Constrained and Unconstrained Optimization
Prof. Debjani Chakraborty
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 45
NLP with Equality Constrained – II

In continuation to my previous class today I am going to explain Lagrange multiplier method for multi dimensional non-linear programming problem, where we are having the non-linear programming problem with the constraints which are all of equality type and after that we will proceed to the case non-linear programming problem with inequality constraint in the next class.

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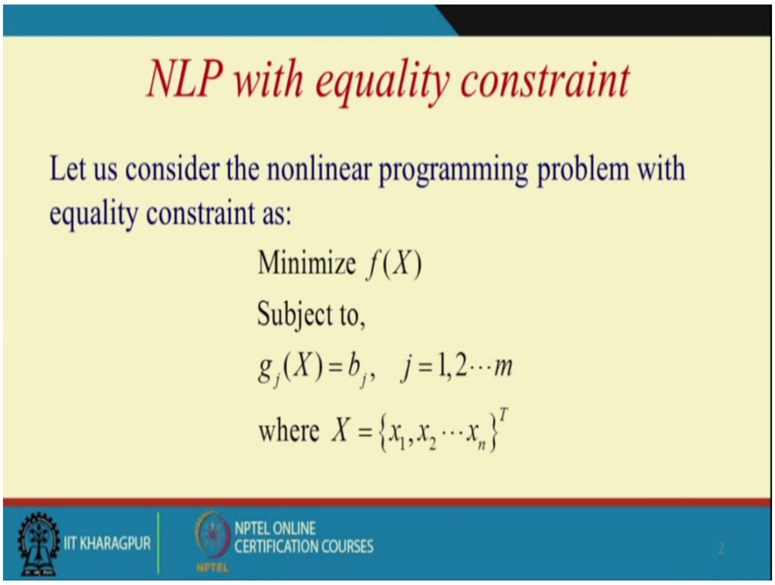

And today we are considering the general model of the non linear programming problem where we are having m number of constraints, all constraints of equality type and one restriction is there m is lesser than n. N is the dimension of the decision variable dimension of the non-linear programming problem and m is the number of constraints ok.

Let us extend the Lagrange method I explain to you for n dimensional problem with single constraint, the same method to be we will just extend it, but with different Lagrange multipliers.

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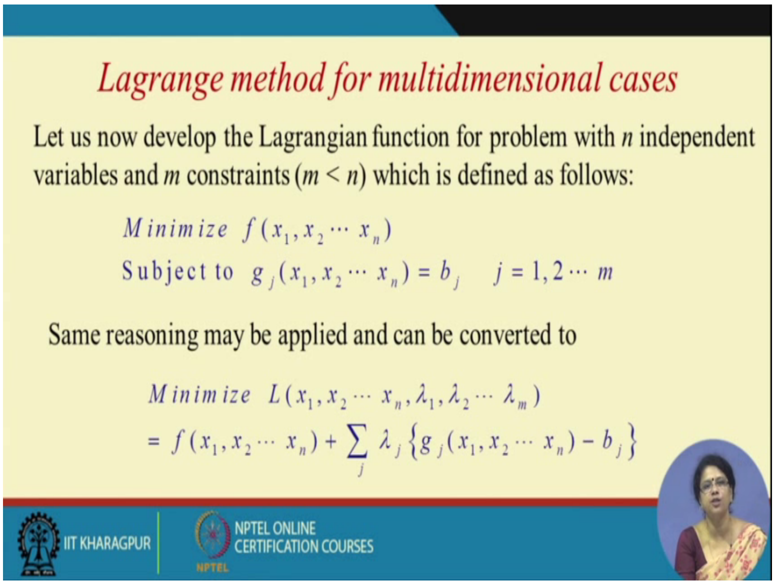

You see what I have written in this slide that, for the given problem multi di; n dimensional problem we can construct 1 Lagrange function, the function will have m plus n number of variables. Where n variables are the decision variables and m numbers are the Lagrange multi number of multipliers and as many number of constraints are there we are, we have that many number of Lagrange multiplier we will introduce in the problem, but one thing is that you see even if we have a problem with constraints we can convert it to a problem with single objective function.

Objective function may be complicated in nature, but the beauty of the Lagrange multiplier method is that so big thing we can convert mathematically with one objective function, but few things are there different Lagrange multipliers and you will see the different Lagrange multipliers will have different nature in the problem.

One thing you must be understanding that whenever we are going to find out any optimal solution for the non-linear programming problem or linear programming problem all the constraints are not active at the optimal stage. You must have been realized who are the active constraints, active constraint are the are those constraints which are contributing something with respect to the optimal solution; that means, for the linear programming problem you must have been seen that we are getting the optimal solution at the extreme points of the feasible space. Extreme points how we are getting the extreme points? These points were getting through the intersecting points of the constraints sets ok.

That is why the extreme, all the extreme points if I get the optimal solution and at one of the extreme point that does not mean that all constraints are involved there, there must have been at the, at minimum 2 constraints at there which are intersecting and we are getting the optimal solution. In non-linear programming problem also it is happening the same we are getting the optimal solution at the boundary of the region certainly, but all the time all constraints are not active at the optimal stage, but we are doing the calculations, there is no simplex method. In the simplex method we could find we can find who are the axtive, but here there is no methodology, but the Lagrange multiplier method is such a nice thing by looking at the Lagrange multiplier value we can say which is that tip constraint, which is not the active constraint that is a very nice thing for the Lagrange multiplier ok.

