Constrained and Unconstrained Optimization

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Lecture - 35
Unconstrained Problems

Hi. Today I will just solve few problems which I explained in the last class regarding the maxima, minima and the saddle point identification of non-linear programming problem. Now again we are concentrating on one dimensional unconstrained non-linear programming problem. Let me take few example on that.

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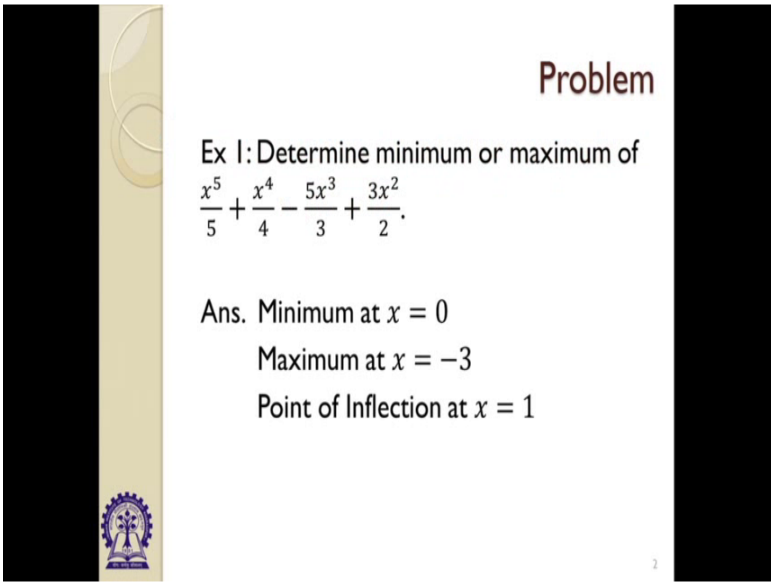

This is one of example. The function has been given as x to the power 5 by 5, plus x to the power 4 by 4 minus 5 x cube by 3 plus 3 x square by 2. We need to find out the minimum maximum and if there is any saddle point or not.

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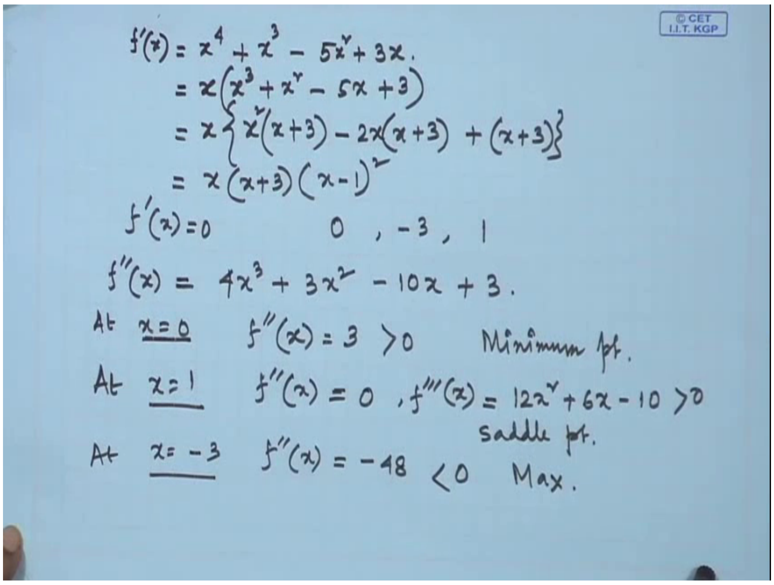

Now, if the function is f x is giving to you, now the process is that now, one thing you have to just see what is the pattern of function. First of all, if you can find out what is the pattern of the function that would be nice. Now for finding out the pattern we have to judge whether the function has in which part of the interval function has the convexity and which part of the interval function has the concavity. Now once the function has been given us a convex function in certain interval, here there is no restriction on the decision variable that is why this is unrestricted in sign x can take any value from minus infinity to plus infinity.

Now if you cons, if you just check in which interval function has the convexity. Then we can declare that within that interval there must be certain minimum point. I explained you in the last class regarding this. And if the function has the concave part in the some other region then function must be having the maximum value within that. And if the function is changing from convex to concavity; that means the pattern of the function is just changing in certain point then that must be the saddle point.

