Constrained and Unconstrained Optimization

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Lecture - 48
Constrained Optimization

Now, today the non-linear programming with inequality constraint. Now before going to the detail of KKT conditions and just to talk more on the feasible direction and other things that is why let me just recapitulate the concept I told you in the last class that is there are 3 constraints g 1, g 2, g 3; is my g 1, this is g 2 and this is g 3.

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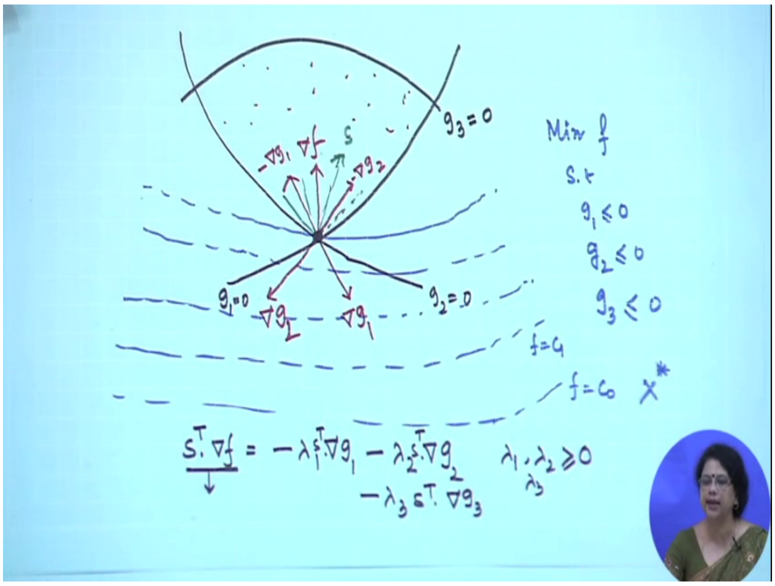

Now, the objective function is moving in this direction our problem is to find out minimization of f subject to g 1 less than is equal to 0, g 2 less than equal to 0, and g 3 less than equal to 0. If this is the level curve of the objective function f that means, objective functional value is increasing in this direction and at this stage once objective function is coming here we are getting the optimal solution x star ok. Now, if this is the optimal solution you see from here we can draw that this is the grad direction grad f direction because objective functional value is increasing in this direction.

If we consider g 1 equal to 0 this one then this must be grad g 1 direction and this is minus grad g 1 similarly, if we consider g 2 direction this one then this is minus grad g 2 and the opposite direction is the grad g 1. Now, we are talking g 2 now we are talking about the feasible direction now this is the feasible space that is why from here if I just move this way then this is the direction of the feasible direction, and if I move within the space anywhere then this is the feasible direction that is why this is the feasible direction this is the feasible these are all the feasible directions, there is no doubt about it these are all the feasible direction; these are all the feasible directions with the green. Now, if I consider that this is one of the direction say S, now in the last class I showed you that if we consider the feasible direction S then grad f is equal to minus lambda 1 grad g 1 minus lambda 2 grad g 2.

Now what else we said that this is this must be positive, why this is positive because the feasible direction and the gradient of f mix the acute angle and we are considering the inner product of 2 vectors one is S that is the feasible direction and another one is the gradient of the objective function gradient vector. Now, on the other hand the feasible direction and the grad g 1 that is the gradient of the first constraint this is making the obtuse angle that is why this is negative and similarly grad g 2 also if a just this is must be S T, S T S and g 2 this is also obtuse angle this is also negative that is why to adjust the sign we have said that lambda 1 lambda 2 must be greater than 0 in this case equal to will be there if we are having the inactive constraint in the group. Since g 3 is the inactive the corresponding lambda will be 0 that is why lambda 1 lambda 2 lambda 3 is greater than 0 where lambda 2 1 and lambda 2 these are all strictly positive, but lambda 3 is 0 here.

Now, with this idea I am going to the formal definition of the feasible direction decent direction etcetera.