Let us do minimization of one with m plus n number of variables that is all, there is no other change only we have considered the summation of all lambda j gj minus bj.

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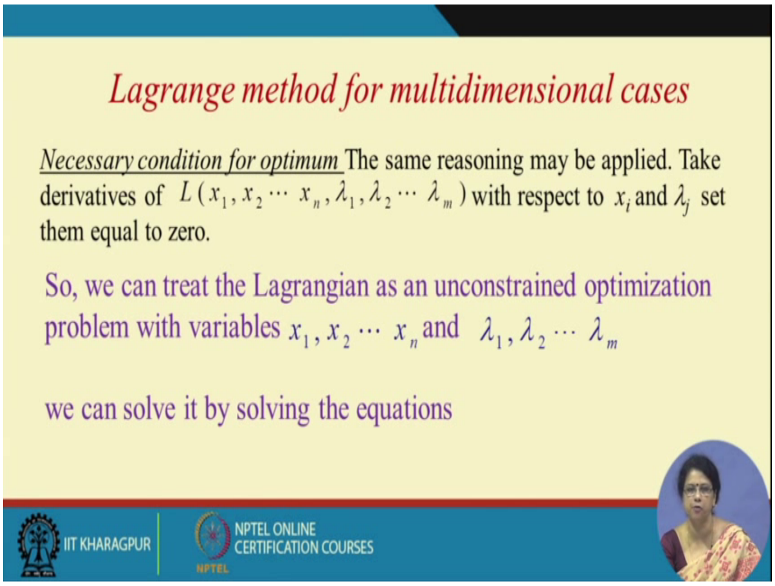

And here also we can say the same thing necessary condition would be the grad of L must be is equal to 0, from there we will get the necessary conditions. Then how many equations we will get from the necessary conditions? We will get m plus n number of the m plus n number of equations and m plus n number of variables from there we can find out easily or ha very complicated way, at least we will get solution of those that is fine that is a necessary condition for that, these are the necessary conditions.

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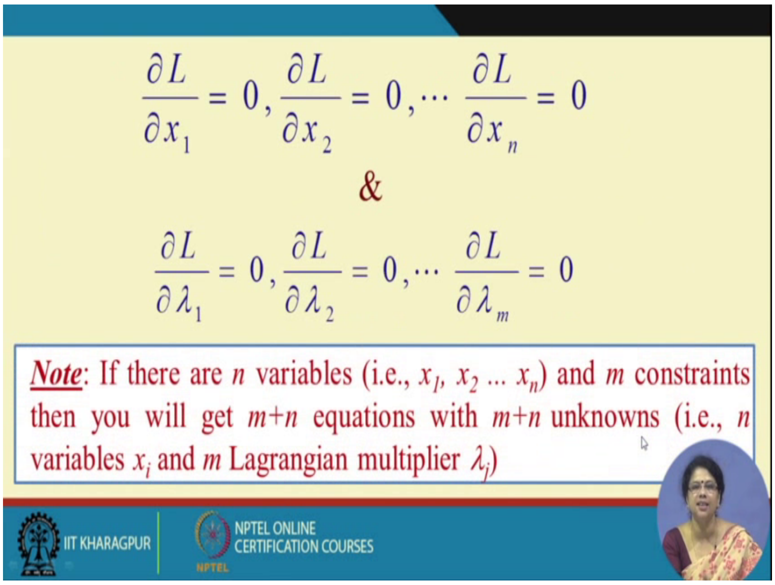

Now we need to check the sufficient condition as well, I need not to explain to you there at we would have to take the first order partial derivative of Lagrange's with respect to the decision variables, x 1 to xn and another set would be with respect to the Lagrange multipliers m depends on the number of constraints, I ex told you many times.

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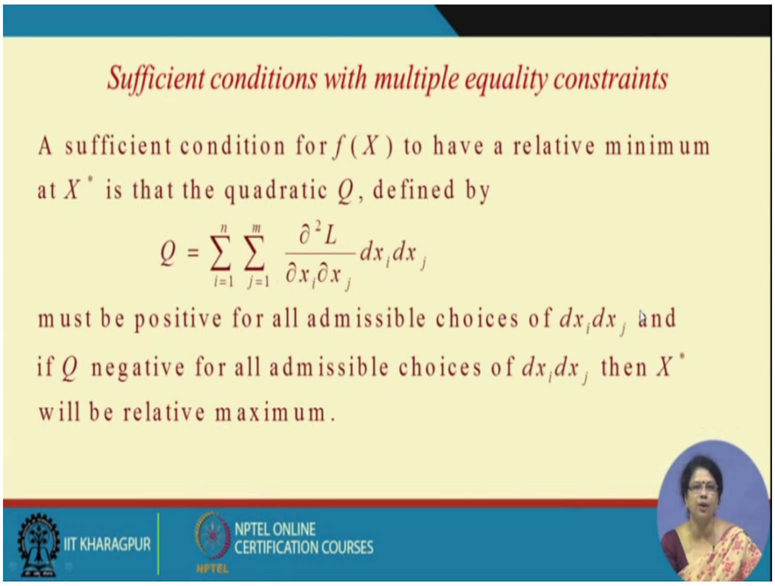

Now, if I just ask you what is the sufficient condition again it comes from the hessian matrix of l, but if I just write the hessian matrix that can be written in a quadratic form Q very nicely like this, del 2 L by del xi, del xj dxi dxi all right. The summation of all I is equal to 1 to n j is equal to 1 to m and since this is the value depending on the value of it we can say whether it has maximum or minimum with a particular choice of xi xj i and j running from 1 to n and 1 to m, clear. This is a sufficient condition mathematically it looks very odd, but this is nothing, but the hessian matrix the determinant value to it clear.

