Constrained and Unconstrained Optimization
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Lecture - 40
Unconstrained Optimization

Now let us consider today again another method that is a interpolation method, this interpolation can be interpolation, the definition of a interpolation I need not to say, you must have some idea you have gathered from your previous knowledge, interpolation means we are trying to find out the trend of a given set of data. Now we are finding out the interpolating polynomial, here also the same thing we are having a function that we need to optimize that is either maximize or minimize.

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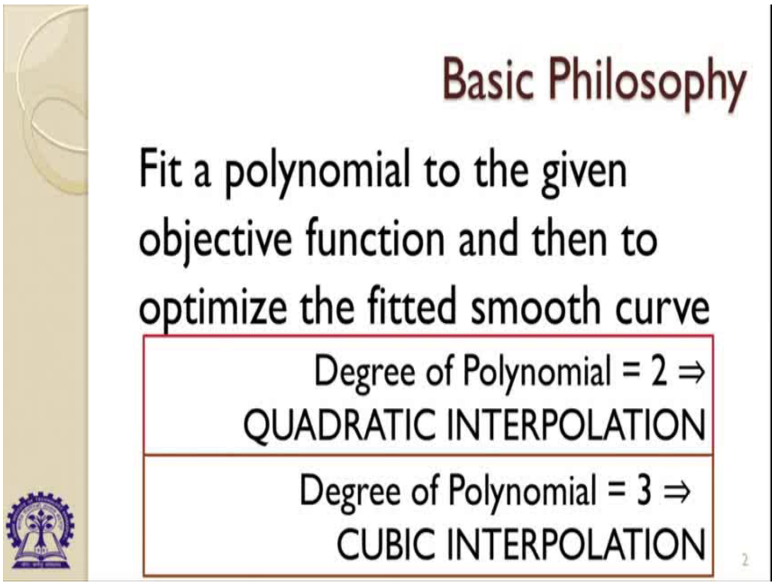

And we will use consider the interpolation technique here, how there are different methodologies available, if we interpolate a quadratic polynomial then that is a degree of the polynomial is 2 then that is called the quadratic interpolation, if the degree of polynomial is 3 that is called the cubic interpolation technique.

Only our idea is as I it is written that to fit a polynomial to a given function and not really we are going to find out the optimum value for the function given, rather we will find out the optimum value of the interpolating polynomial because that is much easier for us to find out the optimal solution.

Once again I am repeating basically philosophy behind, it we are given a complicated function and it has been asked that we need to find out the minimum of the minimization, we have to do the minimization of the function rather the minimum local minimum of function or we need to find out local maximum of a function. The function is so much complicated we are not really able to handle the function, in that case what we do we will interpolate the best fitted polynomial there and our task is to find out that polynomial only. If we fit a quadratic polynomial that is the quadratic interpolation, if we fit a cubic polynomial then that is called the cubic interpolation method.

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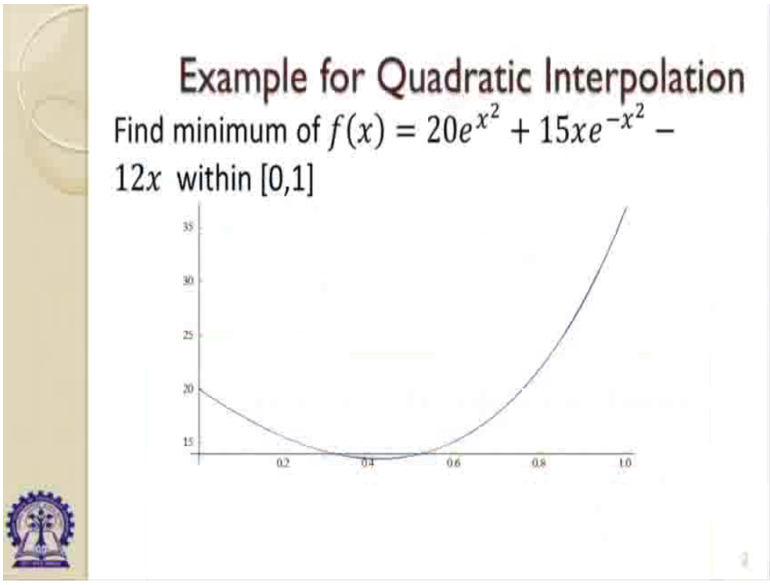

Let me tell you how really we are doing it the methodology I will explain to you as well as I will explain to you the example with examples the methodology. Let us the function is this one, f x is equal to 20 to the power x square plus 15 x, e to the power minus x square minus 12 x and we need to find out the minimum of this function within 0 to 1. This drawing of this function I have done with the software mathematica and through the mathematica we have seen that there is a minimum around point 4, you could see. Now, there is no negative value as such as since the range for y has been given from lower value of 15 that is why its look so that the function is going to the, below to the x region x axis. You can try it also in with some other software how the function looks like, but if I ask you why do not you find out the minimum of this function. You can use different method as you have learned before you can use the Fibonacci, you can use the golden section interval halving anything exhaustive search you can do, but you see with a quadratic interpolation so quickly we will get the solution of it.

