Constrained and Unconstrained Optimization
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Lecture – 36
Region Elimination Technique- I

Now, today we are going to the next type of optimization technique, that is the solving the unconstrained optimization problem using numerical optimization technique. The basic advantage of numerical optimization technique is that that we need not to have the function is continuous within the domain of interest. Now as we have seen in the classical optimization technique we were finding out the first order derivative and the second order derivative. We had to depend on those values that is why there was a necessity to have the function is continuous.

But here we do not have that necessity function can be discontinuous, it can be piecewise continuous, it can be discrete for the discrete optimization problem also, very nicely we can find out the optimal solution. Now today we will tell you are series of techniques for solving numerically the non-linear programming problem with single optimized single objective function, with single variable and there is no constraint. Now for solving this for applying the numerical optimization technique, we need to have on assumption that is necessary that is must that function must be uni modal within the given domain.

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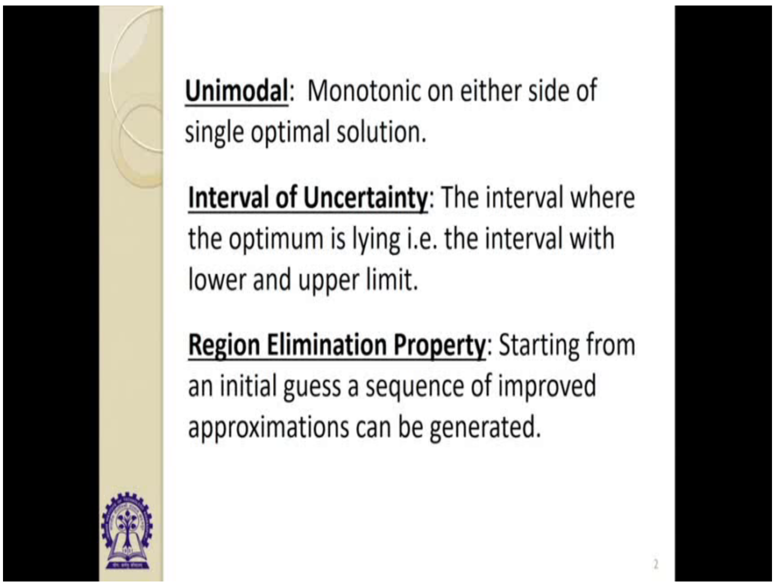

Now, we should have the prior knowledge about the pattern of the function then only we can applied, if we do not have the prior knowledge of the functional values then we have to check the convexity or concavity of the function within the range and there we have to guess that how function is behaving and then only we can say the function is unimodal within a certain race then only we can apply the numerical optimization technique, that is the in detail. I will talk on that later on, but one thing is that we need to have the interval of uncertainty and function must be unimodal.

Now, what is interval of uncertainty, interval of uncertainty is referring to that interval where the optimum may lie either maximum or minimum we are trying to find out, that is why if we have the information that a maxima or minima can lie within this range in the given x, we are considering the function of single variable that is why within the real line if we are having the lower limit and upper limit of the interval where the optimum may lie, then that is being named as the interval of uncertainty. That is why 2 things should be known beforehand one is that interval of uncertainty and second is that we need to know the function is unimodal within that interval of uncertainty.

Another thing is necessary for few cases like fibonacci method and few other methods we need to know that how many iterations we need to do, actually the numerical optimization technique is the thing we are using the iterative process we are starting from a guess point, we are checking whether the point is optimal or not if the function is not optimal we are going to the next point. That is the approximation again for optimal if it is not even optimal if there is a provision to improve the value of the objective function, then we approximate the third one in that way apply the iterative process, this is a process of iteration.

And once we will find out a sequence of values of the approximate value of the optimal solution, then we will see that function is the values are converging to a point and that point will be declared as the optimal solution of the given function. But the whole of numerical optimization technique is very much depend in on the region elimination process, region elimination process I have given certain idea in my last class on interval halving method there you got certain idea; what is the region elimination technique?

But just today, I will formally introduce that technique, to you how mathematically we are establishing the region elimination process. What is that we are starting from a guess point and we will find out the sequence of improved solution after eliminating the region; that means, were reducing the interval of uncertainty and we are reducing further and further we are trying to reach to a smaller interval of uncertainty and at the end we will get the interval. So, small that that interval can be declared or the middle value of that can be of that interval can be declared as the optimal solution, that is why we are applying that method.