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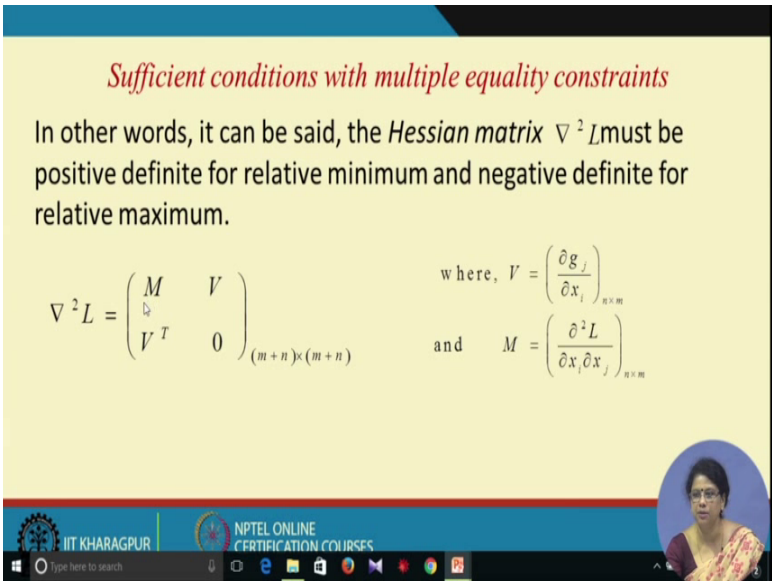

And this is the del 2 l, this can be if we just write the hessian matrix you will see there will be 4 parts to it you can just a hessian matrix you can partition it in 1 partition we will have m these are all of second order values. We will be our first order differentiation and there will be a set of 0s like this, we can formulate hessian matrix like this. If we see that the hessian matrix is positive definite minimum, negative definite maximum, nope. Otherwise no pattern, no conclusion we will see we will solve few problems in the class and few problems I will give you with different combinations in the assignment. So, that you can have a, have experience on it we can interact with each other even on that part, but I hope this is clear to you if we consider a 2 dimensional problem then we will get this one just this part you will understand I will solve 1 problem today itself, all right.

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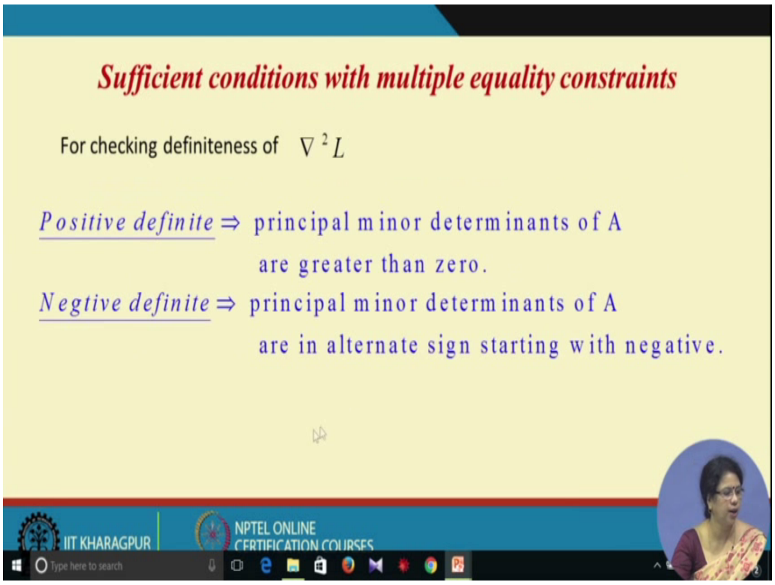

And this is the positive definite and the negative definite definition that depends on the principle minors and if we do not want to calculate the principle minors because it may happen that it will happen that the principle minors are very big values that is why we may go for the eigenvalue calculation wherever you are more comfortable go by that.

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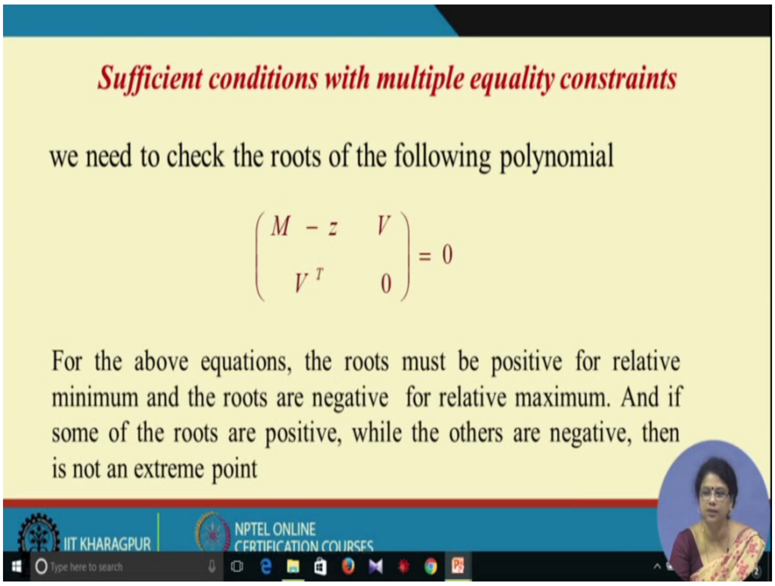

If eigenvalue are all positive, positive definite that way eigenvalue can be calculated from the characteristic equation that is why character equation can be taken as a minus z that part I need not to tell you just this is the simple part from the matrix algebra you can have the thing of it.

