Constrained and Unconstrained Optimization
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Lecture – 39
Unconstrained Optimization

In continuation to my previous lecture on Fibonacci method, today again I am going to cover another method there is a very well known method as a golden section method that is again a religion elimination technique. Now, with that I am starting today now since this is again a region elimination technique that is why in the method always the initial interval of uncertainty will be given to us and based on that we have to find out the optimal solution.

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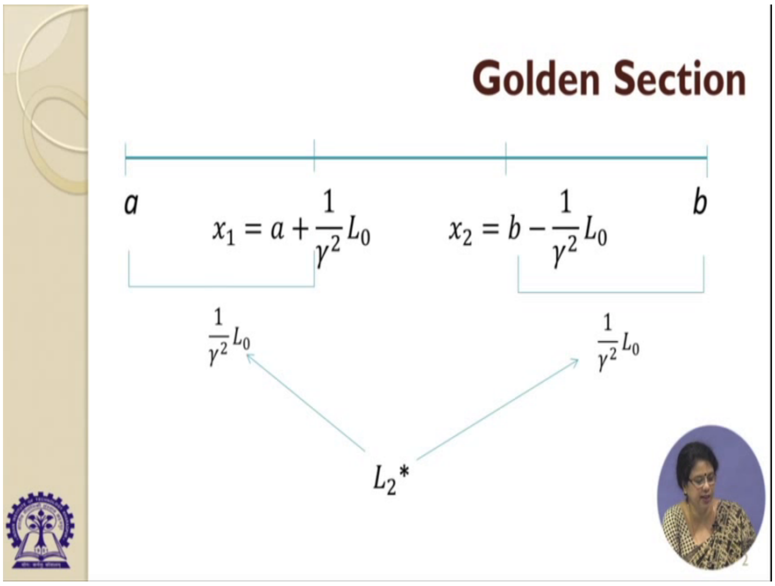

But there is one difference with the Fibonacci method is that, in the Fibonacci method we need to know how many experiments we are going to do that we need to know beforehand, but here in the golden section method that is not needed we are expecting that that we will do a large number of experiments, and based on that we will find out the optimal solution.

Now, let a to b the interval of uncertainty is given to us, and we do not know how many experiments we are going to do, but we will see when the convergence will reach we will stop the iteration. Now, but the basic principle is based or just is just like the Fibonacci method that we will find out 2 points in between, which is equal distant from both the ends and from there we will find out the optimal solution. This is the first point that is the first approximation of the optimal solution that is x 1, and this is being calculated with the a plus 1 by gamma square L 0. L 0 is the initial interval of uncertainty is a b minus a and here one parameter we have introduced that is gamma the Fibonacci method was based the Fibonacci numbers, but golden section method is based on the parameter gamma.

First point we will consider that is x 1 and the second point would be equal distance app distance apart from the point b that is b minus 1 by gamma square L naught. That is why we could see that the distance is from a to x 1 is 1 by gamma square L naught, and the same distance x 2 is from bs 1 by gamma square L naught. And this is the value since we are going to the next iteration which these values, that is why we are starting from a to b we are going to the next experiment that is why the naming is L 2 star. Just like your Fibonacci method, but here the question arises what is the value for gamma how really gamma will be calculated that is a beautiful thing in golden section method, this is called the golden ratio gamma is the golden ratio and that is a very nice property it has let me tell you the property of gamma first.