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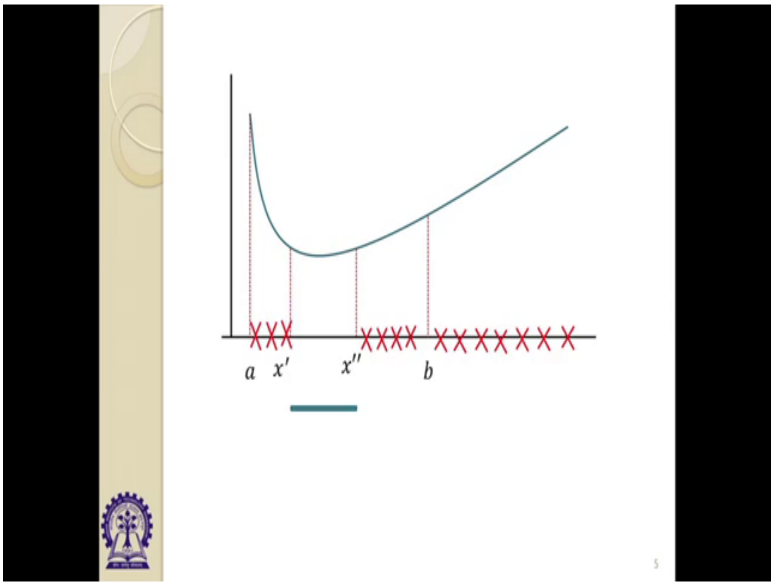

Now, let us consider this function, this function is this one it is clear that within a and b the function is unimodal, that is why my interval of uncertainty is a to b, that is the lower limit is a upper limit is b and we are having the point x naught and this is my initial guess point and the next guess point through the method we will find out the next guess point and the next guess point exact is double prime it is this one. Now why what we could see that functional value is increasing from x double prime to b, that is why if we are searching for the minimum of the function, we know the function has a mode within the range and we are trying to find out the minimum of the function.

That is why we are sure that within x double prime to b the function cannot have the minimum value that is why we will eliminate that region that is why the process is being named as the region elimination technique. We will element in the next iteration we will have the lesser sized interval of uncertainty, that is why the next interval of uncertainty would be a to x double prime all right, this was initial and this would be the next.

Now, if this is the next level of uncertainty then let us move to the next function next case. What we could see here that from now, I am changing the values I have renamed the values x double prime, I have given as b x prime is my initial approximation again that is x double prime from the previous iteration. Now here the x double prime is my new finding in this iteration all right, that is why again we are finding out the functional value, what we could see that from a to x prime function is increasing and we are not able to see the difference we really do not know what will happen from x double prime to b, that is why better not to discard that interval. Let us in discard the interval a to explain because we are sure minimum cannot lie there, that say we are discarding this and my next interval of uncertainty would be this one all right clear.

Now again we are renaming x prime as a b as it is x double prime x and we are moving to this, that was my initial now in the next what we could see here that x prime a now my a is here the previous 1 all right. What we could see that we are having the same value at x prime and x double prime, that is why we are not we are sure that there must be certain ups and downs within x prime and x double prime, beyond x double prime function was is increasing because we could see the value that. So, we will declare that the minimum is lying within the x prime and x double prime; that is why my region elimination would be I have eliminated the region and my next interval of uncertainty would be x prime to x double prime.