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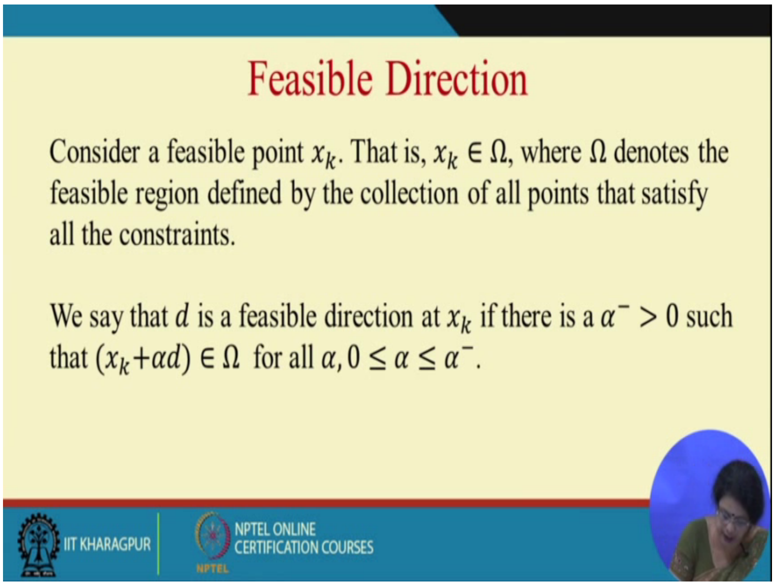

Now, we can say that if x k is a point which is the point within the feasible space here I showed you the feasible space that if we consider the feasible region as row then, collection of all points satisfying the constraints can be named as the feasible space. Now we can say that if I move from x k to any direction d in such a way that a x x k plus alpha d alpha is the step length alpha d will remain with the feasible space then d can be said as the feasible direction that means, from the figure if I just want to tell you same this is my x k.

Now I am moving in this direction this is the directions say d and how far I will move I will move with the step length alpha that is why I am moving from x k to x k plus alpha d, where alpha will remain from 0 to alpha, bar then this direction is the feasible direction. From x k if I just go in this up to this point such that alpha is beyond this alpha value just put me beyond the feasible space then direction is the feasible direction, but we are not reaching to the feasible point that is the definition of the feasible direction.

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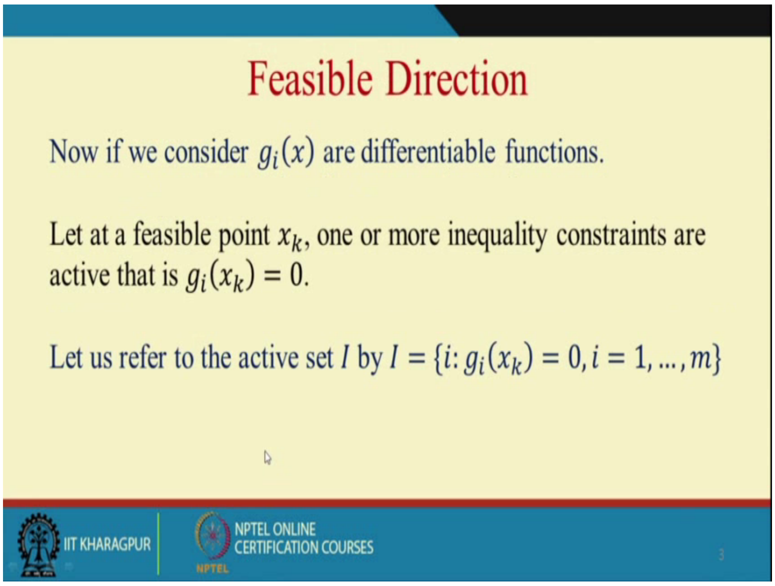

Now, if I just say that gi x is the constraint where that is the differentiable function then at the feasible point one or more inequality constraints at the if x k is the optimal solution, then one or more inequalities will be with the equality side because these are all the active constraints. That is why if I consider a group of active constraints as I showed you the example here the active constraints are g 1 and g 2 g 3 is not the active constraint that is why if I consider g 1 and g 2 then the we can say at the point optimal point g 1 equal to 0 and g 2 is equal to 0 then only these are all the active constraints.