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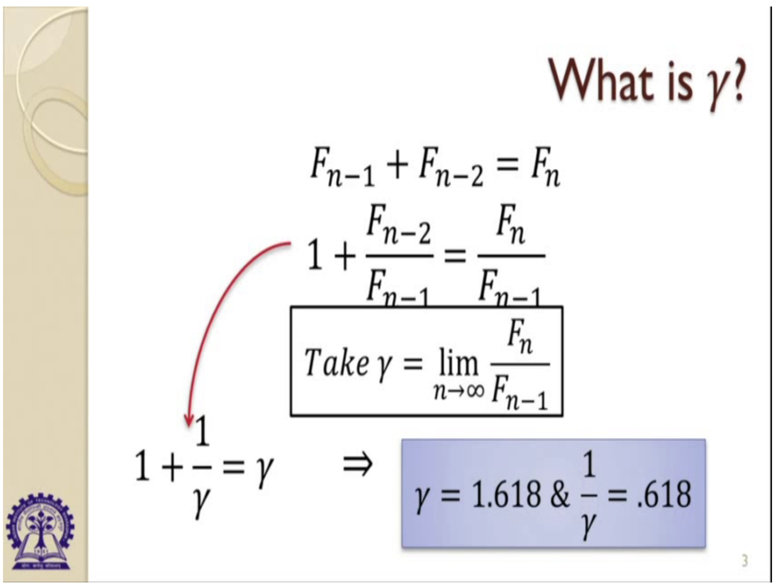

Now, we know from the Fibonacci number that F n minus 1 plus F n minus 2 is equal to Fn, that is for the Fibonacci series any number in the current position is the sum of the numbers in the previous 2 positions we know. That is why if we do some simplification here by just dividing with F n minus 1, we could see that 1 plus this one, 1 plus F n minus 2 divided by F n minus 1 is equal to F n divided by n minus 1 what is in n is the number of experiments we considered in the Fibonacci method. Now as I said in the golden section method this n is not given to us, we will always depend on the convergence criteria what is the convergence criteria? Whatever value we are getting in one iteration and whatever value we are getting in the previous iteration, the difference must be very small then we will say that the method has converged and we will stop there that is the idea.

Now here I said that in golden section method, we are thinking that that the number of experiments would be infinite; that is why what we will consider? Limit n tends to infinity F n by F n minus 1 and truly speaking this is the value for gamma that is the golden ratio, but there is another significance of golden ratio as well. Because this golden ratio has come the architect, they are also using this golden ratio that is the beauty of it because it has been the Greek architect they says that if we consider a room.

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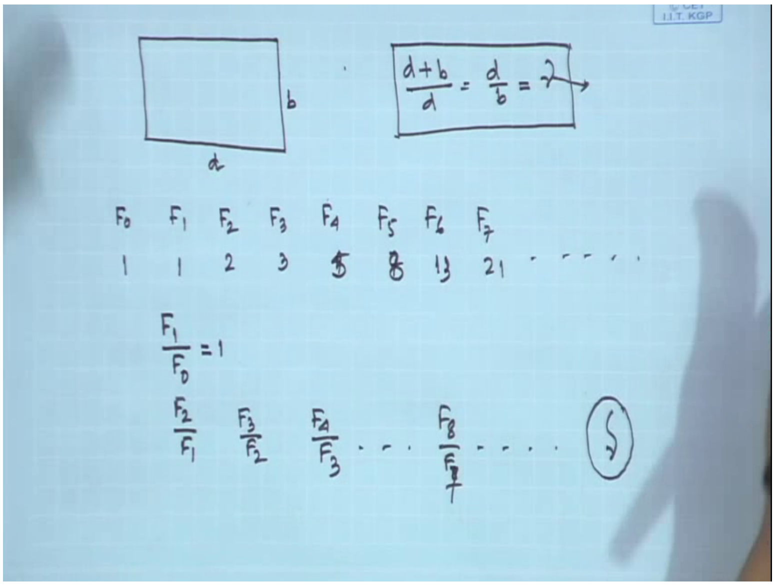

If we design a room in such a way that that is of rectangular form and there are 2 sides of the room one is d and another one is b. If we just consider d plus b divided by b is equal to d by b is equal to gamma, if we just design the room in by just considering this ratio we will see that always the room will have a pleasing property, and this gamma value has come from there only. We have seen after this gamma has been associated with this optimization technique, not only that this gamma has been associated with a Fibonacci number in this way.

Why did we say gamma is equal to limit n tends to infinity F n by F n minus 1? Because we know it starting from F 0, the value for F 0 is 1, F 1 is 1, F 2 is 2, F 3 is 3, F 4 is 4, F 5 is 7 oh sorry this is 5 this is 8 F 6 is 11, 13, F 7 is 21 this way it is going. If we consider F n by F n minus 1 that is F 1 divided by F 0 take the ratio of it this is one take the ratio of F 2 by F 1, take the ratio of F 3 by F 2 again just you consider F 4 by F 3 n F 8 by F 7 if I just proceed in this way we will see that these ratio will converge to the value of gamma.