Now if I ask you then what should be the polynomial the, of degree 2 which is best fitted for this function, how would really we will start our journey? We will see that let us the polynomial is a naught plus a 1 x plus to x square what is unknown to us, unknown is a is a a naught a 1 and a 2, that is why if we need to find out the value for a nought a 1 and a 2 which is best fitted to this polynomial. This is the target of us for today's class, to fit a quadratic polynomial to this function and we will find out the minimum of that polynomial.

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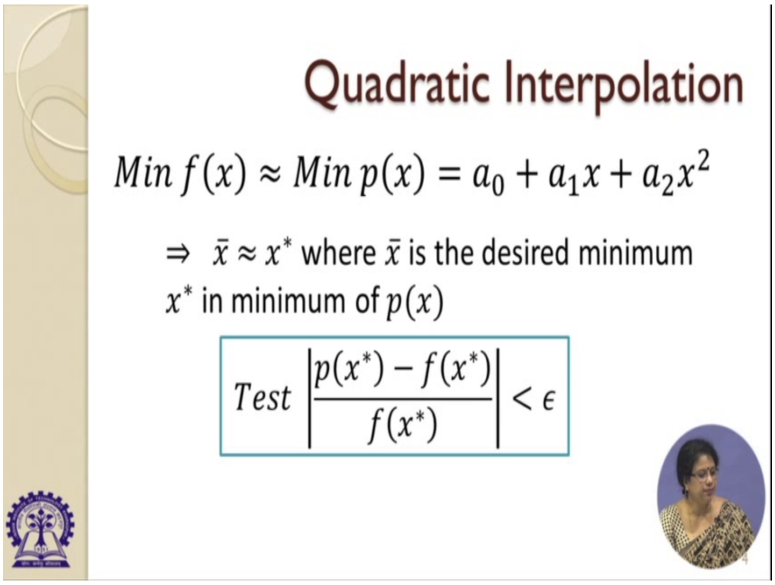

Now, if this is so then in the quantitative interpolation we can say that after we are interested to determine the value of a naught a 1 and a 2 such that minimum of function f x must be almost same as the minimum of the p x, p x is the fitted polynomial all right.

If x bar is our desired minimum then that must be very closer to x star that is x star is the minimum of the fitted polynomial all right and what how we can test it, we can just take the ratio of this difference and if this value is very small, lesser than very small value epsilon then we can say that the fitted polynomial is accepted. That is why this is the target of our methodology to fit a quadratic polynomial.

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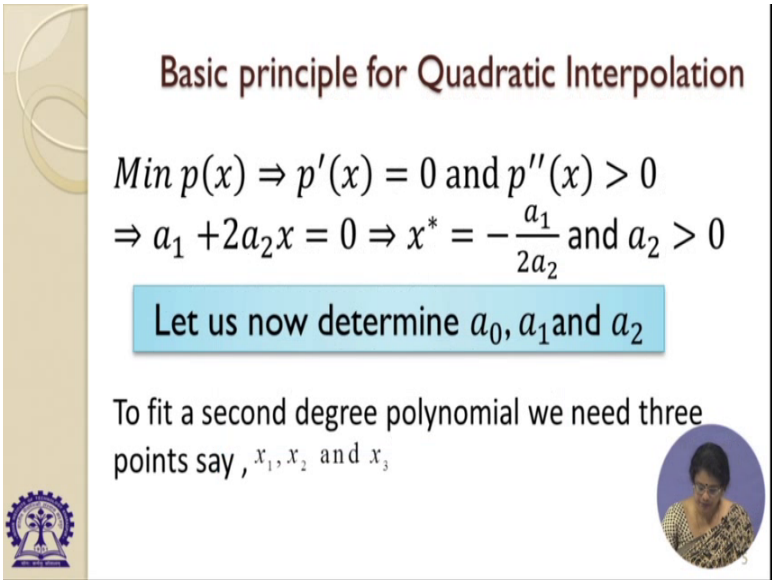

Now, you see we are having a function that is the function for us all right.