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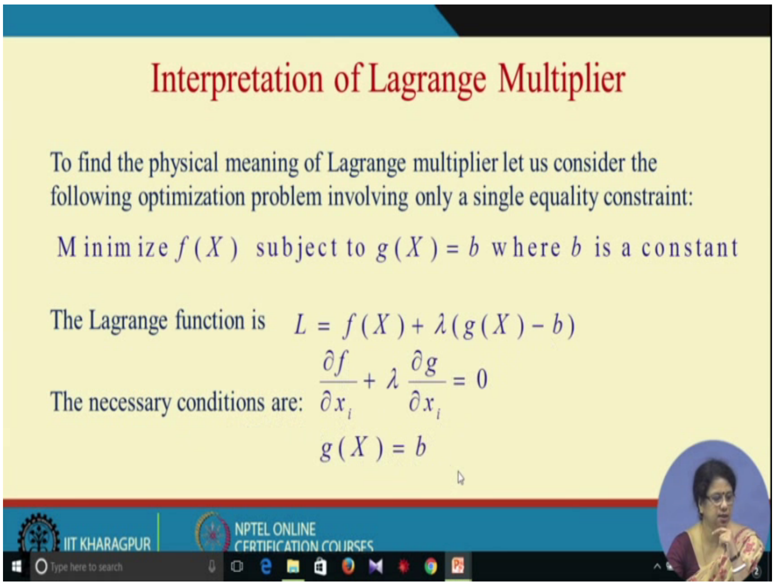

Now this I will explain you this methodology with 1 example in the next, but before to that I want to tell you what is the physical significance of Lagrange multiplier, just you see.

I am taking 1 simple problem minimization of f x subject to g x equal to b, to explain the Lagrange about the Lagrange multiplier, then the Lagrange function would be this one from there we can calculate the necessary condition as this one, all right all of for all I just I will do few calculations from their.

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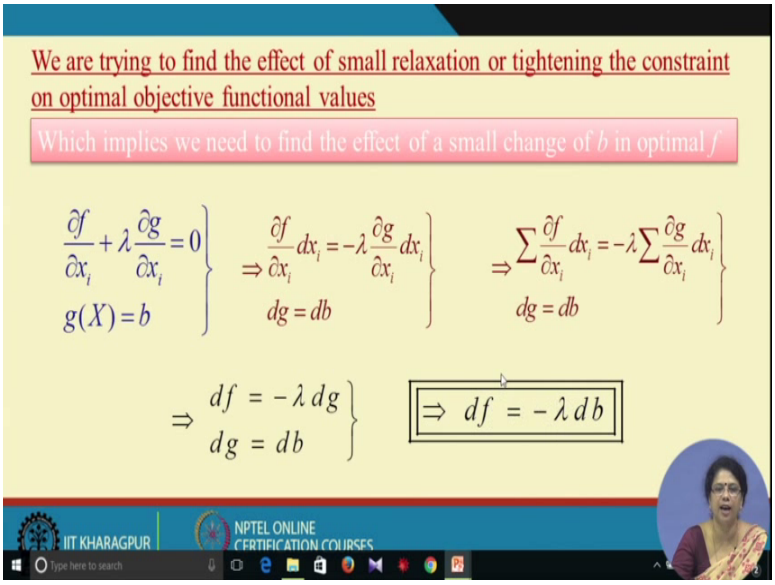

There just do you see what are these, these are the necessary condition, from here what we can say that if I just multiply d x i then we are getting del f by del xi d x i plus equal to minus lambda del g by del xi dxi. What is the meaning of it df is equal to minus lambda dg, all right and from here since g is equal to b we are considering dg is equal to db. You may ask me that b is a constant why we are differentiating, g is a function of several variables we can differentiate if we can do, but why we are considering the b because the significance of Lagrange multiplier is there when we we consider the sensitivity of the say it non-linear programming problem.

What do mean by sensitivity analysis of a of an optimization problem, you must have been done for the linear programming problem the sensitivity analysis what means that, if I change the feasible space little bit. If I just reduce the feasible space rate little bit or if we just extend the feasible space little bit how the optimal solution is being affected, the objective function is same, but we are changing only the feasible space, we are changing the feasible space means what? Your feasible space is constructed with ga g is equal to gx is equal to b if I make b is equal to b plus delta; that means, we are relaxing the feasible space. The feasible space will be expanded if I consider b is equal to b minus delta then my feasible space will be will reduce; that means, we are considering a smaller portion of the feasible space clear. That is why by changing of b if I take the objective function as the same 1 how it is affecting to the optimal solution, that sensitivity we are going to do depending on the value of lambda we will see a great deal of it we will get out of i,t that is why we are changing b that is why we can say that g is equal to b is a given constraint, but we can take the small increment in the both sides. That why if we have written dg is equal to db and we got df is equal to minus lambda dg, that is why what exactly we are getting df is equal to minus lambda dg rather df is equal to minus db that is the nice part of it.

This is the whole fact from here we can conclude what is the meaning of lambda; that means, if I change b it is affecting the objective functional value did you see, but if I change b in a positive direction by keeping lambda constant functional value will decrease, did you see that fact I am going to tell You in the next.

