the line one and this one is the line 2 line 2 is going from 2 0, and it is intercepting at 0 5.

So, what is happening once you are drawing this one your feasible region becomes the intersection of these 2 lines x 1 axis and x 2 axis on the positive side; that means, on this side only the lines bounded by this 4 lines this line 1 line 2 x 1 axis and x 2 axis. Or we can say that the feasible region is this thing. So, first what you have to do this was the step one in step 2, you have to find out the extreme points. As you remember your extreme points are which one the boundary points; that means, the boundary intersection of 2 lines. So, one is this point 0 that is 0 0, second one we are saying this is a point a say this is the point a. So, point a the coordinate is 2 0 already we have done it 2 0. Then if I see this is the point b. So, your point b if you calculate from here by solving you will get 11 by 6 and 5 by 12. And the last point is this one because this is the feasible region from 0 2 I am going like this then from here from here and coming back to this.

So, the last intersection point or extreme point is the point c, again c is x 1 0 and x 2 will be actually it is not 2 it is a little less than 2 2 will be sometimes, here it will be 0 and 4 by 3. So, these are you see these are the extreme points. So, what you are doing please note this thing this is the extreme points. So, first you are getting the equations from the equations you are writing it in the standard intercept form you are drawing the lines once you have drawn the lines, after that from there you are finding out 0 a b c the extreme points.

Now, at each extreme point what is the value of the objective function z equals x 1 plus 3 x 2, that you have to find out now which becomes easy. So, at the point 0 0 the value is 0, your z a if you calculate it will become 2 your z b. This will become 37 by 12 and your z c this will become 4. So, from here it is quite clear that the optimum is attended at the extreme point 0 and 4 by 3. So, therefore, your z max is the value is 4 and z max equals z c, and this is equals 4 and the corresponding vertex is corresponding vertex is 0 4 by 3.

So, the solution of your problem by using the graphical method is at the extreme point 0, 4 by 3 you will obtain the optimum value that is maximum value in that case and maximum value of the objective function is 4. I hope it is clear and I am repeating once more the steps the first steps are first step is first convert the inequality into equation type then for each equation draw a line on the x 1 x 2 axis just we have drawn the lines one and line 2. And once you are drawing the line you find out what is the feasible region feasible region is intersection of the equation lines whatever you have drawn from the constraints. And the x 1 axis this is your x 1 axis here this is your x 2 axis here and then find out the extreme points that is intersection of this 4 lines for this case you got it o a b and c.

So, these are the 4 points at these add 4 points, you find out what is the value of the objective function z, which you derived like this. And from there whatever is the maximum value of z that extreme point gives you the optimum value. So, therefore, optimum value can be obtained at 0 4 by 3 and the optimum or maximum value of the function is equals to 4. Now let us see how we can solve the same problem using Iso-profit approach.

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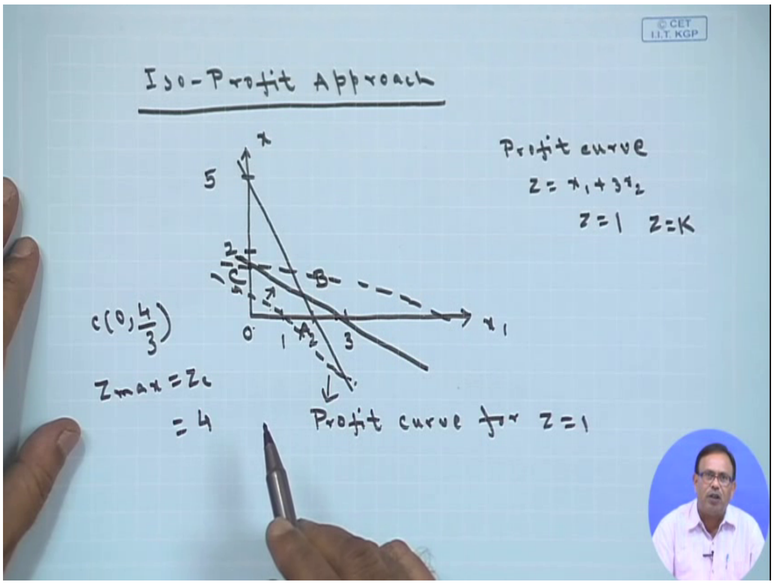

So, next one is Iso-profit approach. In Iso-profit approach as I mentioned earlier the first 2 steps are same. That is the same figure you draw and from the same figure you find out what are the extreme points. So, I am just drawing once more the figure. This is your x 1, this is your x 2. So, say this is your 3, somewhere here you have the point 2. Here it is say 2. So, just below 2 one point will come. It is something like this and there will be another point if it is 5 then 2 and 5. So, it will be something like this. So, this was your 0 this is you're a this is your b this is your c. And already you know what is the feasible region.

