Constrained and Unconstrained Optimization

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Lecture – 04
Graphical Solution of LPP - I

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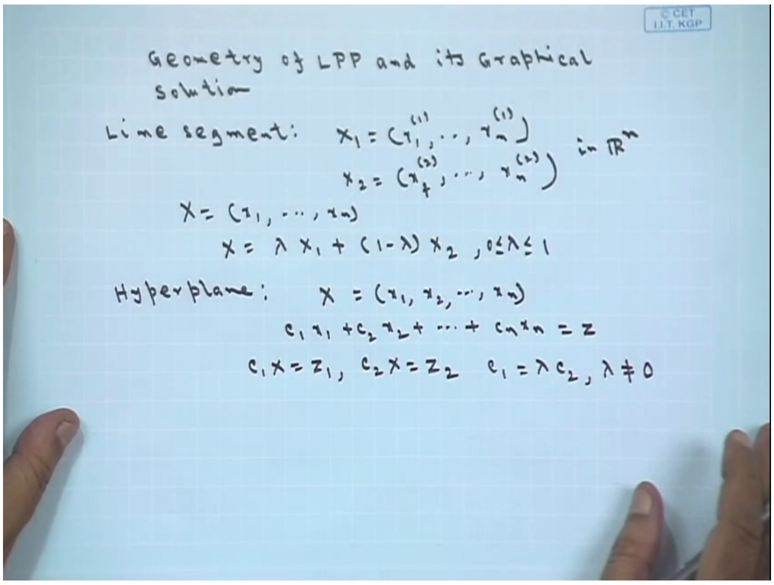

Now, let us start the geometry of LPP and it is graphical solution. Geometrically how LPP looks like what objective function actually looks like geometrically or the constraints, and what about the feasible solution all this things let us discuss first. And then we will go to the graphical solution of LPP and we will only go for solution graphical solution of LPP for 2 variables only because 3 variable or more variables it becomes difficult to visualization. First let us go with certain definitions one is line segment suppose you have 2 points x 1 like this way. X n 1 and x 2 equals x 1 2 like this way x n 2. You have 2 points in r n x n x 1 and x 2. Then the line segment joining this 2 points x 1 x 2, is the collection of points x is the collection of points x which can be written as x 1 x 2 x n, such that x equals lambda x 1 plus 1 minus lambda into x 2, where lambda lies between 0 to 1.

So, for any 2 points x 1 and x 2 in r n that is if you have x 1 1 up to x n 1 and x 2 equals x 1 to up to x n 2 in r n. In that case the line segment joining this x 1 and x 2 is a collection of points of the form x where x is x 1 to x n which will satisfy the equation x equals lambda x 1 plus 1 minus lambda into x 2 lambda lies between 0 to 1. Hyperplane, next one is hyperplane. You have a point x which is nothing but x 1 x 2 x n. If this points satisfies this one C 1 x 1 plus C 2 x 2 plus C n x n equals z. Then this line this equation represents one hyperplane for a given values of cis and j please note this one, for a given values of cis and z this equation will define one hyperplane. Or in other sense basically whenever I am supplying some values of C 1 C 2 C n and the value of z, then the objective function is representing one hyperplane and in 2 dimension how it looks like then in it will look like a line and that we will see whenever we are trying to find out the solution of the graphical solution of the problem of 2 variables.

You may have the parallel hyperplane also, that is say C 1 x equals z 1 and C 2 x equals z 2. This 2 planes will be parallel sorry, this 2 hyperplanes will be parallel if C 1 equals lambda C 2; obviously, lambda is not equals to 0. Or n other sense 2 hyper planes will be parallel if they have same unit normal. If they have same unit normal C 1 by C 2 equals some constant.

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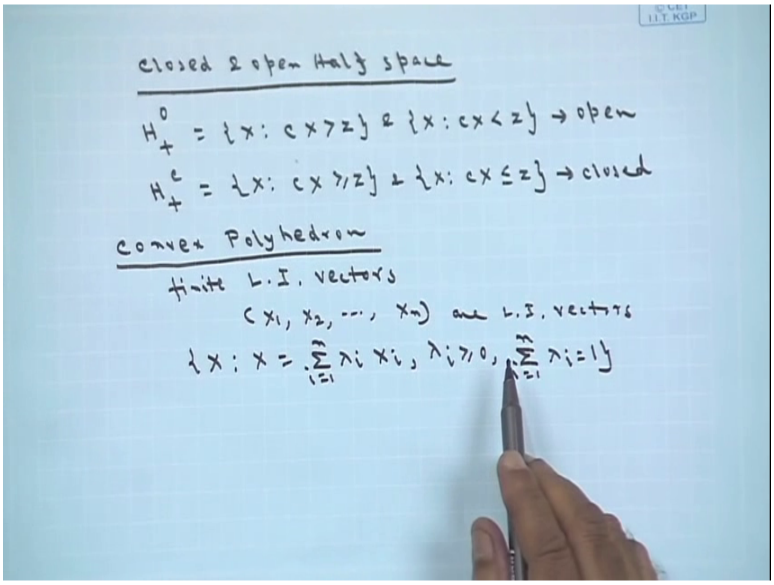