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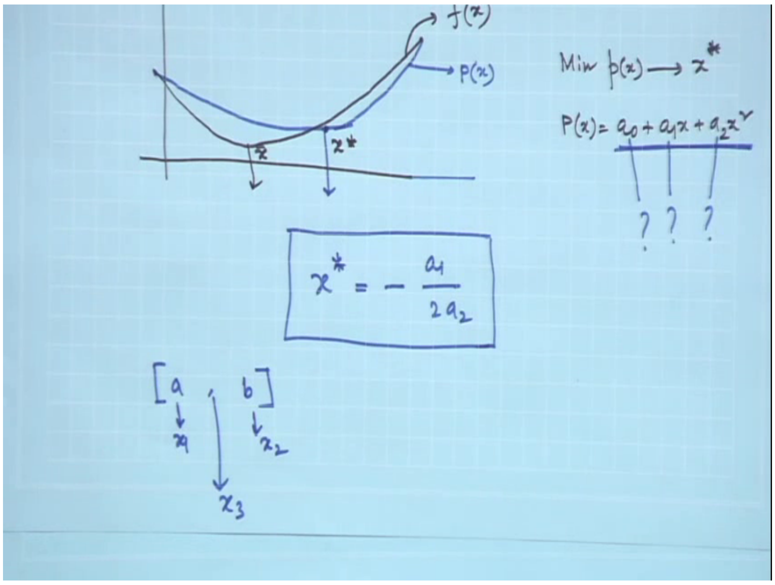

We are interpolating a polynomial, this is within these 2 points we are interpolating this polynomial say a quadratic function, all right. Quadratic function can be considered as a parabola a simple quadratic function and if I just find out the minimum of this polynomial this is my fitted polynomial p x and this is my given f x all right.

What we can see that minimum of p x is coming almost here, but minimum of f x it is expected here that is why we cannot say that this is my x star and this is my x bar, we cannot say that these are closer, very much closer that is why this fitting is not accepted at all, all right. We need to find out another polynomial with, which must fit to this one very nicely that is our target, that is why what we will all work with p x we would not do any work with f x. If we say that minimum of p x is there at x star then what is the necessary and sufficient condition, the necessary condition would be first order derivative of p with respect to x must be 0 and the second order derivative of p with respect to x must be positive. That is why it has been written here, that is why if we consider the first order derivatives of p x your p x is a naught plus a 1 x plus a 2 x square.

Then the first order derivative if we consider it is coming a 1 plus 2a 2 plus into x is equal to 0, in other way we can say that x star is equal to minus a 1 by 2a 2, that is the important thing we will just use everywhere. That for the fitted polynomial that would be the optimal solution and what else we need to check, we need to check always the second order derivative of p that is coming if it is positive then a 2 must be positive, otherwise it cannot possible, it is not possible in this way we will determine. Now if we are having a function of this kind where 3 things are unknown then how many points we need to passed to this function, we need 3 points then only we will get 3 equations with 3 unknowns we can get the solution, let us consider 3 points are given that is x1, x2 and x3.

Generally in the given problem if the initial interval of uncertainty is a to b we consider 3 points as a is the, a as x 1, b as x 2 and the middle point of a and b as x 3 this is the, this we considered, here also we will do the same thing just.

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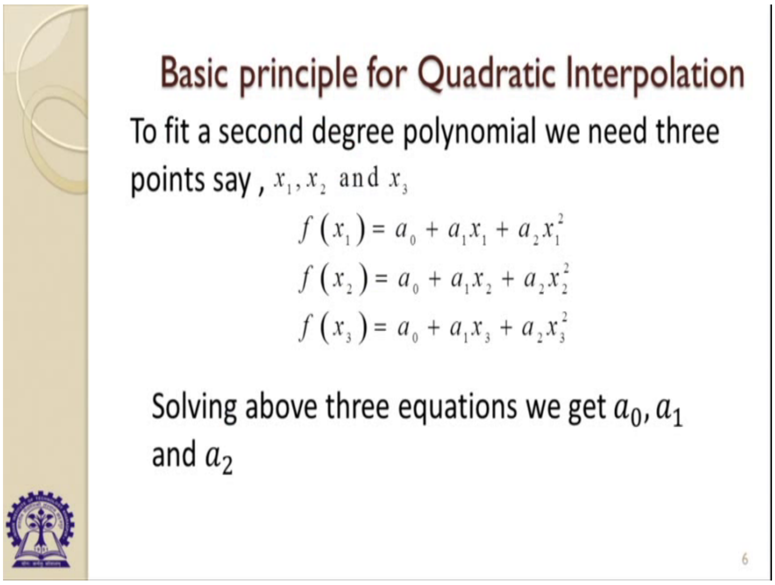

Let us see how the algorithm for this as I say that to fit a second degree polynomial x1, x2, x3 and since we are fitting a polynomial to the function that is why we will expect that f x 1 must be almost same as p x 1, f x 2 is almost same as p x 2, f x 3 is almost same as p x 3. Since we are considering 2 end point certainly f x 1 will be same as p x 1, f x 2 must be same as p x 2, but the middle point will vary that is the thing all right.