Now, you have to draw a profit curve. For some value of z such that the profit curve lies in the feasible region. Your function is z equals x 1 plus 3 x 2. I am assuming my profit curve initially is z equals 1, z equals 1 means if here it is 2, it is one here. So, it will be something like this. So, this is your profit curve. This one is the profit curve for z equals 1. This value z equals k basically it is z equals k k can take any arbitrary value. This we have taken arbitrarily in such a manner that whenever, I draw the line z equals 1 or x 1 plus 3 x 2 equals 1 that line lies portion of the line lies in the feasible region.

Your next step is you move this line z equals 1 away from the origin, away from the origin by changing the value of k such that whenever it goes to maximum distance it gives the value as a maximum value for z. So, from here it is quite clear, if you move in this direction if you move along this, line then it will be at some point of time I will get some line like this. So, which basically crosses the point c. So, once it is crossing the point c that is c is your 0 4 by 3. There you will obtain the maximum value that is z max equals z c and whose value is equals to 4.

So, the basic idea is move the line away from the origin. So, that your profit becomes optimum. And at some boundary point it will cut where you will get the maximum value of z. For this particular problem you are obtaining the maximum value of z at the point c. Therefore, your optimum value remains the same. So, this is the Iso-profit approach. Similarly, if you want to use the iso minimum approach that is for minimization problem this profit curve, should move towards 0. Please note this one for maximization problem the profit curve should move away from 0. Whereas, for profit for profit curve for minimization problem, should be at the towards the origin. So, that you can obtain the minimum value object. So, I hope it is little bit clear to you let us take one more problem in this direction.

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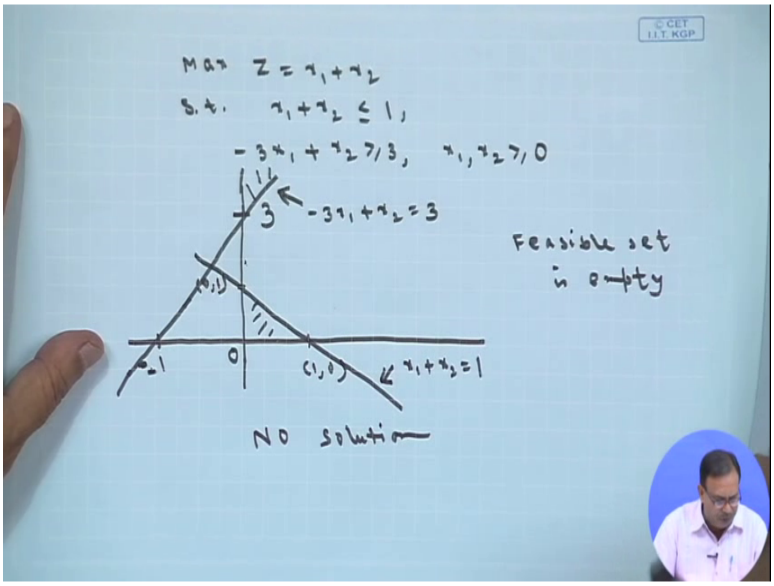

So, that it becomes easy for us the next problem let us take maximize z equals x 1 plus x 2 subject to x 1 plus x 2 less than equals 1, minus 3 x 1 plus x 2 greater than equals 3 and; obviously, x 1 x 2 greater than equals 0 these are the 3 conditions whatever you are using.

So, I am not writing this one first you have to write down x 1 plus x 2 equals 1 and minus 3 1 3 x 1 plus x 2 equals 3. And after that you have to draw the you have to draw the lines. So, I am just directly drawing the curve over here. So, this is the point 0. There will be a point 1 0, for the first one. There will be another point that is 0 1. So, you will get the line like this. So, effectively this is x 1 plus x 2 less than equals 1; that means, it is direction will be towards the left hand side of this line towards the downwards side. So, x 1 plus x 2 this is equals 1. This arrow indicates that the region will be on this side below this. One since it is less than equals 1, because here if you note that in the constant we have one less than equals constant we have another greater than equals constant.