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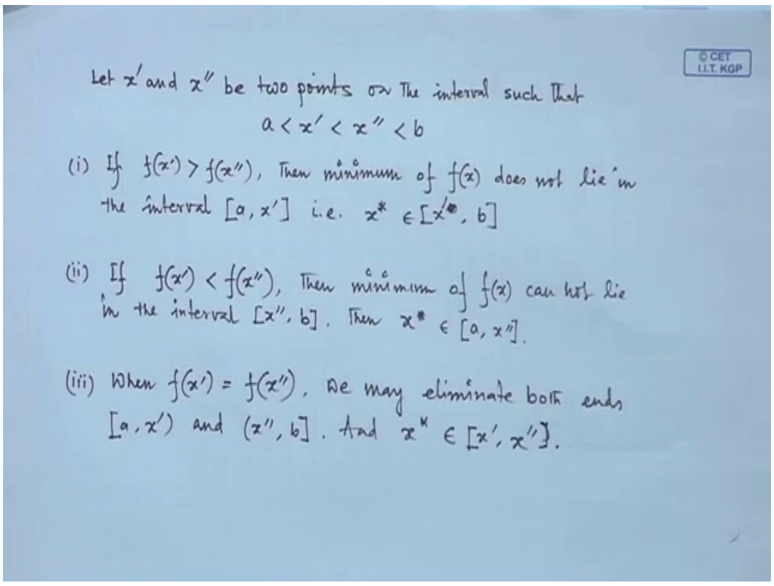

Now, the whole process can be just mathematize in this way, just to see we are considering 2 points x prime and x double prime and these are within the range of interval of uncertainty a to b. Now if x double prime is greater than f x double prime we are discarding the interval a to x prime and we are taking the new interval of uncertainty rather we are guessing them optimum may lie within x prime to b that was the first figure. The second figure if f x prime is lesser than f x double prime we are declaring the new interval of uncertainty as a to x double prime and in the third case if x prime is equal to f x double prime then we are eliminating both the sides of the interval that is from a to x prime and x double prime to b and we are declaring a new interval of uncertainty as x prime to x double prime. That we will proceed further and further in the similar way now few things are there 1 thing is that, how to select the first initial guess point that you need to know and the second is that how to reach to the next point. Once we are having the initial guess point how to reach to the next point that is the unknown thing for us.

Now, how to solve this kind of problem that is why we are learning a series of methodologies and as I was mentioning this part.

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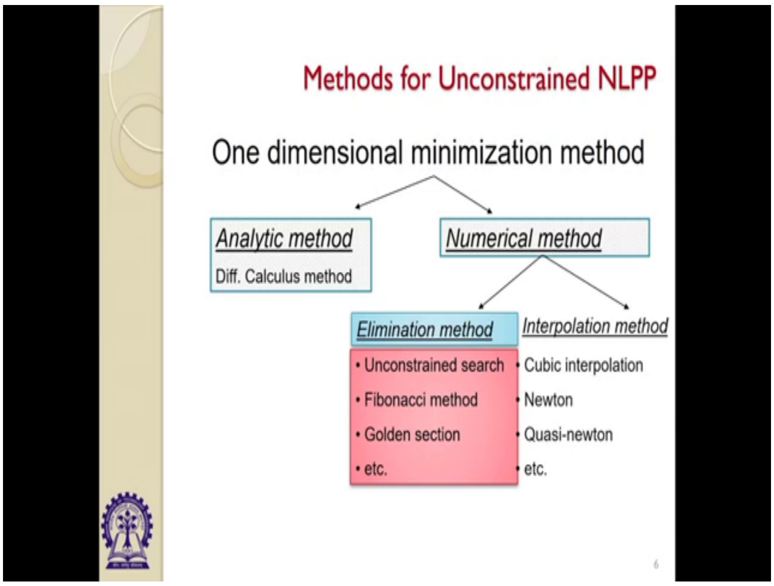

In the last to last class that this is the series of methodologies available and in the numerical method, I will explain to you the unconstrained search Fibonacci golden section etc and there is another series of methods are there on interpolation, I will mention 1 of that in my classes. Now I am coming to the hint interval halving process I need not to explain this method to you.