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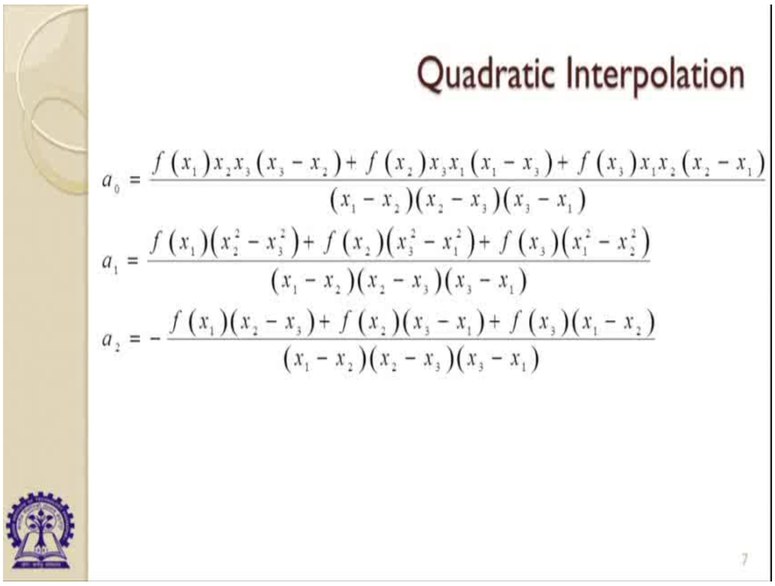

And from there if these are 3 equations just you do certain calculations even by writing a program or by using the software mathematica also, you can find out the unknowns of 3 where 3 equations are given to you. Otherwise very difficult to remember this formula, if you can you can use this formula as well this has been we got it after simplification after simplifying 3 equations, previous 3 equations just look at 3 equations look at this and from here we found out the value for a naught a1 and a2; that means, we will if we pass 3 points and we will the value for a1 a naught a1 and a2, let us try to apply to this function.

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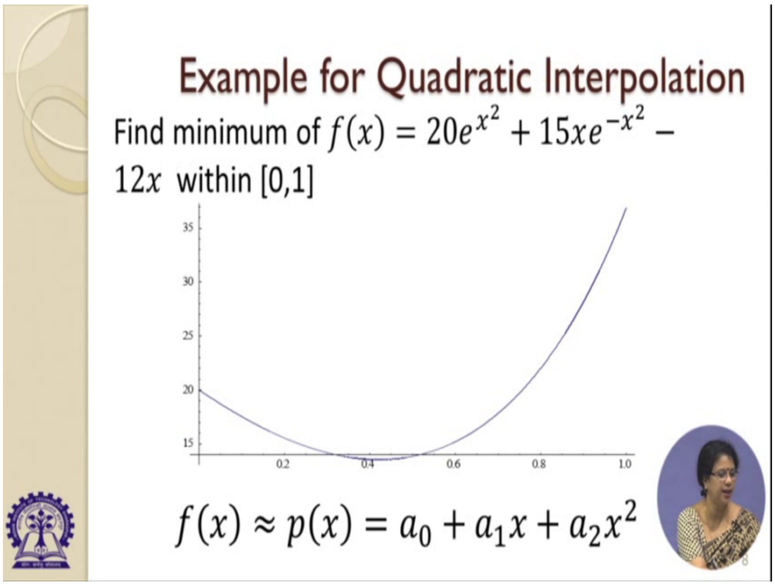

Let us see what are the values for a naught a1, a2, but you need to know what the algorithm for this function as well.

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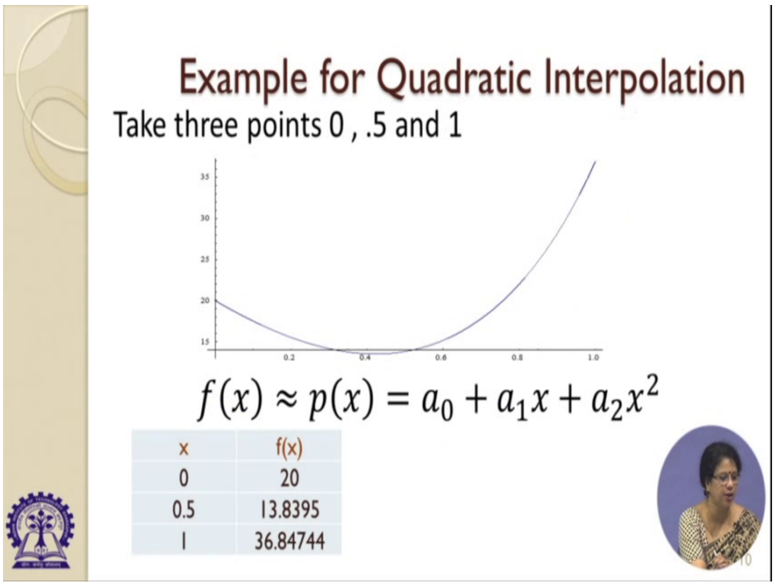

But just by considering 3 points a 0 is the given, 0 is the given value yes 0 to 1 is the initial interval of uncertainty.