So, for less than equals it will be below the line, whereas for greater than equals the region will be above the line. For the other one it will be, I think one is minus 1 say and the other point will be say 3. So, that the line will be something like this. This is the minus 1 point this is the point 3. So, this line is your line is x 1 minus minus 3 x 1 plus x 2 this is equals to 3. So, it is going in this direction. So, if you think about this one, one is going like this another is going like this. So, the feasible set that is the intersection of this 2 lines and the positive x axis and y axis that is positive x 1 and x 2 is empty.

So, here that feasible set if I have to say, this feasible set is empty. Please note this one. Because there is no intersection point one. Portion is this side another portion is this side there is no common portion. So, feasible set is empty and once feasible set is empty means as you know the solution whatever will be there which will satisfy the constraint and the non-negativity condition always must lie in the feasible region, but and the feasible region is equals to the intersection of the lines and the x 1 axis and x 2 axis on the positive direction.

So, therefore, for this case the feasible region, there is no feasible region or we are writing feasible set is empty therefore, this problem has no solution. Please note this one this problem has no solution since the feasible region is empty or feasible set is empty. So, it is not necessary that for every LLP problem, there will have some solution I must have some feasibility region which is non empty and then only I can obtain the solution. So, this is a one case.

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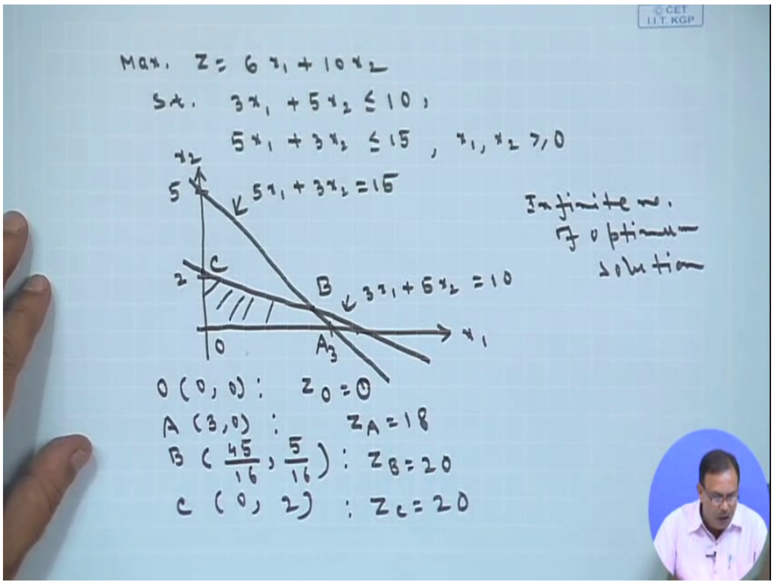

The next see z equals maximize we are in general take maximization problem z equals 6 x 1 plus 10 x 2 subject to 3 x 1 plus 5 x 2, less than equals 10 5 x 1 plus 3 x 2 less than equals 15. And the non-negativity conditions that is x 1 plus x 2 greater than equals 0. So, to find the solution again equate the inequality constraint and draw the equations as it is as we have done earlier. So, this is x 1 this is x 2. So, there will be one line which will be crossing here say this is 2. And somewhere here it will be there if it is 3, it would be little about 3. So, it will be something like this. So, this is the line 3 x 1 plus 5 x 2 equals 10 the other one is 5 x 1 plus 3 x 2 equals 10. So, it will go through 3 and the 0.5. So, that if you see it will be something like this. So, here also both are less than equals. So, I am giving the direction it is 5 x 1 plus 3 x 2 this is equals to 15.

So, this is on this side. So, the feasible region is quite clear feasible region, is this one extreme points are 0 a b and c. Here point 0 is 0 0. Here points a will be 3 0 point b is if you solve it 45 by 16 and 5 by 16 whereas, your c will become 0 and 2. So, these are the 4 points extreme points, whatever you are obtaining now at these extreme points you have to find out the value z dot o is one sorry 0 0 0. At the point a it is value is 18, at the point b z equals b it is 20, at the point c the value is 20.