Now I am not going into that geometry of the function of these let us try to solve this function first. For doing that thing the necessary conditions suggest such we will find out the first order derivative of the function, f dot x. Then we will equate to 0 then we will get the stationary points of these. If this is the if we considered the first order a derivative, then x to the power 4 plus x cube minus 5 plus 3 5 x square plus 3 x.

Now, if I equate to 0. Let me just simplify this, if I take x common then it will be x cube plus x square minus 5 x plus 3. Now for these minus 3 must be one of the root of this that is why let me considered minus 3 as one of the root. Then x is equal to x plus 3 must be the factor of it. Then this is x square x plus 3 x cube 3 x square minus 2 x x plus 3 plus x plus 3. Then x into x plus 3, and we are getting x minus 1 because x square minus 2 x plus 1 x minus 1 whole square.

Now, if we equate f prime x equal to 0, then we must be getting there are 3 saddle points for this one is 3, one is 0 another one is minus 3 and another one is 1. Now for checking the maximum and minimum of the function we will go for the second order derivative of f x. Then it would be 4 x cube plus 3 x square minus 10 x plus 3. Let us see what is happening in individual 3 stationary points. At x is equal to 0, we get that f prime x equal to 0 as well as the f double prime x is equal to if I just substitute 0 everywhere it is coming 3; that means, this is greater than 0; that means, x equal to 0 must be the minimum point all right.

Now, let us go to the next x is equal to let me considered one first. Then we are getting f x then these must be is equal to 0. If we considered f double prime x equal to 0, our sufficient conditions suggest that we will go for third order derivative of x. Now third order derivative is coming as 12 x square plus 6 x minus 10. At x is equal to, 1 this value is coming 12, 22 that is must be positive that is coming 8. That must be positive since you we see that from the sufficient condition that third order derivative is positive. That is why we cannot we cannot go further we have to declare then these must be a saddle point all right, at x equal to 1 fx must be changing is it is pattern.

Now, let us go for x is equal to minus 3, f double prime x is coming as I calculated this is coming as minus 48. That is negative; that means, at x equal to minus 3 there is a maximum point.

Now, if this is the information we are getting from the necessary and sufficient condition, what we can see from the function that minus 3, if I just take the interval from minus 3 to plus 3 we see at minus 3 there is a maximum point; that means, the function must be concave. After that it is going to zero; that means, we are reaching to the minimum point; that means, must around 0 that function must be convex. And after that the function is changing it is pattern.

If I just draw it just we see that we will get this function.

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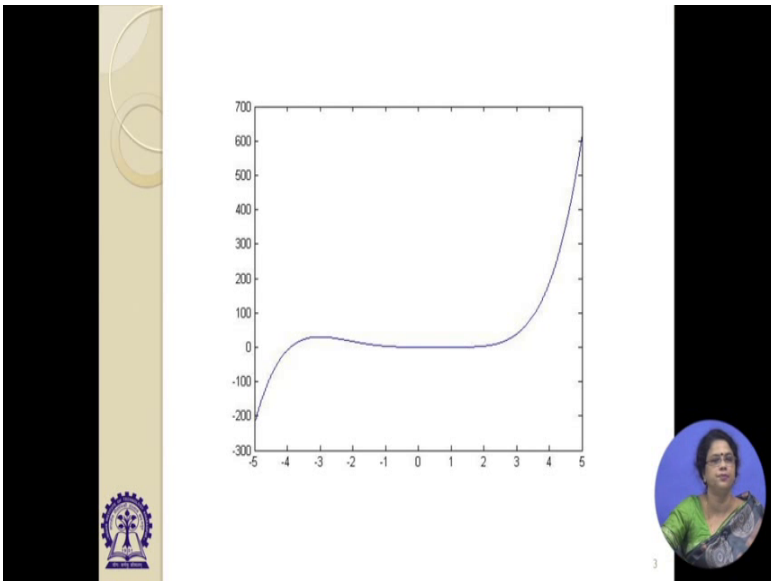

Now, what we see here at minus 3, we are getting the maximum value and at 1, it is changing the pattern and 0 there is a minimum value for the function.