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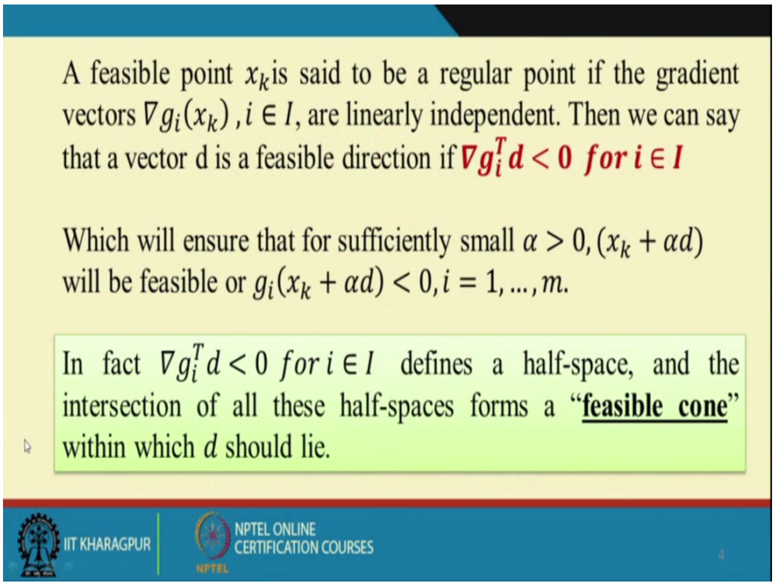

Now then we can say that if d is the feasible direction then grad of g i T dot d must be less than 0 where I is in the group of active constraints I showed you that is why it can be said that if we just take a sufficiently small increment alpha in the direction d that is the feasible direction then g I of x k plus alpha d must be lesser than 0.

Because, as I said that that is the grad of grad of g i of this thing with the feasible direction must be lesser than 0 that is why we can say that grad of g i d is less than 0, if we consider all active constraints together then we can show that every constraint grad g i d grad g i dot d that is a inner product makes less than 0 makes half space.

If we consider the half spaces of with all active constraints then it will form a feasible cone. Here we can say that in this one we can say that here there is one plane that is the half space over g 1 there is 1 plane over g 2 half space and whatever, if we consider the combination of half planes in the feasible space that will form the feasible cone. Now we will show you more about it. Now another part is that now we know that in this direction the objective functional value is increasing that is why we can say that in this direction grad f dot instead of S here we are calling about the feasible direction as d, then the grad f of d must be greater than 0. But if I consider the other way that is the just opposite to grad f direction; that means, if I just consider this direction this is minus grad f that is why we can say that this minus grad f makes an obtuse angle with the d thus the descent direction can be defined as grad f dot d must be less than 0.

Now, why we are talking all about this feasible direction, descent direction, ascent direction, all these directions because in the next phase of my lecture you will see we will deal with the methodology to solve the constraint non-linear programming problem. There we can see that this feasible direction descent direction all this mathematics will help us a lot.

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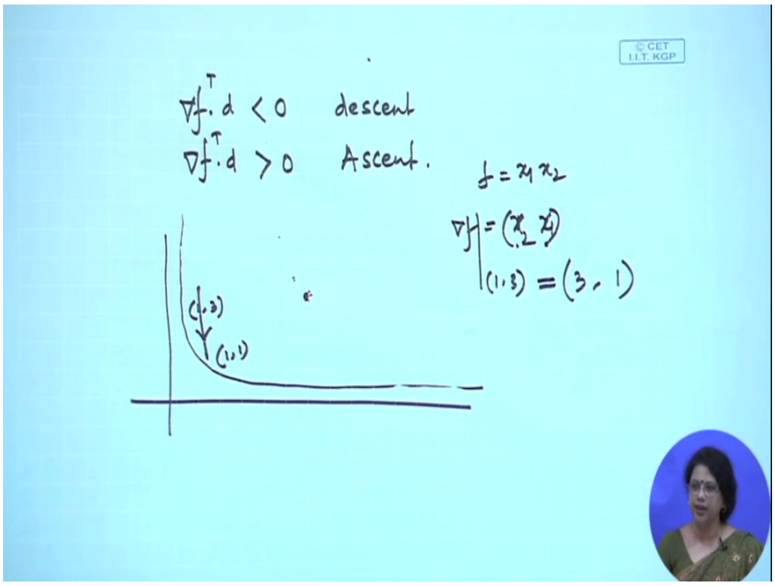

That is why we need to understand that the grad f dot d is less than 0 this is the descent direction similarly grad f dot d greater than 0 is the ascent direction we need to remember this 2 things together. Now, therefore, in that direction if I just move.