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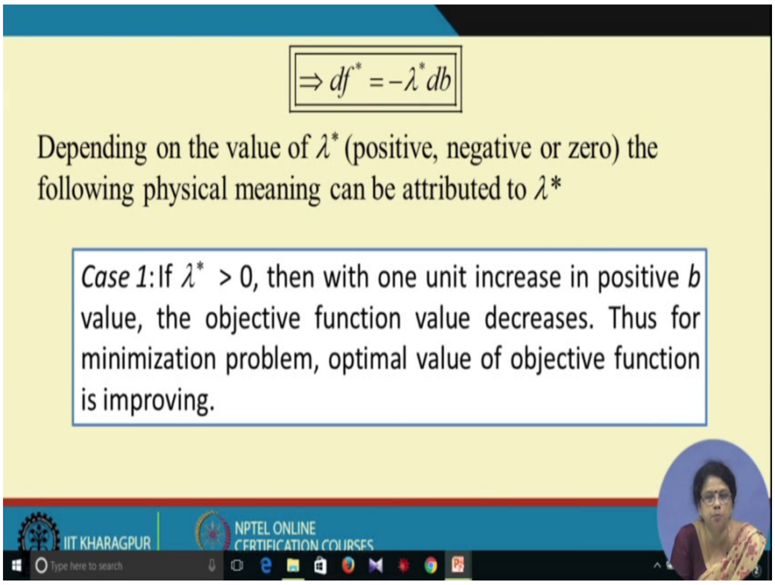

That is why at the optimal stage if I consider the lambda star is the Lagrange multiplier value at the optimal stage then we can say that df star is equal to minus lambda star db.

Let me consider lambda is equal to lambda star is positive if lambda star is a positive value then if I increase b, a 1 increase positive increase in b what will happen object if functional value is decrease, how much it will decrease? It will decrease of amount lambda star, did you see if I change b 1 unit then f will decrease lambda star unit. Do you see the nice connection of the Lagrange multiplier here, what else you can see? You can see that I am changing b, I am changing b means what? I am increasing b means; that means, I am relaxing the objective function then in that case we are getting better minimum value of the optimus of the optimization problem. What does it mean? It means that lambda gives me the gain in the minimum problem, that is why it is being said in economics lambda star is the marginal gain in the minimization problem, that is why always we can say lambda star if it is positive then if I increase b we will get better df ok.

But for the maximization problem the case is just reverse because lambda star is positive I am increasing b f is decreasing, that is why that is not a marginal gain for the minimization problem, maximization problem sorry. That is why it is being named as a marginal loss for the maximization problem whereas, it is the marginal gain for the minimization problem and very nicely the increment of the objective functional value can be associated with the Lagrange value, that is the nice thing of it. Just take the reverse case when lambda is star is less than 0 then if lambda star is less than 0, 1 increase in b positive increase in b what will happen the f value will increase; that means, it is bad for the minimization problem, that is why that is the loss for the minimization problem. But that is the gain for the maximization problem that is why if I just relax the space little bit for lambda star positive the corresponding constraint if I just relax then we will get better value for the maximization problem, but the worst value for the minimization problem.

Did you understand, in both the cases the cases are different, but in the other case neither positive nor negative if lambda star is 0 then what is the thing, df star is equal to 0 into db that is why even if I change b there is no effect on the optimal value of f all right.

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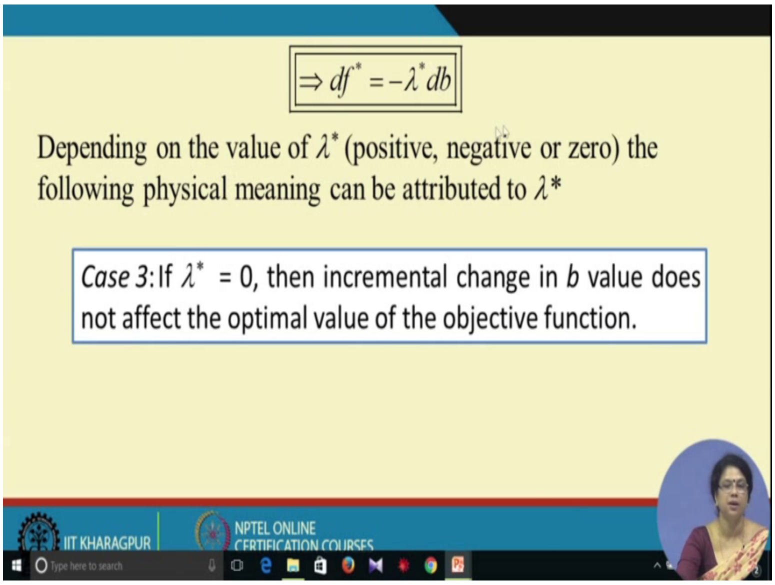

That is why it is being said that for lambda star is equal to 0 it has no effect if, even if I change b hope it is understandable, that is why lambda star is an amount that gives a lot of thing in the economic interpretation t hat is why it is being named as the shadow price in optimization problem. Now that is all about the interpretation of the Lagrange multiplier.