So, the maximum value is attended at 2 extreme points at the point b and at the point c. So, therefore, what happens here, it is not that there is only unique solution there may have more than one solution. And since this is a convex set therefore, if you line join this line b and c any point you take on b and c that also will give you the maximum value of the objective function. Or in other sense for this particular problem you have infinite number of solutions. Infinite number of optimum solutions. Please note this one we have the infinite number of optimum solutions. The reasons we will discuss whenever we are going through the theory of simplex method and other things in details.

So, one problem we have done earlier which has unique solution. We have done one problem which has no solution and if you have considered this particular point this has unique not unique the multiple solutions that is infinite number of optimum solutions and at each point it gives you the optimum value.

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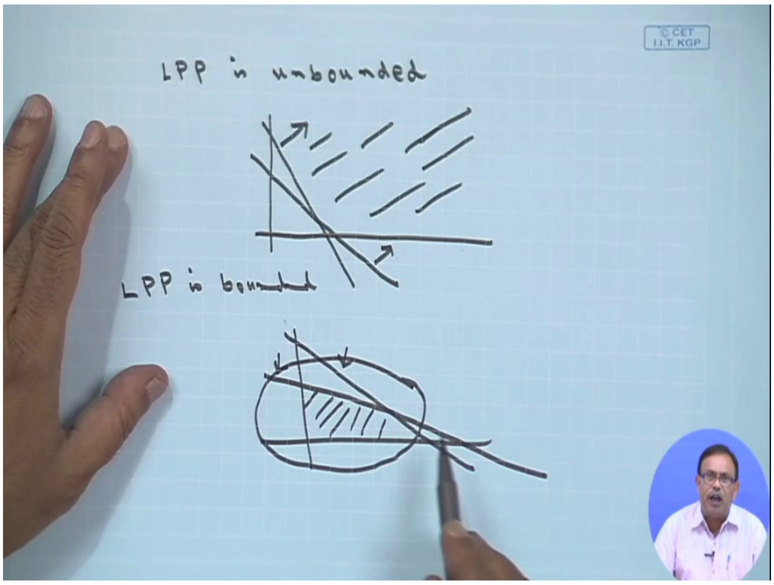

Now graphically if you talk graphically we can say that one LLP is unbounded. If you take graphical case one LLP is unbounded. If the feasible region is increasing indefinitely in the same direction. If the feasible region is increasing in the same direction, please note that one in the same direction; that means, if I draw something like this. And if I have something like this for both cases the direction is on the upper side therefore, if you draw the region will be increasing as you increase the values of x 1 and x 2.

So, feasible region extends indefinitely here increasing and in the same direction then we say that the LLP is unbounded. Whereas, the LLP will be unbounded sorry LLP is bounded for if I say LLP is bounded again. If I draw something like this and if I draw something like this, where in both cases direction is on this side; that means, feasible region is like this. So, if I can draw some circle in such a fashion that the feasible region lies inside the circle. Now the radius of the circle may be as large as possible, but if I can draw one circle of sufficiently large radius, such that feasible region lies inside the circle then we say that LLP is bounded.

So, graphically the LLP is unbounded if the feasible region extends indefinitely in the same direction please note this one it is unbounded, if it moves in the same direction whereas, the LLP is bounded. If we can draw a circle of sufficiently large radius such that the feasible region lies inside the circle.

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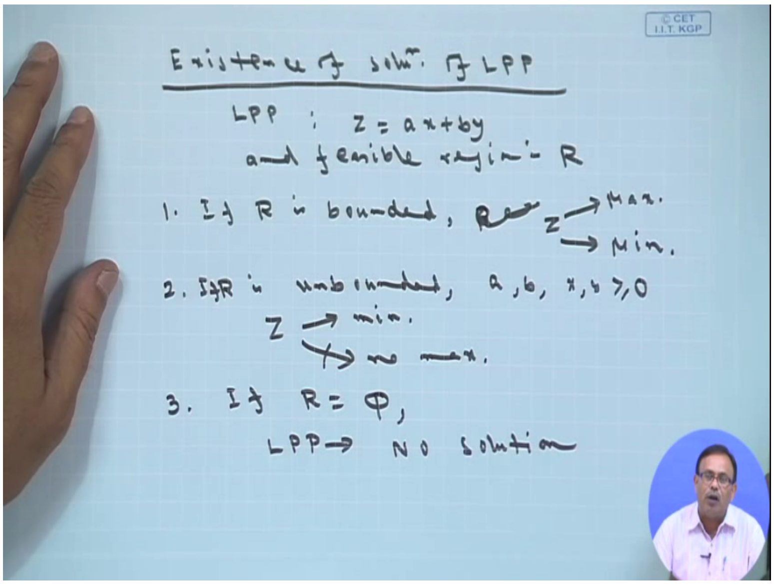