Next one is closed and open half space, closed and open half space. I am denoting it like this means open. So, x such that C x greater than z and x such that C x less than z this we are calling as open half space. And similarly closed will be x such that C x greater than equals z. So, you can understand the difference and x, such that C x less than equals z this we are calling is closed. Just like open set and closed set you are including this point or not. Like this way we are defining it as a open or closed half space.

The next definition is convex polyhedron, you have a finite number of please note this one. Finite number of linearly independent vectors. If you have a finite number of linearly independent vector, then convex combination of all these linearly independent vectors is known as a convex polyhedral. So, if I have a finite number of linearly independent vectors, and if I take the convex combination of them then they will form a convex polyhedron. Mathematically if I have to say if x 1 x 2 say and x n, they are linearly independent vectors. If these are linearly independent vectors, then the set x such that x equals summation i equals 1 to n lambda I x I where lambda I greater than equals 0, and sum of all these lambda is equals to 1 is known as one convex polyhedron.

So, if you have the linearly independent set of vectors x 1 x 2 x n and if we can find a set x, such that x equals summation i equals 1 to n lambda I x I lambda I greater than equals 0, and summation of all this scalars lambda i equals to 1, this will form one convex polyhedron.

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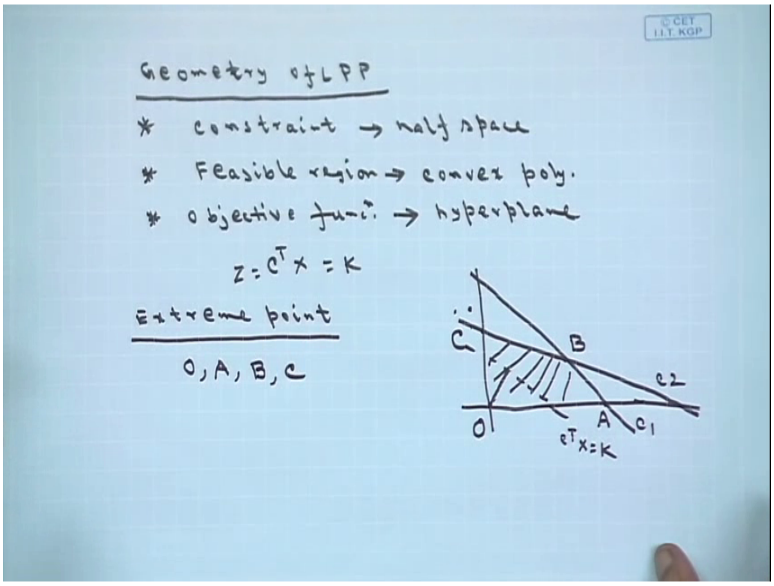

So now you see the geometry of LPP. Here what is happening? If you see number one you have a constraint. Each constraint defines one half space, half space means it may be open up space or it may be closed up space. It may open it may be open up space or it may be closed up space.

So, each constraint is defined as a half space. And feasible region already we have talked about feasible region, this feasible region is nothing but the convex polyhedron. Your feasible region is nothing but the convex polyhedron. Or it is defined as the intersection of the half spaces. Your half spaces are nothing but the constraints feasible region is the region which is bounded by these constraints, and this feasible region is nothing but a convex polyhedron. And the last one is you have the objective function. Your objective function is nothing but a hyperplane. So now, see your objective function is a hyperplane of the form C x equals z, where you will supply the values of C and z. Your each constraint will declare or will be treated as a half space, intersection of all these constraints or half spaces will form a feasible region which is nothing but the linear convex combination of linearly independent vectors, which we are calling as convex basically convex polyhedron.