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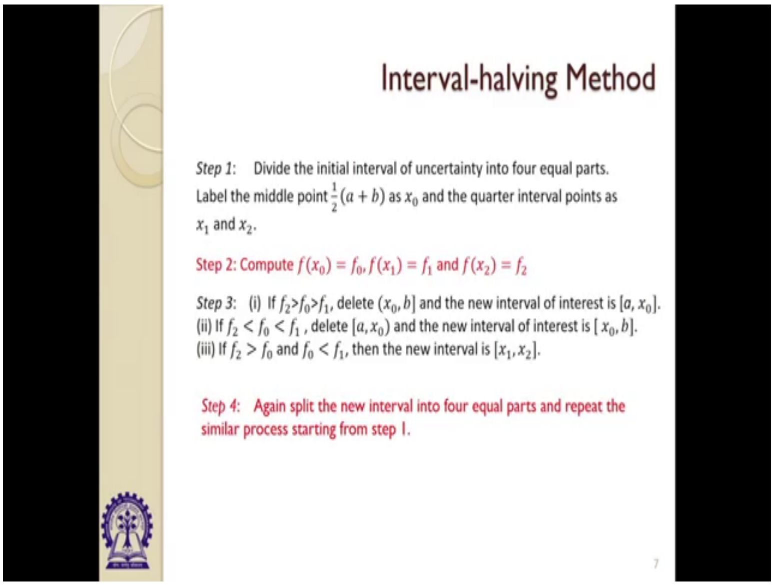

Just let us see how we are eliminating the region for this, we are applying this methodology. We are taking the same function we were considering before, in the last class and that was the function was 12500, that function that is the orange plant problem.

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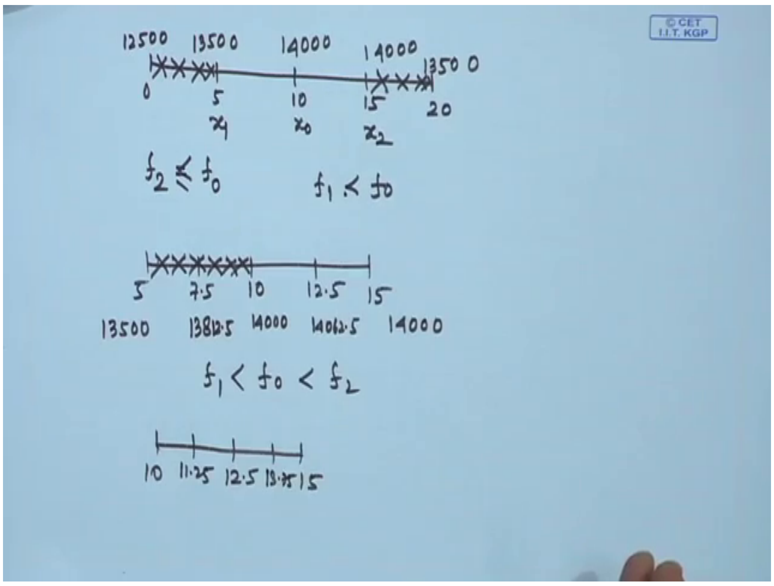

Now, there was a functional values were given to us we are using those function only and from the function itself we are getting that from 0 to 20 function is unimodal, that is why we are breaking into four parts, 0 to 5, 5 to 10 and 10 to 15 like that ok and we could see the functional values at 0 was 12500 at 20 it was 13500, you can get the values are just after substituting the value of x this is x 13500 14000 and 14000. What we could guess here we could see that from here if this is my x naught, this is my x 1 and this is my x 2, you could see that f 2 is same as f naught rather I can say less than equal to the f naught all right, in general what else you could see f 1 is less than f naught.

Then which process shall we consider, we will consider the third 1 because this is a maximization problem. Whatever inequality relation we took for the minimization for the region elimination for maximization problem that is the reverse one that say we are sure the optimum cannot lie from 0 to 5 or 15 to 2. That say we will eliminate this region my new interval of uncertainty would be from 5 to 15, again we will just divide it into 4 parts 7.5 12.5.

Let us write down the functional values here for five we know this is 13500 for 7.5, 5 it is 13812.5, for here 14000 for this point 14062.5 and here in the 15 its 14000 we are finding out the maximum value all right. What we could see here that functional value is increasing here, after that it is increasing further then it is decreasing, that is why from here we can say that f 1 this value is lesser than f naught and lesser than f 2 all right. That is why my new region of uncertainty would be from here to here 10 to 15 I will discard it. I will proceed further in this way, I will proceed further 10 to 15 I will consider 20.5 I will consider 11.25, when here it is 13.75 and we will find out the functional value.

And we will see if we just proceed in this way we will get small interval in the in the region where 12 is lying because through the classical optimization technique. We found the maximum value is lying at 12 all right, 12 or 13 because 12.5 that is why it could be 12 or 13 we are not sure, which 1 gives us the better solution and we will just through this process if we process this is the task of yours to complete this table, otherwise I will put this problem an assignment and I will give you the solution also just take it as a assignment for this.