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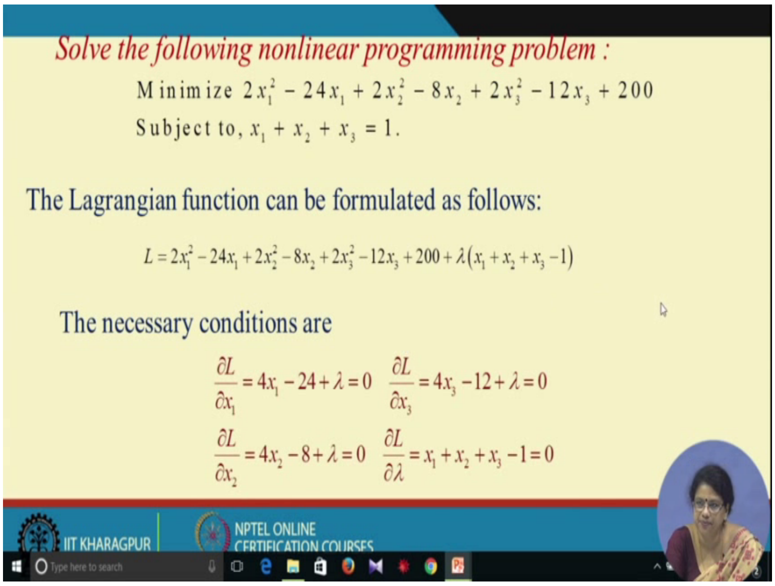

Now, I am applying the Lagrange multiplier technique for a problem where we are having the equality type of constraint and several decision variables, here I have considered only 1 decision variable this is of minimization type and this is a necessary condition for it all right. Now, necessary condition all are having the linear equation, 4 equations, 4 unknown very easily you can find out the values of it and this is the value for lambda, this is the value for this.

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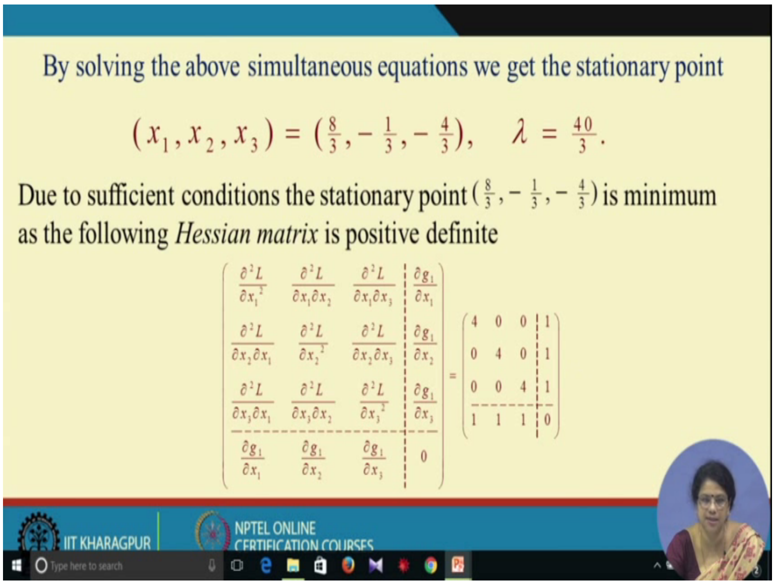

What did you see the value for lambda, lambda for lambda the value is coming positive for the minimization problem that is why if we relax the objective function, relax the constraint we will get better minimum value of the objective function we can see.

Now, you see here we are having only 1 constraint, if we have more than one constraint we will have different lambda values in different for different respective constraints and the effects we will judge we will see the same thing we will see in the Kuhn tucker condition. How it is being related, depending on the lambda value we can say many things. Now after that we will construct the hessian matrix for it and hessian matrix it seems its positive definite because a princip; you see this is the diagonal matrix and the values are all coming as a positive or either you can go for the eigenvalue calculations and if you just find it out we will see that the value will come as a positive 1 and 1 nm that is why the corresponding stationary point is the minimum point all right.

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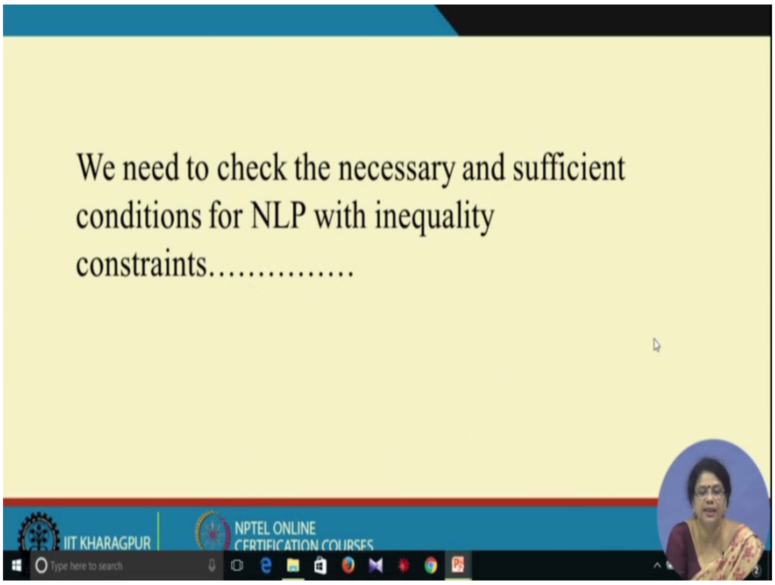

Now, in the next we have to see the non-linear programming problem with the inequality constraints, there also we will use the same Lagrange multiplier method and we will get the optimal solution for the problem. Let us solve for an optimization problem using Lagrange multiplier technique we are considering 2 constraints which are of equality type.