Now, what about this one? Your value of this one objective function can be written as z equals C transpose x equals say k. If very simple form if I have to write down. Suppose you have a something like this, and you have another one which is cutting something like this. If I draw a line like this way, which represents C transpose x equals k. Say C transpose x equals k. So, what is k k basically the distance of the origin on this line that is this one. K is nothing but the distance from origin to this line.

So, if I move this line away from origin I will go towards this side. So, for maximization problems I can move this hyperplane for the objective function, away from the origin and if I have to find out the minimization of a problem. This line will come closer to our origin. Next one is the let me explain from here itself. Extreme point already we have discussed. Extreme point is basically nothing but the intersection of the 2 boundary lines.

So, here if you see, if this is one constant we have defined using equality sign, and this is another constant. So, I am writing this is as a constant C 1 this one as a constant C 2. And since the variables are non negative. So, I have to take this axis and this axis on the positive sign. Then your feasible region is basically nothing but this. And what are the extreme points? Extreme points will be the bound intersection of the boundaries of this 4 lines. So, intersection you can obtain at O, this you can obtain at this point A you can obtain at this point B and also you can obtain at this point C.

So, your extreme points I can say that this will be O A B and C, these are the extreme points of your problem.

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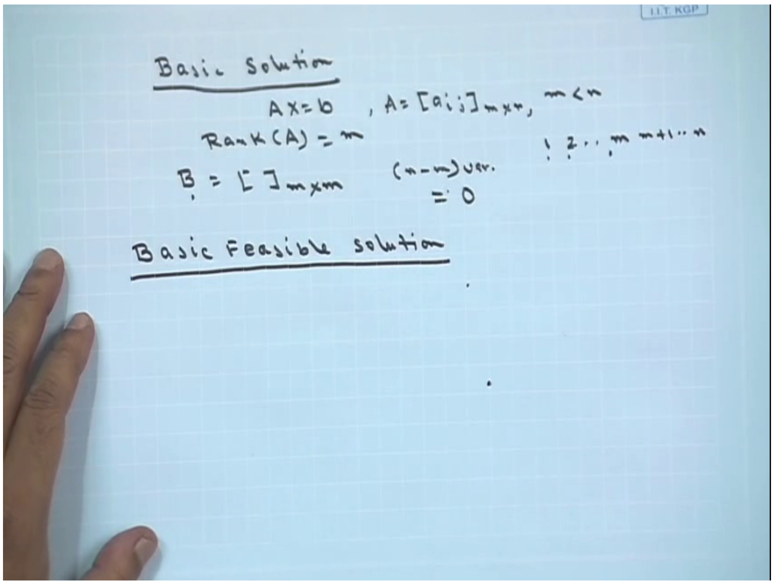

Which we have discussed earlier also. This one also I think, I have discussed the basic solution what is the basic solution. If you have a system of simultaneous equation A x equals b where a equals a i j this is m cross n matrix and m is less than n if I assume that rank of a is equals to A. Then I can form a matrix B which will be of the m cross m. That is I will take only m linearly independent column vectors and I can form a matrix B. Whereas, all other n minus m variables; that means, if I have the rows sorry columns one 2 like this way m and then, again m plus 1 like this way up to n.

So, these are the variables basic variables, and these are the non basic variables. So, these are linearly independent column vectors, from that I have formed matrix B. And the remaining n minus m variables if I make it equals to 0 which I have shown earlier. Then the solution obtained is known as basic solution. In the last class I think we have given one example also. So, you can go through that part also. Till there comes basic feasible solution. So, you have done earlier the feasible solution. We have done the basic solution, should now come to the basic feasible solution. If you remember feasible solution is the solution which satisfies the constraints. So, a feasible solution to a problem to and LPP. If it is also a basic solution, then we call it as a basic feasible solution. So, if a solution is feasible solution of an LPP, and if it is also a basic solution in that case we tell that this is a basic feasible solution.