Therefore, we can say that the existence, existence of solution of LLP. In general, what we can say for the existence of solution of LLP. If you have a LLP of the form z equals a x plus b y and your feasible region is your feasible region is suppose r. So, number one, if r is bounded then r has then z has both maximum and minimum value. Please note this one if r is bounded z has both maximum value and minimum value, here we want to say maximum and minimum means optimum minimal value.

Now, if r is unbounded whereas, a b x and y are all are greater than equal 0 then z has minimum value, but no maximum value. Please note this one if r is unbounded and a b x y is greater than equals 0 z has minimum value, but no maximum value and third point is r r equals null, that is if r is the there is no feasible solution r equals 5 in that case LLP has no solution. So, these are the cases depending upon the boundedness or unboundedness or empty, it will be either as a solution may be unique or infinite number of solutions maybe we can achieve only minimum value for maximization problem and no solution.

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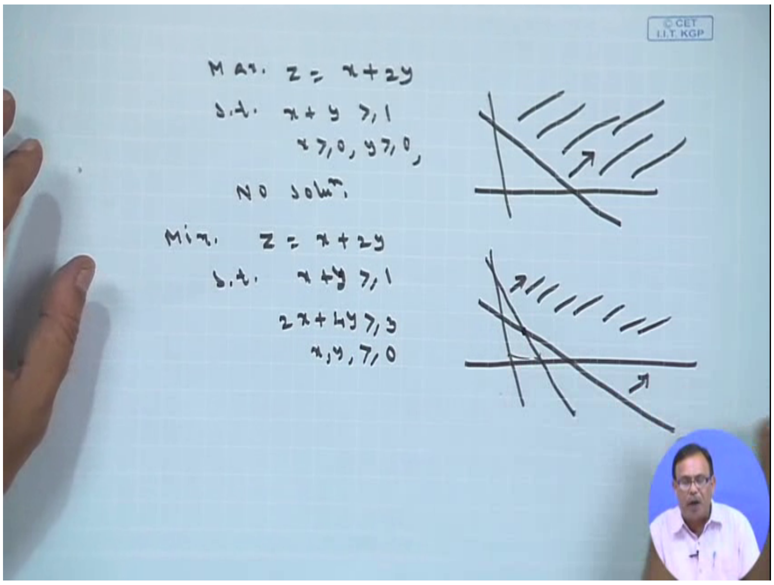

If you consider this one, say maximize z equals x plus 2 y subject to x plus y is greater than equals 1, x greater than equals 0 y greater than equals 0. So, if you try to draw your x plus 2 y something like this it is greater than equals 1 means the direction is this one. So, your feasible region is unbounded therefore, this problem has no solution. Whereas, if you take minimize z equals x plus 2 y subject to x plus y equals sorry x plus y greater than equals 1 2 x plus 4 y greater than equals 3 x y greater than equals 0. In this case you will get one line something like this, and you will get another line something like this. And here the feasible region if you see both are greater than equals 1. Both are greater than equals 1. So, again; that means, the direction of both are like this feasible region is coming like this, but here although the feasible region is unbounded. And this is a minimization problem therefore, at this point you can obtain the minimum value and this is possible only because a b x y all are greater than equals 0.

So, at one of this points you will get the minimum value of the objective function, but in the other cases you will not obtain this value. And the last one is LLP with unbounded solution, please note that the value of decision variables may increase indefinitely without violating any constraint as we have shown earlier. In that case the value of the objective function increases in definitely both the solution space and objective function becomes in that case unbounded. And unbounded solution is equivalent to infinite number of solutions, but no optimum solution please note this one. If the region feasible region is unbounded you can obtain number of solutions which are satisfying the your constraints, but you will not obtain the optimum solution. The examples are given earlier. So, next the simplest method we will start in the next class.