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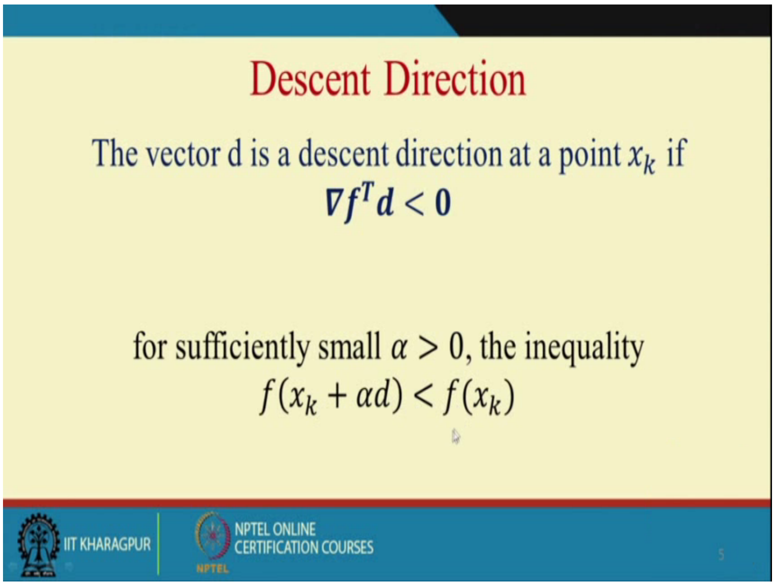

Further we will see that in the descent direction if you move further we will see the functional value will decrease further and further.

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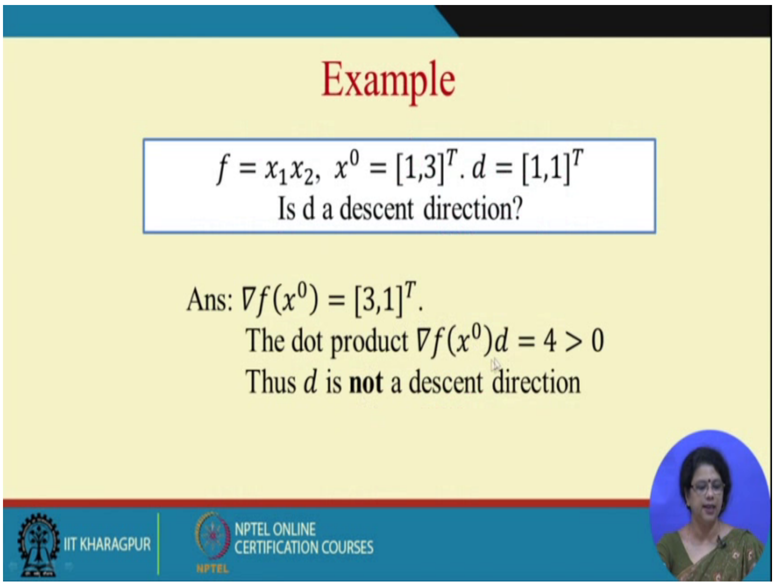

Now I can show you one example now see this is the function given to us X 1 X 2 is the function for us now there is a point 1 3 within the space, now the direction is given as 1,1 direction have you has been given 1 because from 1 3 if you move in the direction 1,1 for a just if I consider the a 2 dimensional space, x 1 x 2 is the function; that means, the function would be like this x 1 x 2 this is one of the level curve of the function.

Now, there is a point that is 1 3 say this is 1 3 this is the point 1 3 now there is a direction 1,1 this is the direction it has been said that we have to check if I move from 1, 3 to 1,1 whether this is a descent direction for the function x 1 x 2 or not. Now the function has been given as x 1 x 2 then in the descent direction just now we have learnt grad of f dot d must be less than 0 in the descent direction.

That is why let us find out what is the grad f for it if f is x 1 x 2 then grad f must be is equal to 1 1 that is why if I consider the grad f at the point 1 3 then it must be yeah 1 1 actually this is x 2 this is x 1 because with respect to x 1 we are differentiating and this is with respect to x 2 we are differentiating that is why at the point 1 3 this point at 1 3 will become 3 1 alright. That is why if we consider the grad f dot d then we can see 3 1 and 1 1 if we just multiply we can get the value of that would be equal to 4 that is why we can say d is not the descent direction certainly this is the ascent direction of x 1 x 2.

Now let us deal with another example there one optimization problem has been said, problem has been given as f is equal to minus x 1 plus x 2 alright and t 3 constraints are given one constraint is g 1 is equal to given x 1 square plus 4 x 2 square minus 1 less than equal to 0, g 2 has been given as minus x 1 less than is equal to 0 rather x 1 is greater than 0 to make it in the same direction all the inequalities we can consider this way.