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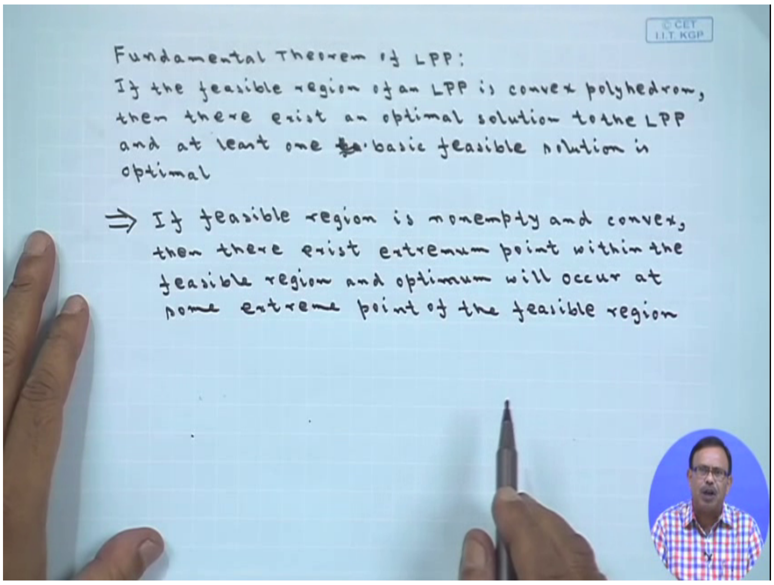

Now, let us go through quickly some theories which is important, which we have to use afterwards. First one is fundamental theorem of LPP. What it says? If the feasible region of an LPP is convex polyhedron, please note this if the feasible region of an LPP is convex polyhedron. Already we have told the feasible region will form the convex polyhedron. Then there exits an optimal solution to the LPP. And at least one basic feasible solution is optimal. So, in other sense this theorem says that if I can show that feasible region of the LPP is convex polyhedron, in that case you will obtain optimal solution and at least one of the basic feasible solution will be optimum.

If the feasible region is non empty and convex, then there exit extreme points within the feasible region, and optimum will occur at some extreme point of the feasible region. Already we have defined the feasible region, we have define the extreme point. So now, what we are saying that if the feasible region is nonempty and convex. Then the extremum point one of the extremum point will give you the optimum value. So that means, inside the feasible region I do not have to check for all points, but I will check only for the extreme points, and at the extreme points I will find out the value of the objective function. And from the value of the objective function only we can tell which extreme point is giving me the optimum value.

So, please note this one if the feasible region of a LPP is convex polyhedron. Then it ensures that at least one of the basic feasible solution will give you the optimal value. And from the feasible region you can obtain the optimum value only at the extremum points. So, which makes your life simpler, and this theory we will use afterwords to find out very large LPP problems.

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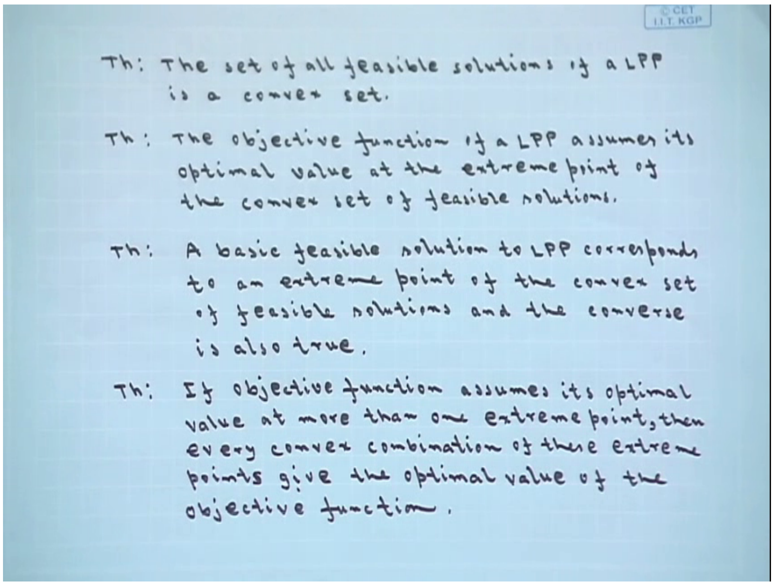

So, this is one theories. Some other theories related theories are also there the set of number. One the set of all feasible solutions of a LPP is a convex set.

So, this proves that if I have formulated one linear programming problem, where the objective function and constants are linear. And they satisfy the other assumptions of LPP, then the feasible solutions will form a convex set. Similarly next theorem says the objective function of a LPP assumes it is optimum value at the extreme point of the convex set of feasible solution. So, there is mathematical proof is there what I told you just now that the objective function of an LPP assumes it is optimum only at the extremum point or extreme point of the feasible region.