That is why 0, 1 and 0.5 if we consider these 3 then you see the functional values there if this is so we will get 3 equations with 3 unknowns, what are the 3 equations and 3 unknowns?

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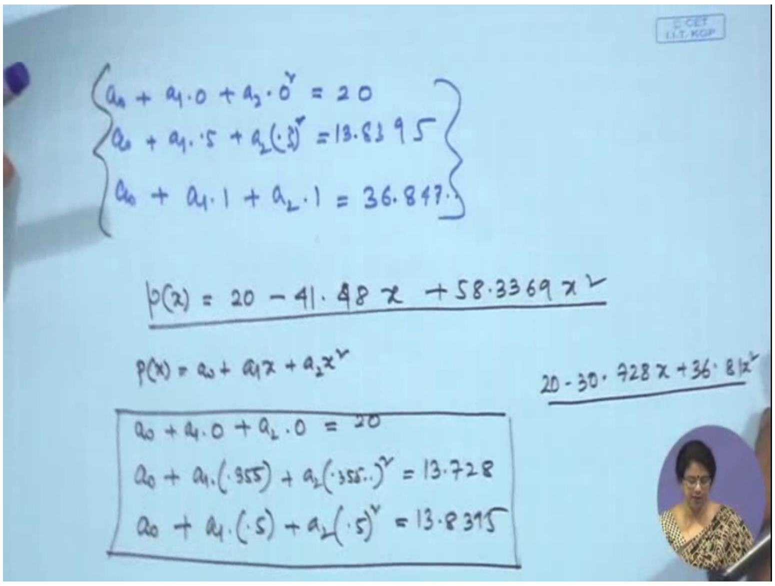

Certainly it will be a0 plus a1 into 0 plus a2 into 0 square is equal to 20 and a naught plus a 1 into 0.5 plus a 2 into 0.5 square is equal to 13.8395, a 0 plus a1 into 1 plus a 2 into 1 is equal to 36.847 etcetera. With these 3 equations and 3 unknowns we will get values for a naught a0, a1, a2 I will show you what are the values and how we are going to fit the polynomial in the next part.

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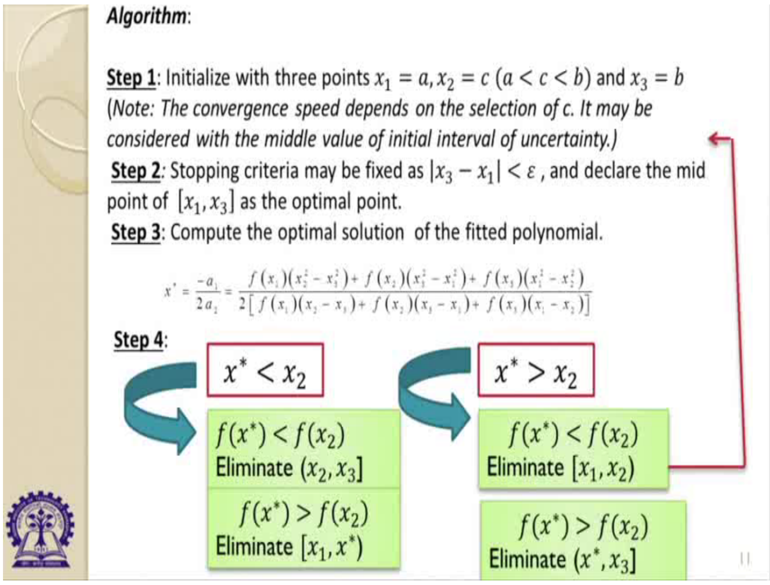

But before to that let us me tell you the algorithm whatever I say that had been, that has been given in the algorithm nothing more than this has been given you that initialize the 3 points x1 is equal to a x 2 is equal to c and x3 is equal to b and the stop what should be the stopping criteria here. Again that question of convergence is coming if the it will final interval of uncertainty smaller we will stop there and we will declare the midpoint of that final interval of uncertainty as the optimal solution.

That is why here also the same thing since we have considered that left hand point as x 1 and the right hand points are x 3 the difference between x3 and x1 the absolute value must be lesser than epsilon. Then we can say we can stop our iteration, if we are not happy with the this difference if you are if you want the difference must be much smaller than this you can proceed further. Now for computation of the optimal solution as I said x star is equal to minus a1 by a 2 and this value we are getting as I showed you.