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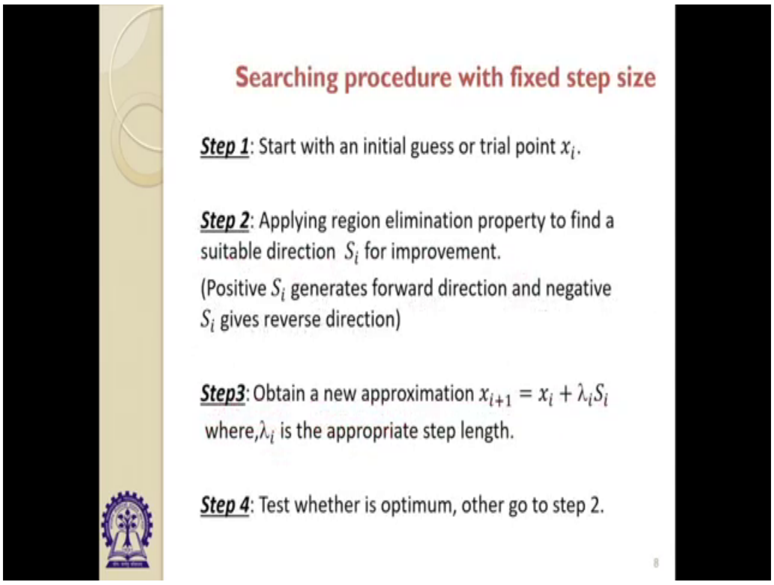

Now, I am moving to the next method that is the searching methodology with the fixed step size, what is the assumption for this method again the assumption is that function must be unimodal and we must be having the initial interval of uncertainty and the initial guess point must be given to us then only we can proceed otherwise we cannot. How we are proceeding this just you look at the process we are starting from the point xi initially we are setting i is equal to 0.

Now, whether I will move to x 1 in the left or x 1 in the right that is why we have to select that, that is why we are moving to the step 2. In the step 2 it is being said that find out the suitable direction si, si is positive if I move to the right si is negative if I move to the left. Now one thing is there the fixed step size I will move, I will jump with the fixed step size if I move to the right with and with the with the safe size lambda I will move with the plus xi plus lambda i si or I will move to the left, in the at way we will get a sequence of x naught x 1 x 2 x 3 x 4 and we will get the solution.

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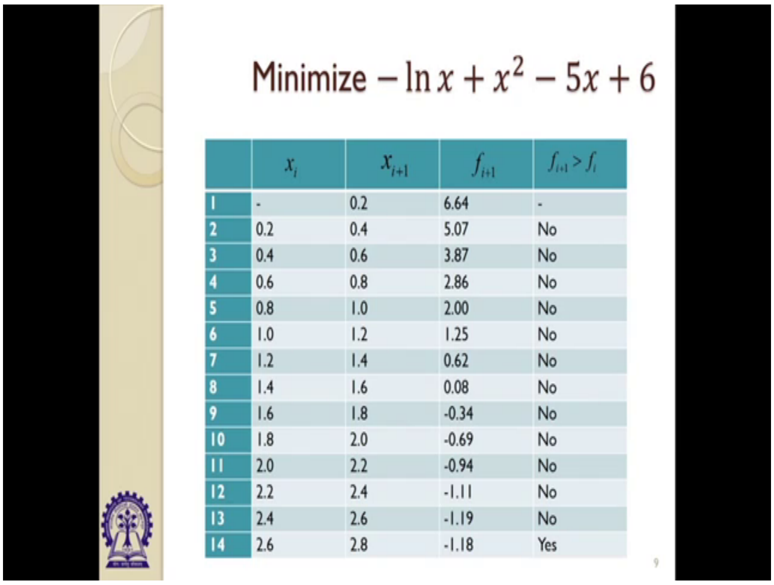