Now, the question comes what should be the value of gamma? This gamma value we can consider by considering these ratio only, if we just find out the ratio values from here we will see that the value of gamma the value is converging to a certain decimal point and that is the value of the gamma that is the beauty of it, but from here we can also find out the value for gamma as well. How really it is from here? Since if we just take the limit n tends to infinity in both the sides then we will see that this is one, the next term is one by gamma because limit n tends to infinity F n minus 1 divided by F n minus 2 is equal to gamma, and the right hand side would be again a gamma.

If we just do some simplification this will become a quadratic equation what is the quadratic equation? The quadratic equation would be gamma square minus gamma minus 1 is equal to 0. Since this is a quadratic equation we can get the value for just like a minus v plus minus root over b square minus 4 ac divided by 2 a, with that if we just consider the value for positive value we will see that the gamma value will come 1.618 and 1minus ga 1 by gamma rather will come 0.618 that is a beauty of it.

Here also you will see the same thing, if we just consider the ratios of the Fibonacci numbers in this way F n by F n minus 1 limit n tends to infinity, even you need not to wait for greater value of n you will see after this only it will converge to 1.618 that is the beauty of gamma and this gamma parameter we will consider in our golden section method and what is the method altogether method is just like your Fibonacci method, we will consider the initial interval of uncertainty, we will consider 2 point that is equal distant apart from the end points we will see the functional values in these 2 points, after that we will consider the region elimination property. Since the function is unimodal if we are going to find out the minimum of the function, then accordingly we will proceed, but we will consider gamma value this value I will show you the example in the next.

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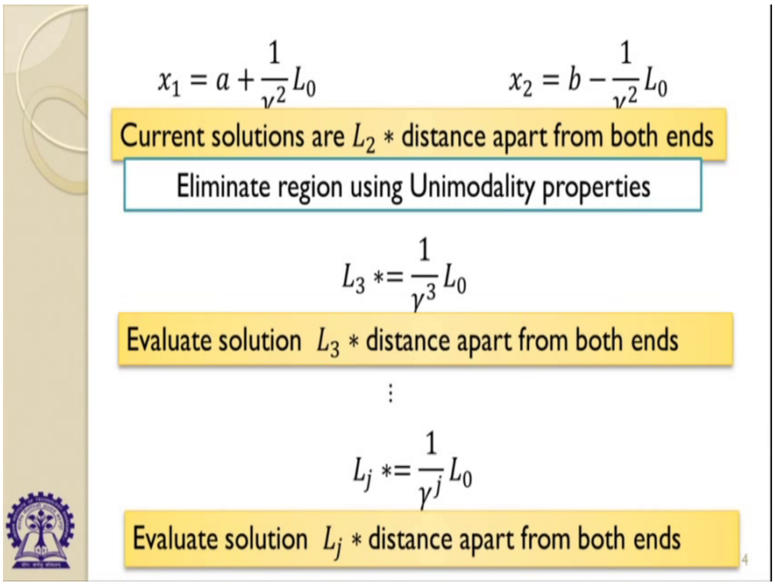

Now, you see the first solution x 1 that is the L naught by gamma square distant apart from a, x 2 is the same and you have to manage the region you have to consider a region by eliminating the region be by using the unimodality property, that is why initially we are having 4 points a x 1 x 2 and b. We will see the trained of the function if we see that the function is not having the minimum value, there is no possibility to have the minimum value within x 2 to b then we will eliminate that region.

We will consider new interval of uncertainty as a 2 x 2 and even if it is the other way we will do the same thing and once we are getting again another interval we are getting 2 endpoints of that interval again we will consider x 3 star. What is the value for x 3 star? The L 3 star rather L 3 star what is the value for that? That is L naught divided by gamma cube, and again we will take 2 points that would be equal distant apart from the new interval of uncertainty and what is that equal distance? That is L naught divided by gamma cube, that is L 3 star we will get any again 2 points within the endpoints again we will use the unimodality property again we will just discard eliminate a portion of the interval and in this way we will proceed.