If you remember the formula you can if you can remember the formula just take those formula and substitute otherwise you do the calculations on your own, that is another possibility, but whatever it is we will get another x star that must be in between a to b. But we do not know where the where x star is situated it may happen that x star is a less than the mid value; that means, in between a and x star and it a and x 2 or it may happen x star is in the right part of the mid value, there are 2 possibilities. If x star is lesser than x 2 there again 2 possibilities may happen, that f x star must may be lesser than f x 2 or f x star may be greater than f x 2 if x star is lesser than f x 2; that means, we are finding out the minimum of the function.

There we can say that there is no minimum in between x2 and x3 because the functional trained value showing that x star is lesser then x2 value is more and then f x 3, it must be more value that is why function is unimodal, first of all then only otherwise not all right and if f x star greater than f x 2 very easily we can say there is no minimum in between x x 1 and x star we can eliminate that region. And similarly for the other 2 cases if x star is greater than f x x 2, but f x star is lesser than f x 2 there is no minimum in between x 1 and x 2 we can just eliminate that. There is no minimum between x1 and x2 [FL] it is there is no minimum in between x1 and x star all right and in the next case when f x star is greater than f x 2 there is no minimum in between x star and x 3 ok.

After elimination what we will do, we will again go back to the step 1 because we are defining new interval of uncertainty we are defining the interval where the left point points we are considering we are redefining right hand point we are redefining and we are having another point in between as well again we will repeat the process.

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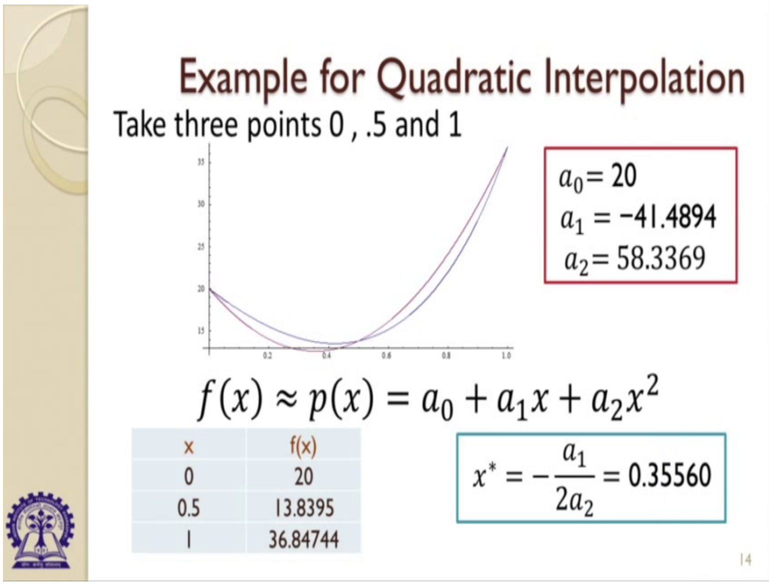

Let us see with the example the same thing, this is the function and we this is the function rather as I showed you previously all right and we are, we found the value for a naught is 20 a 1 is minus 41.4894 and a 2 is 58 point this is after solving these 3 equations only we got it, because there are 3 unknowns and 3 equations from here we are getting the values for a naught.

Once we are getting the values for a naught a1, a2 then we can say what is my p x then, p x must be is equal to 20 minus 41.548 or so, x plus 58.3369 x square, this is my polynomial. I have drawn the polynomial just you see with the blue the red is the function and blue is the fitted polynomial, I hope you are not really satisfied with this polynomial because this is not fitting function very nicely and if I want to get the minimum of this polynomial then we could see the minimum of the polynomial is 0.355 and that cannot be the desert minimum at all that is why it is not accepted.

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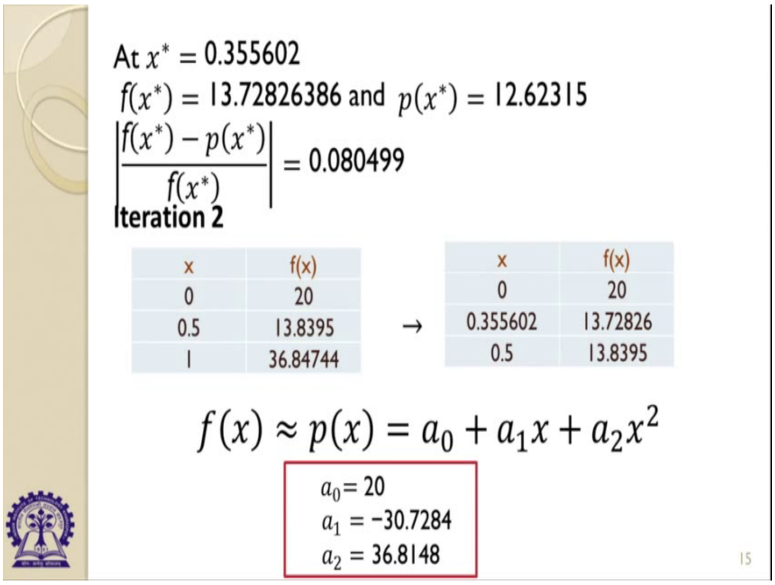

We are going to the next, how really we have checked mathematically a entire thing we found that x star is equal to this f x star we calculated, p x star we calculated and we were testing f x star minus p x star divided by f x star is a large value that is 0.08 we expect the value must be 0.0001 or like that.