The third theory says that, a basic feasible solution to LPP corresponds to an extreme point of the convex set of feasible solution and the converse is also true. So, basically these 2 theorems are we are saying if one is if only if has been proved by the second part. That the feasible solution to LPP corresponds to an extreme point of the convex set of feasible solutions. And the opposite one is true. The last theorem says if objective function assumes it is optimum value at more than one extreme point. If objective function assumes it is optimum value at more than one extreme point, then the points then every convex combination of these extreme points gives the optimum value. Every we are saying that every convex combination of these extreme points also give the optimum value of the objective function. Or in other sense if I have to say I got optimum point at this point and at these value; that means, at 2 points I am getting the optimum value same optimum value of the objective function any convex combination; that means, if I join this point at each point of this line, also the value of the objective function will be same and it will be optimum. That is there will be multiple solutions for that particular problem.

So, this theories are useful for us.

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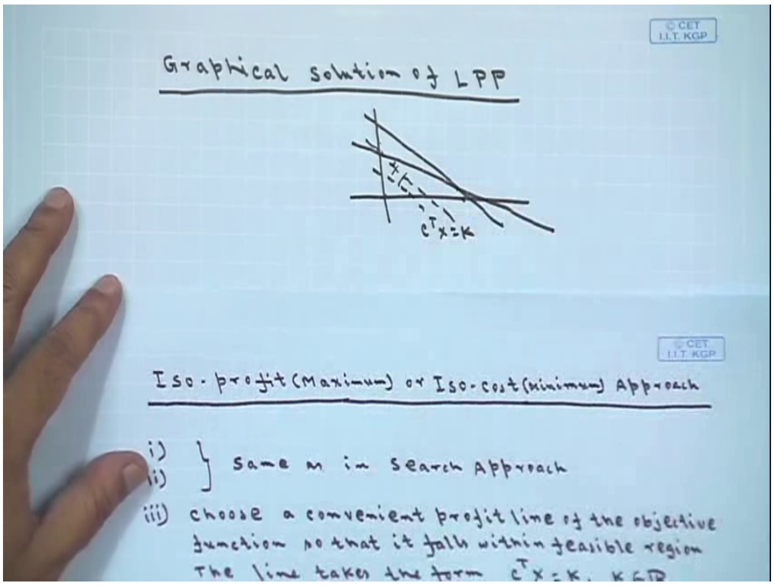

The next one is graphical solution. The graphical solution of an LPP. As I told you earlier we will go for the LPP consisting of 2 variables only. Because if we go beyond 2 variables visualization will not be proper. So, for finding the solution of an LPP graphically we will use the objective function which consist of 2 variables only. There are 2 methods basically one we call as the search approach and another one we call at the ISO profit or ISO cost function. So, 2 methods or 2 approaches we will use one is search approach and the other one is the ISO profit or ISO cost approach.

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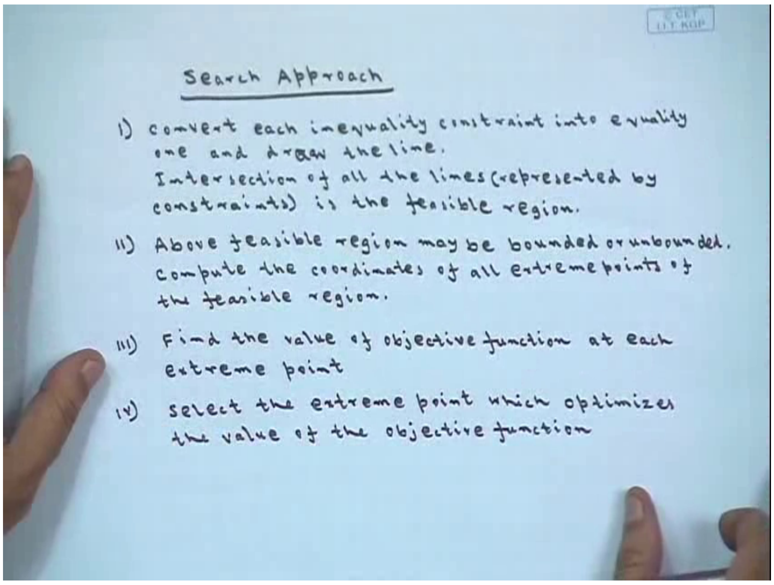

So, first let us come to the search approach, at first what we will do we will discuss all these approaches, and then we will see how to find out the solution. The first one is convert each equality constraint into equality constraint and draw the line. Because in the LPP you have seen the constraint may be equality may be inequality that is constraint may be greater than equals or less than equals. So, you have to treat all the inequality constraint as equality. So, that you can draw a corresponding line to that particular constraint. Intersection of all the lines will give you the feasible region, as we have told intersection of this hyper planes of this open up spaces half spaces we will give you the feasible region.