How long we will go as I said in the golden section method, number of experiment depends on the convergence criteria that why if we just proceed in this way we will see at the jth experiment L j star will be considered, that is the pattern of the golden section method and we will take 2 point second that is L j star distant apart from both the ends in this way we will proceed that is the golden section method.

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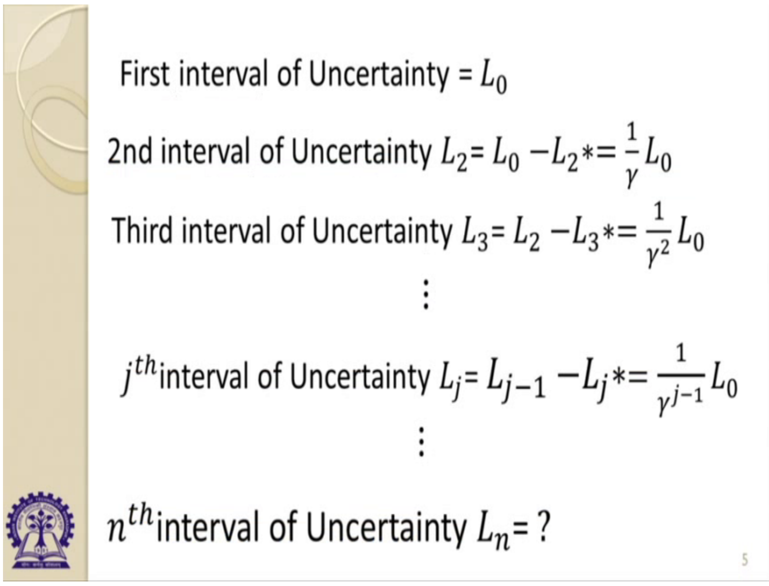

That is why if we just calculate the length of the interval of uncertainty. Initially it was L naught because that is b minus a that is given to us we know the function is unimodal fro between a to b, we know that property. Unimodal if either this is say it has a unique maximum or it has the unique minimum within that interval that is the unimodality definition.

And second interval of uncertainty that would be L naught minus L 2 star, because we have considered 2 points that is L 2 star distant apart from the both the ends after using the region elimination technique one part will be eliminated that is why the new interval of uncertainty would be L naught minus L 2 start that will be L 2. In this way we will proceed and jth would be L j minus 1 minus L j star and if we to see the pattern the values are coming like this L naught L naught by gamma, but L 2 star is here your L naught divided by gamma square and L 3 would be L naught by gamma square in this way we will proceed and in the jet experiment L naught by gamma to the power j minus 1, but L j star is equal to L naught divided by gamma to the power j I showed you. In the previous that is why if I ask you what should be the length of the in nth interval of uncertain very easily you can say that must be Laughter n, and the value of L n you can calculate that way we will proceed in the golden section method. Other than this nothing is there in this method and we will just see the convergence.

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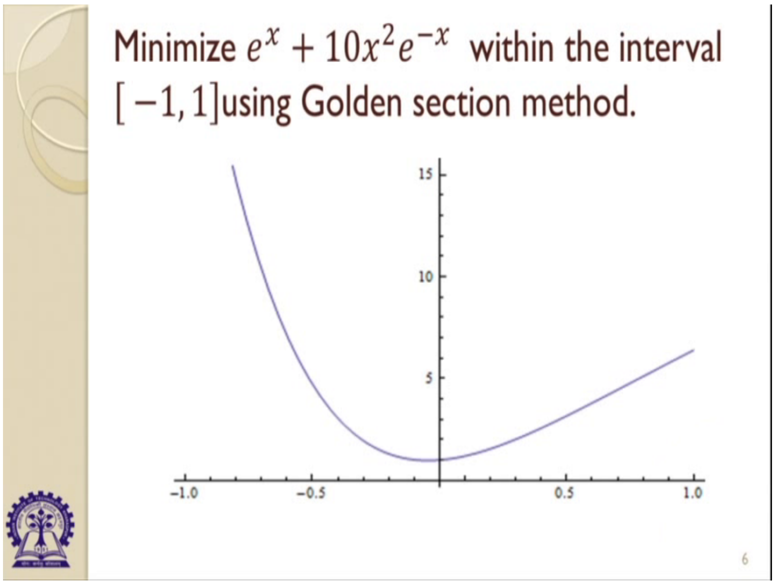