Then only we can say this is a good fit otherwise not a good fit, that is why we have decided we will go to the next iteration once we are doing that we could see the value for function is 20, here is 13, here it is 1 and at the extract the value of f x star is coming 13.728. That is why we can say that x star is lesser than x2 lesser than 0.5 f x star is lesser than f x 2 that is why there is no need to consider the interval 0.5 to 1 because there cannot have any minimum. Function is unimodal within that that is that is why they are cannot have any minimum that is why let us consider the new interval of uncertainty as from 0 to 0.5 all right and the value the new value x star we have considered as the next point.

Not the mid value of these 2 endpoints all right, then if we have again 3 equations with 3 unknowns how really because in the next iteration this is my x1 this is my x2 this is my x3 we will just substitute in the function here like this. We are having the polynomial p x is equal to a naught plus a1 x plus a2 x square then if this is so then a naught plus a1 into 0 plus a2 into 0 is equal to 20 a naught plus a1 into 0.355 plus a 2 into 0.355 square is equal to 13.728 the third equation is a naught plus a 1 into 0.5 plus a 2 into 0.5 square is equal to 13.8395; that means, again we are getting 3 equations 3 unknowns and we will get.

The values for a naught a1 and a2 from these equations these are the new values of a naught a 1 and a 2 all right; that means, we are getting another polynomial, what is that polynomial? Polynomial is 20 minus 30.728 x plus 36.81 x square this is my polynomial.

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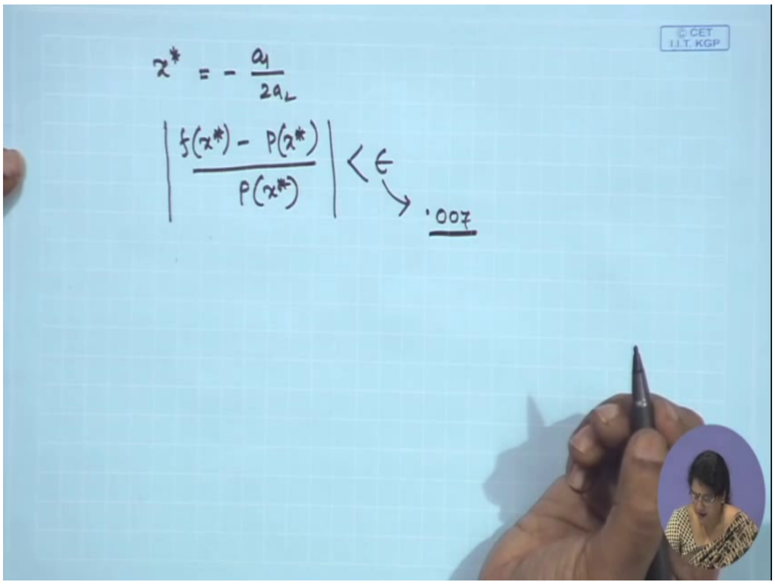

Let us see whether this polynomial is fitting the function or not, for checking that thing you need to find out the x star value, what is your x star value? That is equal to minus a1 by 2a 2 and we need to find out that f x star minus p x star divided by p x star must be less than epsilon ok.

Then only we can say that if the epsilon value is very small then we can say that this is a good fit, just we found for these values it was coming 0.007 or so. We got that value, you can calculate also all right and if we calculate this value if you are happy you can stop the iteration if you are not happy again you can go for the region elimination technique and you can get the better polynomial.