Minimize $-\ln x + x^2 - 5x + 6$

	x_i	x_{i+1}	f_{i+1}	$f_{i+1} > f_i$
1	-	0.2	6.64	-
2	0.2	0.4	5.07	No
3	0.4	0.6	3.87	No
4	0.6	0.8	2.86	No
5	0.8	1.0	2.00	No
6	1.0	1.2	1.25	No
7	1.2	1.4	0.62	No
8	1.4	1.6	0.08	No
9	1.6	1.8	-0.34	No
10	1.8	2.0	-0.69	No
11	2.0	2.2	-0.94	No
12	2.2	2.4	-1.11	No
13	2.4	2.6	-1.19	No
14	2.6	2.8	-1.18	Yes

How to get it I have just given one I have just find out one problem for you this is the function for us, function is unimodal from 0 to 1 and it is given the initial starting point is 0 and we are considering the step size at 0.2 that is why we are moving from x 0.2 then we are moving to initial interval of uncertainty 0 to 1 that is why since I am starting from 0 I have to move to the right I am going from 0.2 to 0.4 to 0.6 0.8 in that way I am proceeding further and further.

Just you see the difference between the functional values, you just look at once I am moving through xi the functional values are decreasing we are trying to minimize the function decreasing and decreasing further and further, but there is a up where at 2.6 function is again going to up that is why there is a check, whether if I plus 1 is greater than fi or not. Once it is that we are sure that around that pointer there must be the optimal solution that is why we will go from 2.4 to 2.6, then again 2 points up to that the function is increasing, but beyond that there is a I am sorry function is decreasing beyond that there is increase of the function, but I do not know where they increase this, that is why we can declare at the final interval of uncertainty as 2.6 to 2.8.

And if you ask me what is the optimal solution? I will say take the mid value of that 2.7, but if you want to get the further approximation; that means, further better approximation for the optimal solution what you can do you can take up a new problem as take the function this 1 take the initial interval of uncertainty at 2.6 to 2.8, then you take a very small step size take 0.0001 as the step size, and you just run the iterative process again and again again and again then you will see that interval of uncertain will reduce further and further and you will get a better approximation there. In that way we can proceed and we will get the better optimal solution and you could guess even though the function is discontinuous if I do not get the functional value at certain point I will just select the next will search for the next point, where the functional value can lie in that way I can proceed further and further that is the nice part of this methodology beauty of this methodology.

This is with the with the step size fixed step size, again it may happen that we are proceeding point 2 to 0.4, this is another variation this is called the random step size we can proceed even for searching we can do the searching. We are not fixing the step size as 0.2 we could see that 0.2 to 0.4 its decreasing 0.42 to 0.6 decreasing we can take a risk. Instead of taking the step size responds to I will just jump I will take the step size as 0.3 then again we will see the function is decreasing, I can take another risk by taking the step size is 0.4.

But by chance I could see somewhere the functional value is increasing then I have to read I have to stop there I will go back to the previous step and I will just take the smaller step size further to get the better solution that is why what is suggest to you, though you are learning the methodology with a fixed step size, but if you apply in the practical problem do not consider as a fixed step size. Even if you consider the fixed step size write the program for it take a very small fixed step size and run the iteration many times 10000 times let the computer do it otherwise you would you consider the random step size increasing step size you will get our get the result better ok.

Otherwise as a beginner if we proceed with the fixed step size at least you can guess the optimum will lie from 2.6 to 2.8 fine I am moving to the next methodology.

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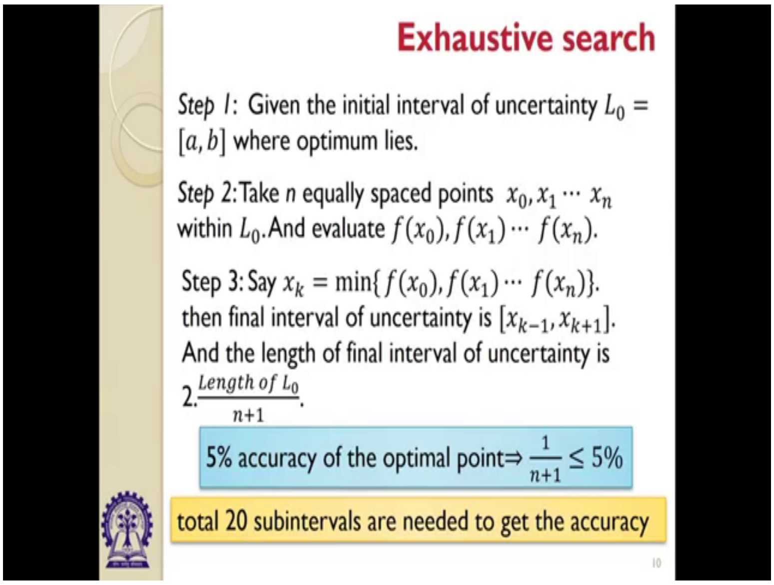