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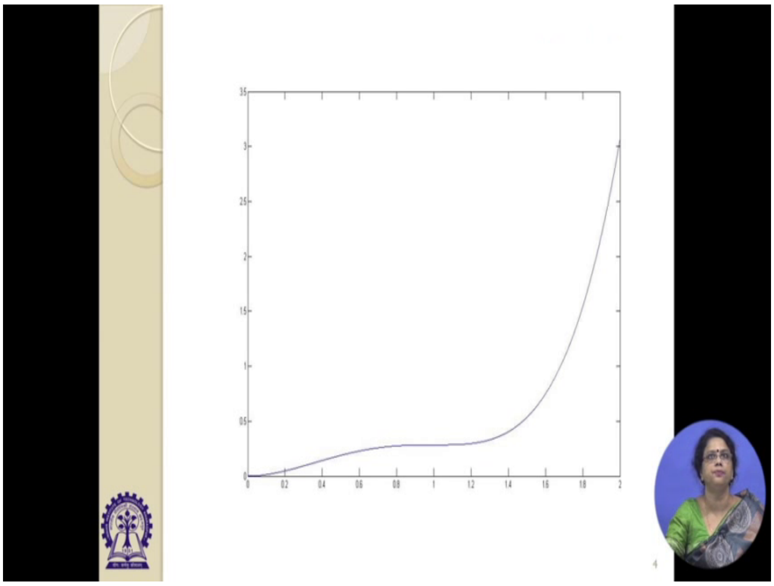

Now, full for looking this one in a better way, I have given another function for you just to see the function. Around 0 the function is around one the function is changing it is pattern from concavity it is changing to convexity.

Now, this is the function all together. Now you see one thing one thing you should notice here that if I, we are finding out the minimum maximum and saddle point from after looking at the function picture, we could see that function at 2 from 1.8 to function is increasing further and further. There is no information about the restriction on x that is why x can varies from minus infinity to plus infinity, but x is changing it is pattern and the function is going up and up after 2. I do not know to where it is going if we draw we can find it out. That is why though we have we found that at point minus 3 function was having the maximum value, that was a pattern of the function maximum value.

But at by looking at the picture we could see the functional value is maximum at 5, but you see we are not able to get it. Why? Because that is the drawback of this methodology as I was mentioning classical optimization technique we are going for the we are calculating the differential coefficients first order derivative second order derivative, if these are all exist that, but one thing is that at the end point of the function we are not able to find out the limit value of the function. Rather we are not able to find out the first order derivative of the function. That is why this point points are not coming in to the picture when we are come we are calculating the maximum minima. That is why by looking after the looking at picture of the function, we can see these are all the minimum maximum these are all the local these are not at all the global optima for the giving function. Now we have to since we have identified the problem in solving the classical optimization technique, we have to see the solution how to get read of these problem.

Now, that is why the next level of problem next level of calculations are coming that call the numerical optimization I was mentioning. Now let us take another problem. Now for this problem we are trying to sketch the function.

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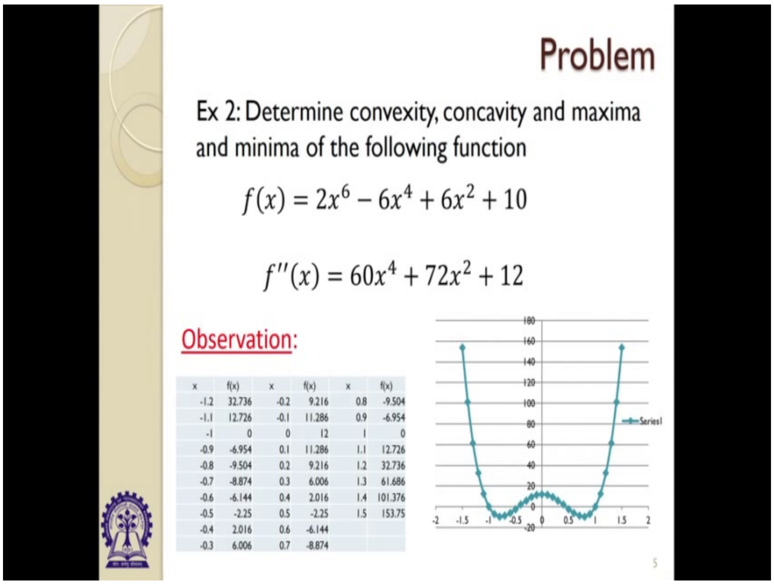

First then we will go for the maximum minima of the function this is the function for us, f x is equal to 2 x to the power 6 minus 6 x to the power 4 plus 6 x square plus 10. Now if I just ask you just tell me how to case this function. One thing we can do that we will first find out the concavity part and the convexity part of this function.