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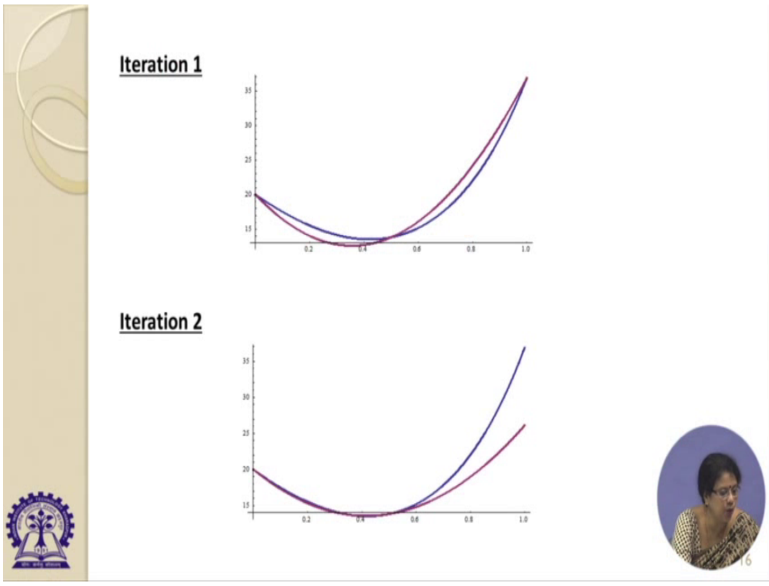

But if we just look at the graph of it the, that here the red 1 is the original function and the blue 1 is the fitted polynomial, in the first iteration that was the fitted polynomial and in the second iteration the blue is blue and red just to see all are almost mixing together. If you just zoom it maybe you may not be satisfied with that, but at a glance if you are satisfied you can stop the iteration here all right here, since the value is 0.007 you can stop ok.

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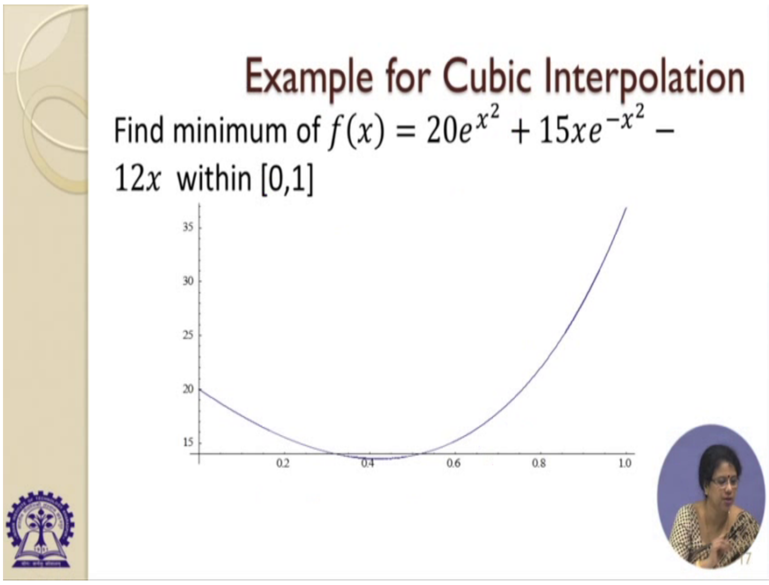

In this way we just do the quadratic polynomial interpolation, but if you want to do the next.

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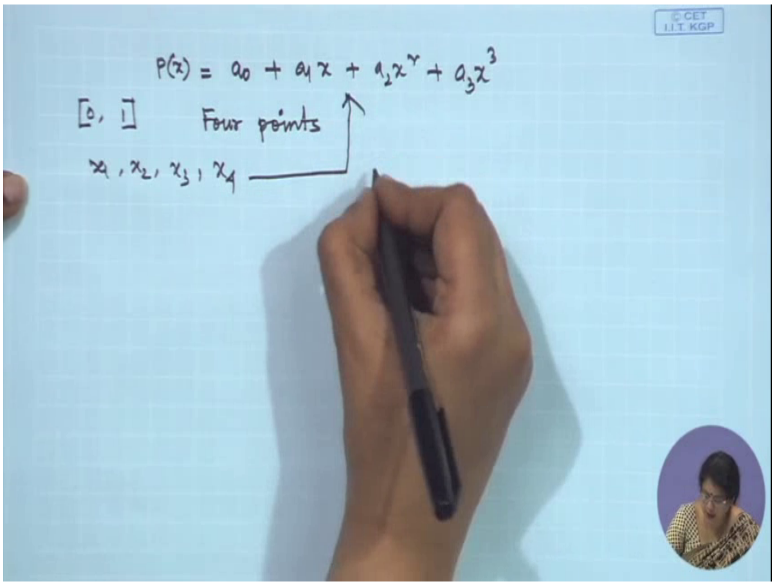

We if you want to just interpolatic cubic polynomial there for the same function that also we can do, how really we can do it? We will take the polynomial as a 0 plus a1 x plus a2 x square plus a3 x cube all right. Now we are having 2 points, 0 and 1 for finding out the values for a0, a1, a2 and a3 how many points you need you need 4 points, then only we will get 4 equations with 4 unknowns you can find out the value for a0, a1, a2 and a3 and again also the we can have x1, x2, x3, x4 and if I just proceed in this way then we can again fit a nice cubic polynomial here like this that is all for today.

Thank you.