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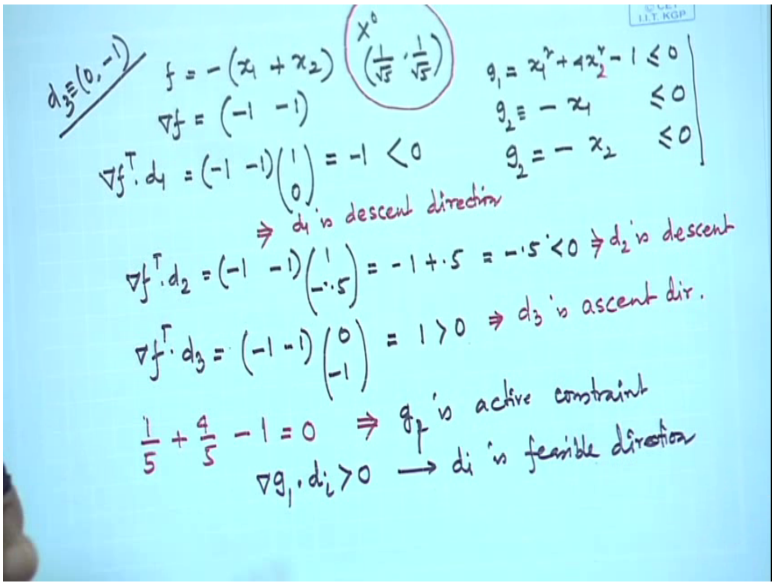

This basically, the non negativity constraints for the decision variables, now there is a point has been given 1 by root 5 and 1 by root 5. Now there are 3 directions are given we need to find out the that all the directions are the feasible and or descent direction not the direction can be feasible the direction can be infeasible the direction can be descent the direction can be ascent with that idea we are just going one by one. If we consider the grad f value certainly the grad f will come as here if I just differentiate with respect to x 1 partially, then this is minus 1 and differentiate with respect to x 2 this is minus 1. Now, if I consider grad f t dot d 1 now my d 1 direction is 1 0 that is why it is becoming minus 1 minus 1 0 that is why this value is coming as minus 1. Now we can say that if we now one thing is that the point has been given here as x not has been given 1 by root 5 by 1 by root 5, but there is no effect on it because the gradient of f is not a function of x 1 x 2 that is why always the grad f would be minus 1 minus 1.

Now, if I just move in this direction then this will become less than 0 then what is the conclusion from here we can conclude that this direction is the descent direction because this is negative alright. Let me do the same calculations for d 2 d 3 as well - grad f d 2 minus 1 minus 1 d 2 direction is my 1 minus 0.5 then this is becoming minus 1 plus 0.5 that is why again this is coming as minus 0.5 less than 0 that is why again we can conclude that d 2 is the descent direction alright.

Now if I do further calculations for d 3 grad f d 3 minus 1 minus 1 and 0,1 if we just consider this is becoming 0,1 0 minus 1 rather because d 3 let please consider d 3 as 0 minus 1 not 0,1 alright 0 minus 1 if we consider this is becoming 1. That is why we can consider we can say that d 3 is ascent direction not descent direction alright. Now this is all about the descent or ascent direction, but we did not say about the feasible direction yet now we need to check the feasible direction. One thing is that you see if this is the point is there now let us see which is the active constraints because feasible direction can only be defined with respect to the constraint set descent direction or ascent direction can be defined with respect to the function objective function, but feasible direction can only be defined with respect to the constraint set.

If I just show you this one you could see that feasible direct feasible direction can only be can be defined because this is collection of feasible points that is the collection of points which satisfy all the constraints together that is why feasible direction with respect to the constraint set. Now, if we see the constraint set what we could see that this point first of all g 2 is not be is not an active constraint, g 3 is not an active constraint, is that knows because minus 1 minus 1 by root 5 is not being we cannot say this is equal to 0 that is why this is not active this is not active, but let us see whether this point is being satisfied with this or not.

If I just substitute the values here then we can say that 1 by 5 plus 4 by 5 that is x square x 1 square plus 4 x 2 square minus 1 this is equal to 0 that is why we can conclude from here that g 2 is the active constraint, g 1 is the active constraint. If g 1 is the active constraints if we can prove that grad g 1 dot d; that means, grad g 1 is making the obtuse angle obtuse angle with the direction of d then we can say that just you see this is my grad g 1. Now grad g 1 with this direction its making the obtuse angle that is why if we can show that grad g 1 is making the obtuse angle that is greater than 0 then we can say that d 1 is the d i is the feasible direction if this is true then we can say that d i is feasible direction alright let us see.