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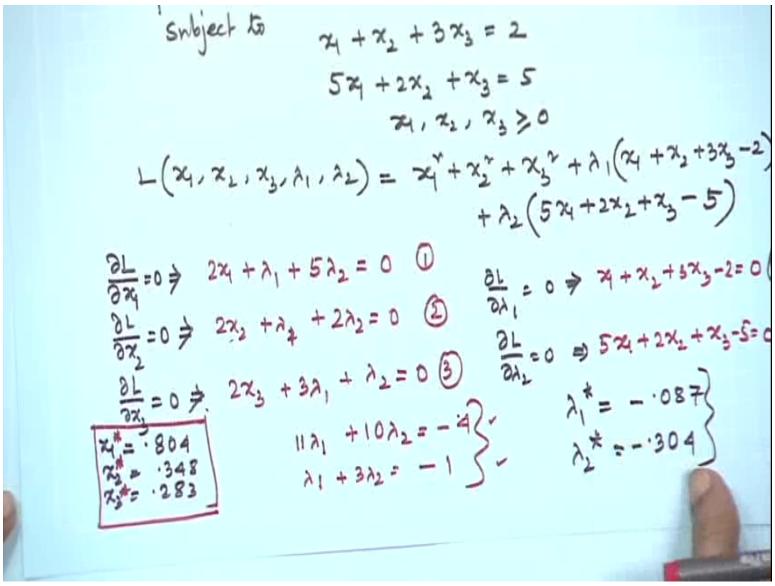

That is why the problem is optimized subject to x1 plus x2 plus 3x3 is equal to 2 and the second constraint is 5x1 plus 2x2 plus x3 equal to 5 and we are having the non negativity constraints of the decision variables all right. Now for solving this problem using the Lagrange multiplier technique we need to formulate the Lagrange function.

Now the Lagrange function would be the function of the decision variables and the Lagrange multipliers as well, we are having 2 Lagrange multipliers here, one is lambda 1 one another one is the lambda 2. Lambda 1 is attached to the first constraint, lambda 2 is attached to the second constraint and the Lagrange function would be x1 square plus x2 square plus x3 square plus lambda 1x1 plus x2 plus 3x3 minus 2 plus lambda 2, 5x1 plus 2x2 plus x3 minus 5. Now if this is the Lagrange function for us for finding out we have to optimize this Lagrange function.

That is why let me consider the first order necessary condition, first order needed necessary condition means del l by del x1 is equal to 0, it gives us let me use the other pen this gives us 2x1 plus lambda 1 plus 5 lambda 2 equal to 0 all right the second 1 del l by del x2 is equal to 0 gives us 2x2 plus lambda 2 plus 2 lambda, lambda 1, lambda 1 plus 2 lambda 2 equal to 0 and the third 1 del l by del x3 equal to 0. This gives us 2x3 plus 3 lambda 1 plus lambda 2 equal to 0 and we are having another 2 by considering del l by del lambda 1 equal to 0 del l by del lambda 2 equal to 0 and we are having 5 equations 5 unknowns with us minus 5 equal to 0 we are having 5 equations and 5 unknowns for this problem.

Now, if we just solve it how we can solve you see let me consider 1, 2, 3, 4, 5. Now if we just substitute 1, 2 and 3 into 4 and 5 then it will become 2 equations with 2 unknowns all right. If we substitute that then the first equation will come as 11 lambda 1 plus 10 lambda 2 is equal to 4 equal to minus 4 and the second equation will come as lambda 1 plus 3 lambda 2 is equal to minus 1. These 2 equations will come once again I am saying by substituting the value of 1, 2, 3 because you see we are having x1 is equal to minus lambda 1 minus 5 lambda 2 divided by 2, x2 would be is equal to minus lambda 1 minus 2 lambda 2 divided by 2, x3 equal to minus 3 lambda 1 minus lambda 2 divided by 2.

If we substitute this x1, x2, x3 this in this equation we will get the first equation and if we substitute the same x1, x2, x3 in the second equation we will get the second equation and from here we will get the value for lambda 1 star as this is the optimal solution rather, lambda 1 star is equal to minus 0.087 all right and we will get lambda 2 star is equal to minus 0.304 all right.

Once we know the value for lambda 1 star and lambda 2 star if we substitute in 1 we will get the value for x 1, this is equal to 0.804, x 2 would be is equal to 0.348, x3 would be is equal to 0.283 this is the optimal solution for the original problem. Now we do not know whether this pro, whether this solution is optimal or not that is why we need to formulate the, we need to consider the sufficient condition the second order derivative of l to consider the second order derivative of l you need to formulate the matrix like this.