That is called the exhaustive search method, what it does the whole interval of uncertainty it breaks into several points exhaustively; that means, we are if there are the interval of from 0 to 100 I will consider 10 points within that, I can consider 20 points within that and I will see the function pattern at those points. Since the function is unimodal from there we can guess where the optimal may lie formally let me introduce the process.

Now, here the initial interval of uncertainty L naught is given from a to b where the optimal may optimum may lie, then we will take n equally spaced points x naught to xn in between then how many what would be the each step length rather the interval size it will be 1 by n plus 1. Now if I proceed like this now we are searching for minimum value that is why we will find out the functional value at x naught x 1 x 2 x 4 up to xn then at b we are getting total how many parts n plus 1 plus 1; that means, 2 n plus 2 points will get the function value will find out the minimum where the minimum can be there.

Once we are finding out the minimum xk, we will consider xk minus 1 to x k plus 1 the optimum may lie that is why what interval of uncertainty you are getting? You are getting the interval of uncertainty as l naught size of L naught divided by n plus 1 into 2 because both side I am considering together. Now if you are happy with that consider that otherwise you consider that 1 as the initial interval of uncertainty again you space is take n equally spaced point and concede it the repeat the process and even the beauty of this methodology is that if I want that the approximation must be within 5 percent of the accurate value I do not know what is the accurate value.

But if I want to have the 5 percent accuracy it is very much dependent on, because 5 percent accuracy gives you this inequality 1 by n plus 1 must be lesser than 5 percent from there we could guess that value of n must be more than 20, that is why for getting the 5 percent accuracy we should have 20 sub intervals ok.

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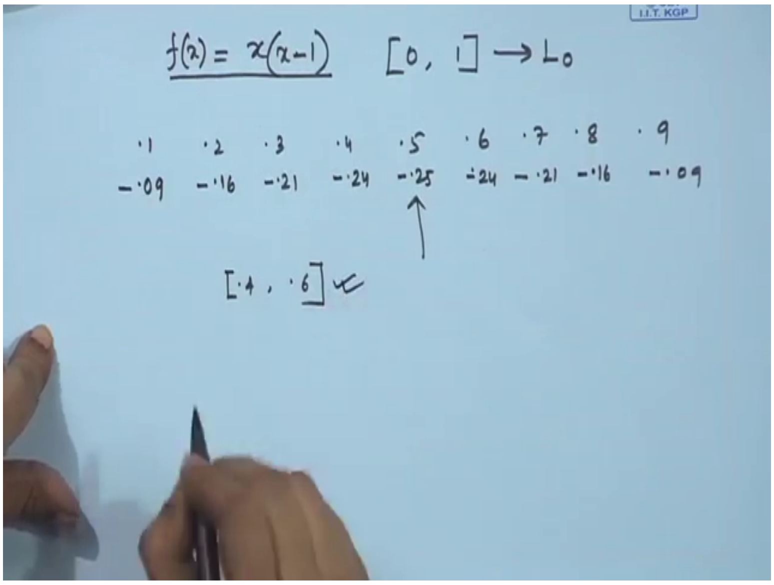

Let us consider 1 example for this there is 1 function the function is very simple this is continuous function within the interval x into x minus 1 and it is a continuous function within the interval 0 to 1 and this is my initial interval of uncertainty all right.

We want to find out that where the optimum is, what the optimum is there; that means, where have we want to get a interval of uncertainty that is much more smaller than this initial interval of uncertainty. For that thing what is suggest you find out through classical optimization technique even other than these searching process. I am breaking this interval from 0 to 1 with equally spaced and if each space may I am considering as point 1, that is why I will get 0.1, 0.2, 0.3, 0.4, 0.5, 0.8, 0.9 here it is 1 here it is 0 ok.