So, whenever I am doing the intersection of all this lines it will give me the feasible. Region above feasible region now may be bounded may not be bounded. Compute the coordinates of all extreme points of the feasible region. So, second step is basically you have to find out the extreme points of the feasible region. Number 3 is find the value of the objective function at each extreme point. And number 4 is select the extreme point which optimizes the value of the objective function. So, I think the procedure is quiet simple the procedure says that if you have the inequality constraint in your problem make it equality constraint, and once I am making it equality constraint after that you draw the corresponding line. Like this way you will get for each constraint one line intersection of all this lines will give you the feasible region.

From the feasible region now compute 2 things one is the extreme points of the feasible region, next will be at each extreme point find the value of the objective function. And the extreme point which gives the optimum value of the objective function that will be your require solution. So, this is the search approach whereas, the other one that is ISO profit which we call as maximum or ISO cost which we call as minimum, those approach that approach is the other one almost similar, But there is a difference is there. Here point number 1 and point number 2 are same as we have discussed in the search approach.

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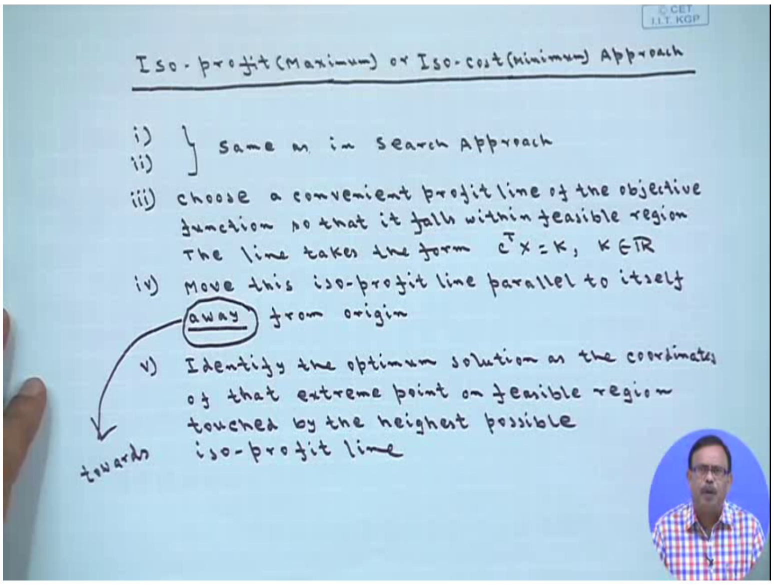

Now, they are same means we want to say that, first you make all the inequality constraints into equality constraint. And once I am making it into equality constraint I am drawing the line. So, for each constraint I will get a line and intersection of those lines will give you the feasible region. And from the feasible region you can calculate the extreme points also like others. Here number 3 is the important one choose a convenient profit line of the objective function. So, that it falls within the feasible region. The line takes the form C t x equals k. Here we are saying you take a line of the form C transpose x equals k, k belongs to some real number. And using these you choose an arbitrary profit line of the objective function. And that line should fall inside the feasible region.

Now, move these ISO profit line parallel to itself away from the origin; that means, as I told you if you remember this thing. We were talking about this one; if I have this as I was mentioning, if it is something like this your feasible region is this one here and here.

I may draw any line like this by using C transpose A x equals k. Once I am doing the C transpose x equals to k. Then since it is a profit function move this line away so that at some place may be it will be like this, I will be finding the solution. So, we are saying that move these ISO profit line parallel to itself away from origin. And this away part will be towards origin. This can be towards origin. If the problem is a minimization problem; that means, I have to minimize the distance from origin to that line and I have to maximize the distance of that line from origin if it is a profit function. And the last one is identify the optimum solution as the coordinates of the extreme point of feasible region touched by the highest possible ISO profit line.

So, this I will explain with an example. So, I think for finding the solution of an LPP graphically we will go through both search approach as well as the ISO profit or ISO cost approach. Same problem we will try to solve in both ways. So, that you can understand how it works, but please note this thing again that whenever we are trying to find out the solution of one LPP graphically we will use the objective function consisting of 2 variables only. Not more than 2 variables because visualization of 3 or more variables becomes difficult. So, in the next class we will do some examples of this and then we will proceed to the solution of LPP using simplest method.