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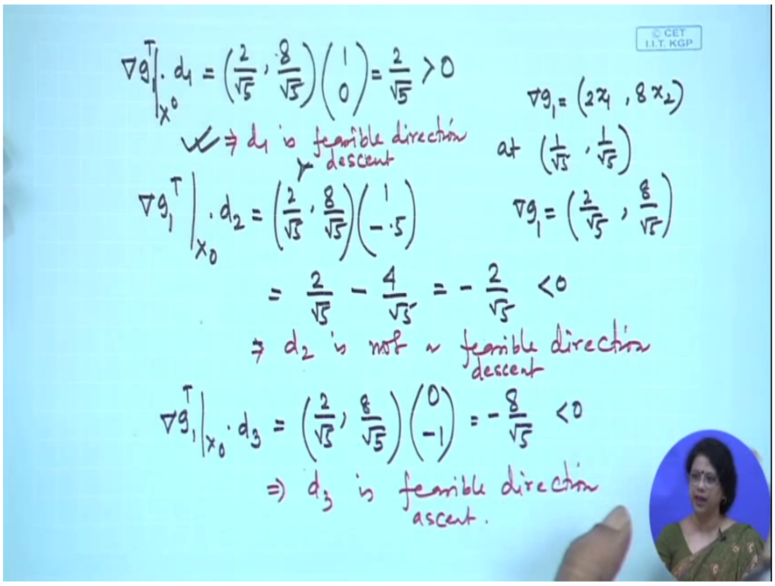

Grad of g 1 dot d 1 now grad of g 1 just now we have we did not find it out grad of g 1 is equal to 2 x 1,8 x 2 now at 1 by root 5, 1 by root 5 grad g 1 value is coming as 2 by root 5 and 8 by root 5 ok.

That is why grad g 1 at point x not d 1 this is at point x not in the d direction if we consider then this would be is equal to 2 by root 5, 8 by root 5 and my d one direction is one 0 that is why we can say his is equal to 2 by root 5 this is greater than 0 that is why what we can conclude that d 1 is feasible direction ok. Now let me calculate the same for other d 2 and d 3, 2 by root 5, 8 by root 5 and 1 minus 0.5 that is why it is coming 2 by root 5, minus 4 by root 5 it is coming minus 0.2 by root 0.5 that is why this is negative. What is the conclusion from here? The conclusion is that d 2 is not a feasible direction let me do the same for d 3, d 3 the d 3 direction is 0 minus 1, if we just do it we can say this is my coming minus 8 by root 5 this is less than 0 that is why again we can say that the d 3 is the is feasible direction. If we keep all together in the previous case we could see that d 1 is descent direction, d 2 is descent direction and d 3 is ascent direction.

That is why we can conclude that only d 1 is feasible and descent direction or other directions are not the feasible and the descent direction that is why for the minimization problem we should move through d 1 we can conclude that.

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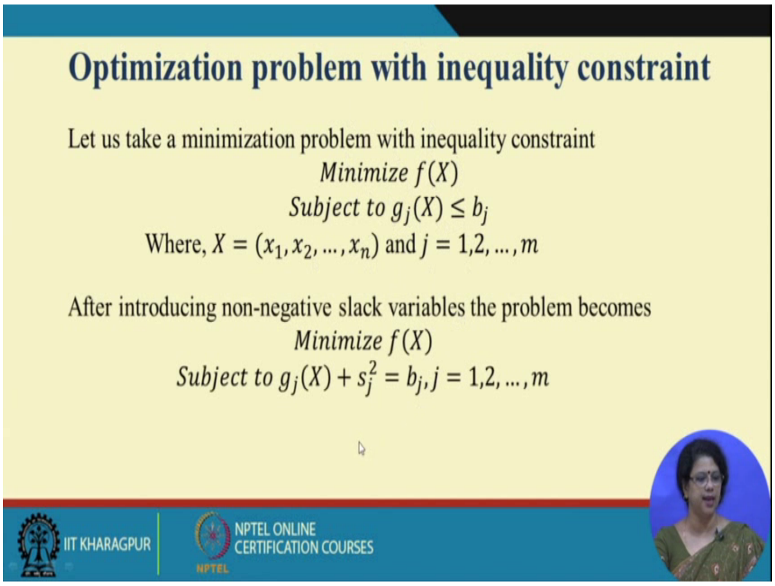

Now, if we just move to again to the KKT condition if we consider the minimization problem minimization of f x subject to g j x less than is equal to b j as I said we can say the KKT conditions are can be defined.