Now, let us try to find out the minimization of the function e to the power x plus 10 into x square e to the power minus x. If we just look at the functional description function is very complicated in nature, because there are involvement of exponential function e to the power x, e to the power minus x and if we just want to apply your classical optimization technique may be little bit difficult to get the values very easily, but we know that function is unimodal within minus 1 and 1let us apply the golden section method here you can apply any other region elimination technique as I explained you before ok.

Let us apply golden section method here how to do it? We will consider a as minus 1 b as 1 and we will consider 2 points x 1 and x 2 that is L 2 star distance apart from minus 1 and 1in this way.

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	x	f(x)
a	-1	27.5507
x1	-0.23604	1.495201
x2	0.236036	1.706214
b	1	6.397076

	x	f(x)
x1	-0.23604	1.495201
x2	0.236036	1.706214
x3	0.527834	3.338719
b	1	6.397076

	x	f(x)
x1	-0.23604	1.495201
x4	0.055785	1.086801
x2	0.236036	1.706214
x3	0.527834	3.338719

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We will proceed and we will considered that we will see that the minus 1, then the next point is minus 0.2 3 6 0 4 and the other point is 0.2 3 6 0 4 and the next point is one given to us, then what is L 2 start here? Certainly the value must be 0.23604 now we will see the functional values here what is a trained of functional values? If we see that the functional value is coming 27.5507, next one is 1.49, next one is 1.7 and next is 6.3 all right.

We must see that there is no minimum between a to x 1 all right that is why we are processing to the next by considering x 1 has the ne the left hand new interval of uncertainty, and 1 as the another right part of the interval of uncertainty. Again we will repeat the process, we will again find out 2 points in between that is L 3 distance apart from x 1 and b and what is the value of L 3 star you know that L 3 star distant apart and again we will find out the functional values and from there we could see that between x 3 and b, there cannot have any minimum value that is why we you we must exclude the point, will exclude the region x 3 to be in this way we will proceed.

Still we could see functional value is not converging that is why we need to do further iterations for this problem.

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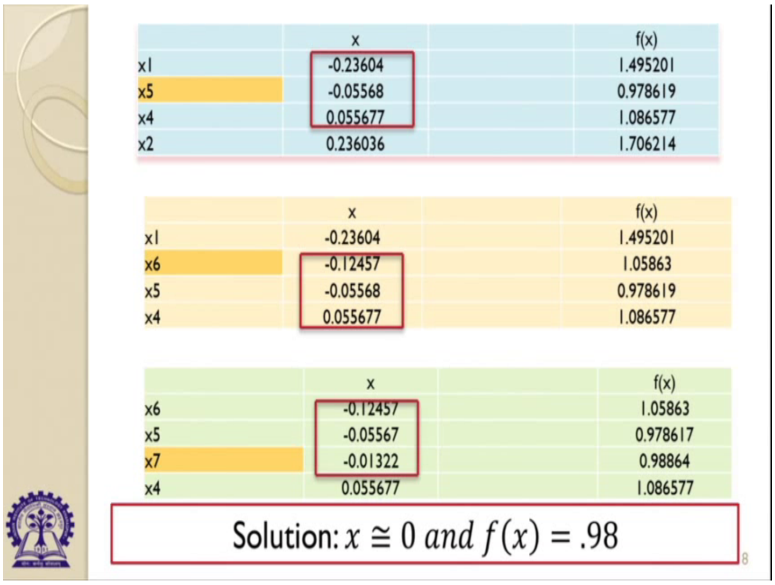

	x		f(x)
x1	-0.23604		1.495201
x5	-0.05568		0.978619
x4	0.055677		1.086577
x2	0.236036		1.706214

	x		f(x)
x1	-0.23604		1.495201
x6	-0.12457		1.05863
x5	-0.05568		0.978619
x4	0.055677		1.086577

	x		f(x)
x6	-0.12457		1.05863
x5	-0.05567		0.978617
x7	-0.01322		0.98864
x4	0.055677		1.086577