Let us calculate the value of the function at each point, we will get the if we calculate we will get the functional value as point is 0.09 0.16, 0.21, 0.24, 0.25, 24 2 1 you could guess where the minimum is lying when is the minimum? Here is the minimum minimum must be lying here somewhere that. So, if this is my xk as I was mentioning that consider xk minus and xk plus 1 as your new interval of uncertainty; that means, that you consider 0.4 to 0.6 as the new interval of uncertainty ok.

Now, what is the length of the new interval of uncertainty here? We are getting the length of the new interval of uncertainty as 0.2 all right this is the solution for it this is another method for getting solution. I will just introduce the next methodology to you and I will elaborate the methodology further in the next class that is another searching process that is called the dichotomous searching process and with that I will just complete my class.

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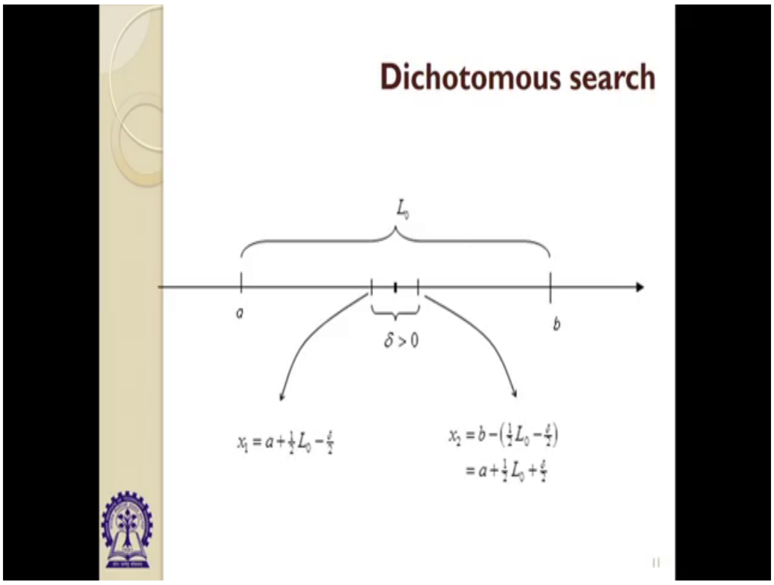

Now, this is the dichotomous searching process, now as I was mentioning that initial interval of uncertainty must be given to us what is should be given the where the function is unimodal in the interval of uncertainty function must be unimodal. How the dichotomous process run? It runs in this way we consider the middle point and it delta never root of that and if we consider a delta never root of that then we are getting the 2 experimental points, one is the x 1 another one is the x 2 we will check that which one is the better candidate for minimization x 1 or x 2 unimodal that is why we will eliminate the region how what is the x 1 value for you the middle value is l naught by 2 that is why x 1 must be delta minus delta to l naught by 2 minus delta 2 because delta is the in the neighborhood I have considered around the middle point that is why, you see with this process what is the advantage if we apply this methodology, with this we are reducing the interval better than the previous one.

Because there because of this delta factor delta factor will reduce at every time delta by 2 or delta regions every time that is the advantage; that means, we will reach to the optimal solution quickly than the previous methodology. As I was mentioning exhaustive search unconstrained search with fixed step length with random step length any one of that. Now, we will we will take x 1 and we will take x 2 that is where x 1 would be is equal to a plus l naught by 2 minus delta 2 and x 2 would be b minus l naught by 2 minus delta 2 within bracket.

And here now my x 1 and x 2 is there, just I was mentioning the region elimination technique x prime and x double prime. Now we will find out the value for x 1 at x 1 what is the functional value x 2 what is the functional value? Depending on that technique there are 3 options only either 1 is lesser than the other or the other one or both should be same accordingly we will eliminate the region further and further and we will just reach to the optimal solution at the end what is. Suggest to you with this idea whatever knowledge you gathered with that you start writing the algorithm for it the steps, and try to apply that dichotomous search on the on this problem f x equal to x into x minus 1 otherwise in the next class, I will elaborate further and in the next class I will do the dichotomous search and the Fibonacci method for you.

Thank you very much for today.