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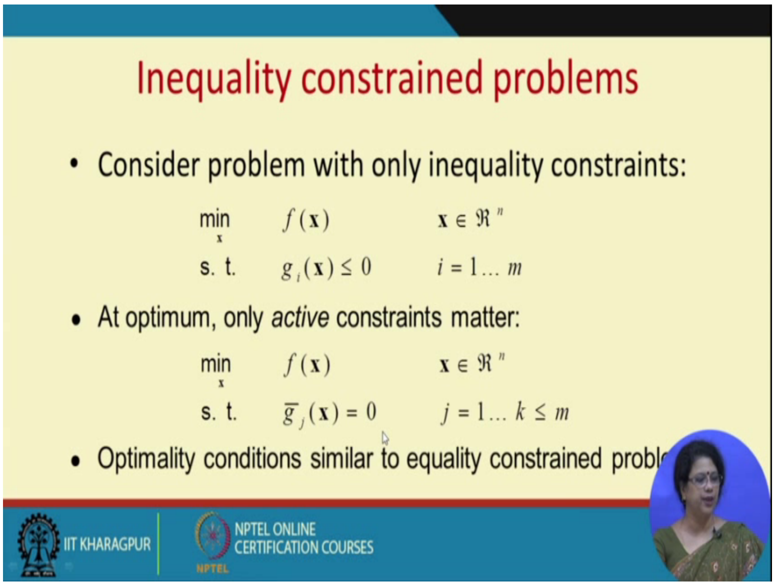

Now here we are again considering there is a set of active constraints and there is a set of in active constraints if we consider that there are few number of constraints k number of constraints which are active constraints if we just make the group of those then we can say that similarly for the inactive constraints also if we just do it we can say this is the first order optimality condition.

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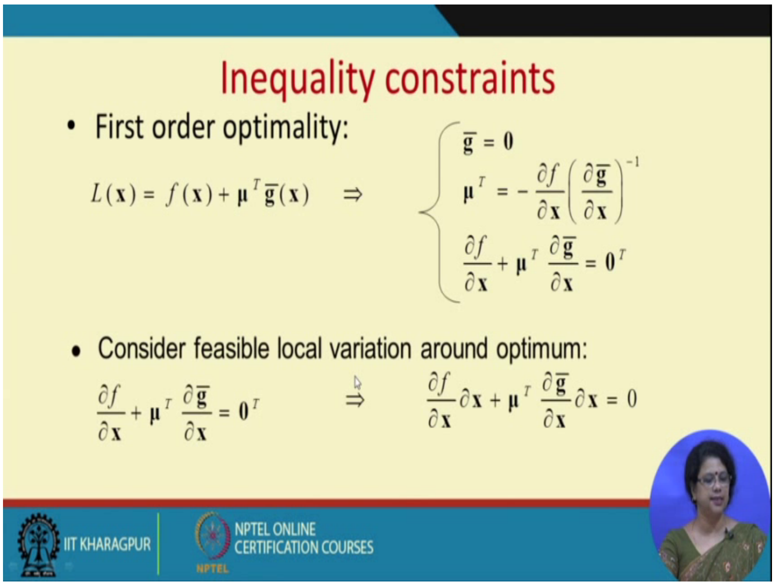

Because we have done the first order derivative of the Lagrange function with respect to decision variable we are getting the first order optimality condition and if we consider these are few calculations I will speak more on this in the next, but one thing wanted to tell you that del f by del x is greater than0, but this is less than 0 in the feasible direction that is why we can say this is positive alright.

And this is the graph of it, if this is my g 1 and this is my g 2 then this is the feasible direction because not only the feasible direction this is ascent direction because functional value is moving further with this direction with the positive value. And this is a grad g 1 and this is my grad of g 2, these are the normal direction at this point and from here we can say that this is the minus grad f direction. And with these today I am concluding my lecture because I will speak more on this in the next class and I will show you the examples how to handle with KKT condition in the next class.

Thank you very much for today.