Solution: $x \cong 0$ *and* $f(x) = .98$

We are again eliminating a portion and what is suggest to you whatever values I am showing to you these are the values I have already calculated before, but I should suggest to you do the calculations on your own. You may use simple you can write a simple program or you can use the excel what calc for calculating the values of these and we can have different iterations this way could you see the functional values are converging somewhere we will proceed further I need not to explain again how really we are eliminating the region from here, we will proceed this way and somewhere you see the va functional values are converging to one, because within this interval functional values are almost one 1.0 0.97 0.98.

If you are happy with these values then you can stop your iteration in at this point, you can say this is we have arrived to the convergence, convergence criteria has been satisfied and what should be the optimal solution then optimal solution would be in between x 6 and x 7 that is the your new interval of uncertainty, take the middle point of that and declare that is your optimal solution. But one thing you must be clear you should know that the method of the function is if you consider the bigger interval, not you are not confined with within minus 1 and 1, if you just proceed for the 2 to 10 to 10 the pattern of the function will be different, that is why do not think this is the global minimum we are getting we are only getting a local minimum point between minus 1 and 1.

Now, till now we have covered many region elimination techniques, now the question come must be coming to in your mind that what is the best method which method will be used so that we will get the optimal solution efficiently. What is the criteria for efficiency of optimality of the region elimination technique that is the number of experiments must be less convergence should reached you know very quickly, then only we can say that we have reached to the solution and the this method is the best one. Let us try to compare that how many iterations really we need to get 10 percent accuracy of the optimal solution or the 5 percent accuracy of the optimal solution. Now I am going to explain you that part in the next, that is why the solution has been declared x as a 0 and Fx is equal to 0.98.

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Elimination Technique	Initial interval of uncertainty	Final interval of uncertainty	Reduction Ratio
Exhaustive Search	L_0	$L_n = 2 \times \frac{L_0}{n+1}$	$= \frac{2}{n+1}$
Dichotomous Search	L_0	$L_n = \frac{L_0}{2^{n/2}} + \delta\left(1 - \frac{1}{2^{n/2}}\right)$	$\approx \frac{1}{2^{n/2}}$
Interval Halving	L_0	$L_n = \left(\frac{1}{2}\right)^{\frac{n-1}{2}} L_0$	$= \left(\frac{1}{2}\right)^{\frac{n-1}{2}}$
Fibonacci Method	L_0	$L_n = \frac{F_1}{F_n} L_0 = \frac{1}{F_n} L_0$	$= \frac{1}{F_n}$
Golden Section Method	L_0	$L_n = \frac{1}{\gamma^{n-1}} L_0 = (.618)^{n-1} L_0$	$= (.618)^{n-1}$

Now, if we just see that different elimination techniques, we have covered the exhaustive search, dichotomous search interval halving Fibonacci method and golden section method let us compare these 5 methodologies. Now in the initial interval of uncertainty everywhere is L naught then only we can compare L naught if this is same. If the initial interval of uncertainty the value is increasing then the situation will be different. If we have the same initial interval of uncertainty just look at the final interval of uncertainty after the nth experiment for exhaustive search, this is 2 into L naught by n plus 1 and the reduction ratio is 2 by n plus 1 if we do n number of experiments.

Similarly, for the dichotomous search we could see the reduction ratio is 1 by 2 the power ni 2. For interval halving is half to the power n minus 1 by 2 for the Fibonacci method we have seen that L n by L 0 is 1 by F n, we could find it out before because we considered L n is equal to F 1 by F n L naught that is equal to L naught by F n that is why the reduction ratio if we considered L n by L 0 what is the L n? L n is the nth interval of uncertainty the length of the nth interval of uncertainty, L 0 is the length of the initial interval of uncertainty. Reduction ratio means after reaching to the nth experiment what is the ratio of L n by L 0 with respect to the initial.