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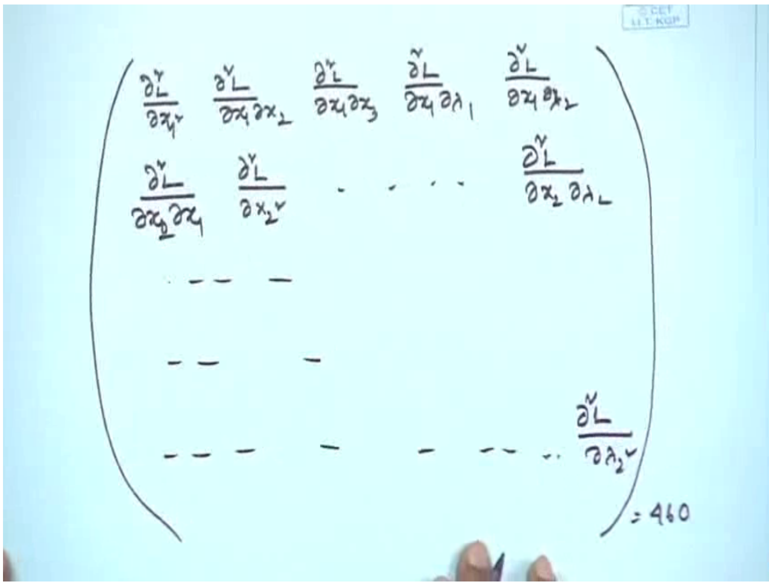

Del 2 l by del x1 square del 2 l by del x1, del x2, del 2 l by del x1, del x3, del 2 l by del x1, del lambda 1 del 2 l divided by del x1 del lambda 2.

Similarly, the second row would be del 2 l by del x1, del x del x2 del x1 del 2 l by del x2 square in this way this term would be del 2 l divided by del x2 del lambda 2 there will be 5 rows and the third row the last element would be del 2 l by del lambda 2 square. If you just find out this matrix that is the hessian matrix for you if we can find out this matrix whether this matrix is gives you the positive the definite, is it a positive definite matrix if it is a for this case you will have the positive definite matrix, then we can declare that it as a this value you will get as 460 actually. This is positive that is why the corresponding x1 star, x2 star and x3 star that gives you the minimum value of the problem all right this way we can solve let me consider another problem.

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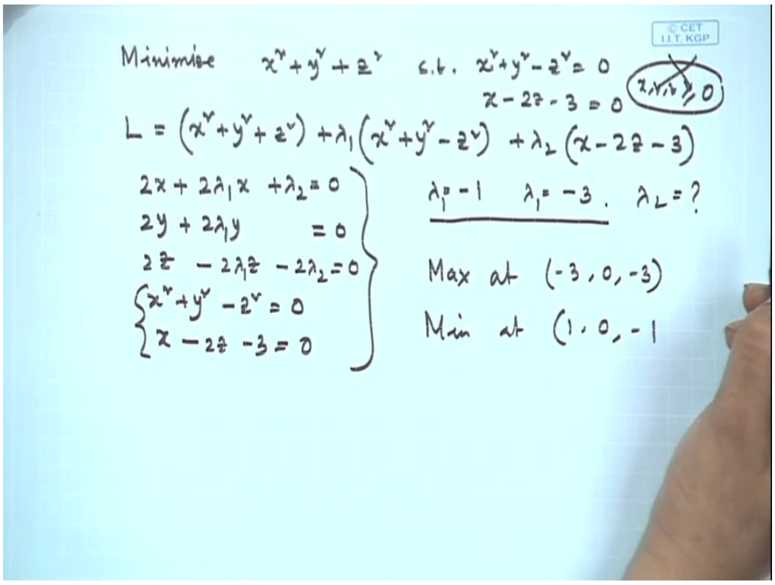

Now, we have to minimize x1 square plus ones 1 square y square plus z square subject to x square plus y square minus z square equal to 0 and another constraint we have x minus 2 z minus 3 is equal to 0 here also we will formulate the Lagrange function. Now the Lagrange function would be x1 plus y squa; x square plus y square plus z square plus lambda 1 x square plus y square minus z square plus lambda 2 x minus 2 z minus 3 all right.

Again here also we will have the first order necessary condition, that is why the first condition will be 2x plus 2 lambda 1x plus lambda 2 equal to 0, the second 1 would be 2 y plus 2 lambda 1 y equal to 0, third would be 2 z minus 2 lambda 1 z minus 2 lambda 2 equal to 0, 4th will be x 1 plus x square plus y square minus z square equal to 0 and the fifth would be x minus 2 z minus 3 is equal to 0.

Now you see we are having 2 constraints together 1 is lambda 1 and another one is the lambda 2. Now if we substitute in the similar way if we substitute the value of x, y and z in this cases we will have different combinations of lambdas in the 1 combination will give us lambda is equal to minus 1 and lambda is equal to minus 1 and another can be another value will have lambda is equal to minus 3. We will have these 2 values lambda 1 lambda 1 we will in the first in 1 case we will have lambda 1 is equal to minus 1 another case we will have lambda 1 is equal to minus 3. Actually for different values of once you are having this one you see if you substitute the value here you will have the equations this is of order 2, that is why you will get different values of lambda 1 and lambda 2 and if you get the different values of lambda 1 lambda 2 few will be possible case and few will be impossible case and that you need to check.

And you need to check whether it is say all the conditions like xyz the non negativity constraints are being satisfied or not. For different cases you have to check and you can conclude that we will have the maximum for this problem we will have at minus 3 0 minus 3 and in the second case minimum value you will get at 1 0 minus 1. Actually for this problem x and y and z there is no restrictions on this all right, only you need to check after getting the value of lambda 1 and lambda 2 you need to check the values whether we are be satisfying the constraints or not that is the check you have to do if the constraints are not being satisfied; that means, you are out of the feasible space the corresponding combination of lambda 1 and lambda 2 you would not consider.

Now, on these I will give you more problems with the solutions as well generally these problems are very big in nature because different cases arise as you have seen I have solved the kkt problem where I will solve it there also many cases will come for combinations of lambda 1, lambda 2 you will consider those and we will provide you solution as well if you have any further query you can ask us.

Thank you.