Now, that is 1 by n and in the golden section method you must have seen that L n is equal to L naught divided by gamma to the power n minus 1, where L n star was L naught divided by gamma to the power n. But here the gamma value you know that is equal to 1.618 and 1 by gamma is equal to 0.618 that is why if we just calculate we will see that there is a golden section method is having this one.

Now, from here you can get some idea for what is the better reduction ratio. Reduction ratio it means that how much we are eliminating in each iteration how much length we are eliminating in each iteration all right that is the reduction ratio, that is why if we to see here then we could see that somewhere Fibonacci method and the golden section method is almost same, but other methods these are taking longer time to reach to the convergence; that means, you need to have large number of experiments, very large number of experiments.

Let us do another analysis where we will see that if we consider 10 percent accuracy by considering a lot as inter initial interval of uncertainty, how many experiments we need to do.

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Elimination Technique Let $L_0 = [0,1]$	If $\frac{1}{2}\frac{L_n}{L_0} \le \frac{1}{10}$ then the number of experiments n (i.e. 10% of exact value)	If $\frac{1}{2}\frac{L_n}{L_0} \le \frac{1}{20}$ then the number of experiments n (i.e. 5% of exact value)
Exhaustive Search	$\frac{1}{n+1} \le \frac{1}{10} \Rightarrow n \ge 9$	$\frac{1}{n+1} \le \frac{1}{20} \Rightarrow n \ge 19$
Dichotomous Search	$\frac{1}{2^{n/2}} \le \frac{1}{10} \Rightarrow n \ge 6$	$\frac{1}{2^{n/2}} \le \frac{1}{20} \Rightarrow n \ge 8$
Interval Halving	$\left(\frac{1}{2}\right)^{\frac{n-1}{2}} \le \frac{1}{10} \Rightarrow n \ge 7$	$\left(\frac{1}{2}\right)^{\frac{n-1}{2}} \le \frac{1}{20} \Rightarrow n \ge 9$
Fibonacci Method	$\frac{1}{2}\frac{1}{F_n} \le \frac{1}{10} \Rightarrow F_n \ge 5 \Rightarrow n \ge 4$	$\frac{1}{F_n} \le \frac{1}{20} \Rightarrow F_n \ge 20 \Rightarrow n \ge 6$
Golden Section Method	$\frac{1}{2}(.618)^{n-1} \le \frac{1}{10} \Rightarrow n \ge 5$	$(.618)^{n-1} \le \frac{1}{20} \Rightarrow n \ge 5$

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If we consider 5 percentage accuracy what is the number of experiments let us see. For exhaustive search we have seen that the reduction ratio was 1 by n plus 1, that is why 10 percent accuracy means half into L n by L 0 must be less than is equal to 10 percent that is 10 by 100 all right. That is why if we just calculate we could see that the exhaustive search is taking more than 9 experiments, for 5 percent accuracy in more than 19 experiments you need to do dichotomous again 6 and 8 and interval having 7 and 9, but you see the Fibonacci you need to have only 4 number of experiments to get the 10 percent accuracy whereas, to get the 5 percent accuracy you need to have more number of experiments that is 6. But you look at the golden section method 5 and 5 in both the cases, that is why as I said it has been considered that Fibonacci method is the best method till date to do the region to implement the region elimination technique because it has to do less number of experiments ok.

Another thing just I would like to point out here you must have been seen if I want to get the 10 percent exact value; that means, and another on the other hand we want to get the 5 percent accuracy, we could see the 10 percent accuracy the interval is larger than the 5 percent accuracy all right that is why we need to do more number of experiments. Though in the golden section method that is the just do be only because of mathematical calculation we could see the both the values are same, but it is not that will happen if I want to get that 1 percent accuracy values may not be same you can calculate how many number of experiments we need to do, if we to get one percent of accuracy that is the task of yours do it.

Thank you for today.