Now, for doing this thing for finding out the convexity or concavity, you know we have to find out the second order derivative of the function. Now if the second order derivative if it is positive, one thing is happ one thing is either convex or concave and of a double prime x is negative some other. What is that? As we know if double prime x is positive; that means, we are getting the convexity of the function. And if f double prime x is negative we are getting the concavity of the function. That is why let us see what is the value for f double prime x.

Now the x is again is not persisted. It can move from minus infinity to infinity. That is why just to excel I was finding out few values for x and the corresponding fx and try to find out the values for f double prime x. What we could see here, if we considered the values just you see from minus 1.2 minus 1.1 minus 1 minus 0.9, if it is moving further at 0 the value is 0, and after it is going to minus then it is going to plus, where it is going the changing it is pattern just look at the excel file you could see the it is changing it pattern at 1. Because before one this is negative after one this is positive at 0, this is the minimum value and at minus 3 we are getting the maximum value, minus 3 is included here yeah. Minus 3 is not included here that is why it is not there.

Now we try to draw the picture of it and we could find out this is the picture of f double prime x. What we could see in the f double prime x that, after minus 1 the function value is the double prime x is coming negative. After that 0 it is positive. And at 0 we are we will see this is not a function f x this is a f double prime x all right. That is why must function must be having the concave pattern convex pattern here f double prime x is negative that is why. After the taking we are getting the negative value and after one it is changing from negative to positive. It is not maintaining before one and after one the pattern is not being maintain from convexity to concavity it is going.

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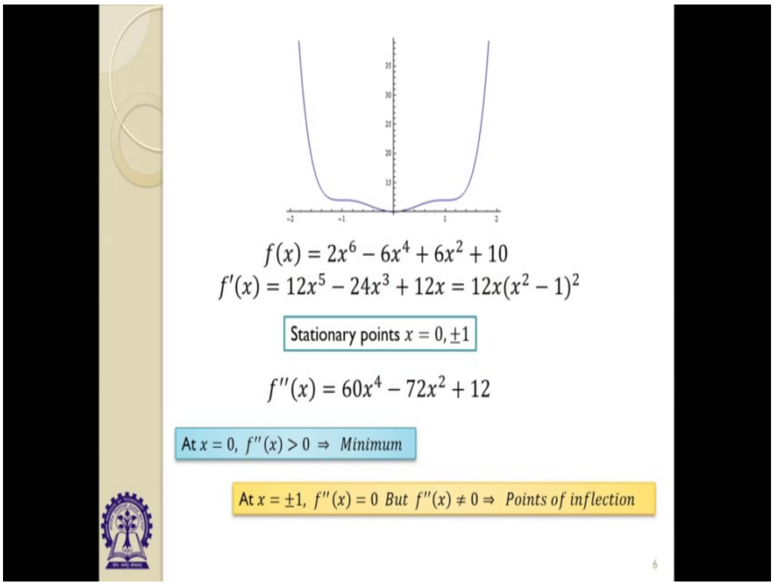

That is why I just draw the function after that just you see. We are getting the convex concave part at minus 3 and at 0, we are getting the convex part of the function.

Again at one it is changing is pattern from minus from the concavity to convexity. That is the thing, now we are applying the classical optimization technique for maxima and minimum. Now if we just look at the function we could see the minimize coming at 0, and maximize coming at minus 1. And after that we are having the changing pattern at one. That is why the first order derivative. We could see now oh no at minus 1 also just we could see the that is it is changing it is pattern, the stationary points are 0 plus minus

1. Now f double prime x is these and we could see at x is equal to 0, if double prime x is positive that is why it is having the minimum value, all right.

And at x plus minus 1 f double prime x is coming 0, but f triple prime x is not 0 that is not double prime that is triple prime x is not equal to 0. That is why we can see we can say that seems the aim is here or that is why that must be the point of inflections at point plus 1 and minus 1. From the picture of also we can see the same things same pattern. That is why see if the function is given by judging the convexity concavity part minimum maximum part, we could find we can find out the we can sketch the function rough sketch we can have of the function all right, but the exact for getting the exact picture of the function we need to study something more than that. That is the asymptote and all other properties, we need to find out then only we can sketch the function better otherwise a rough sketch we will get true this only.

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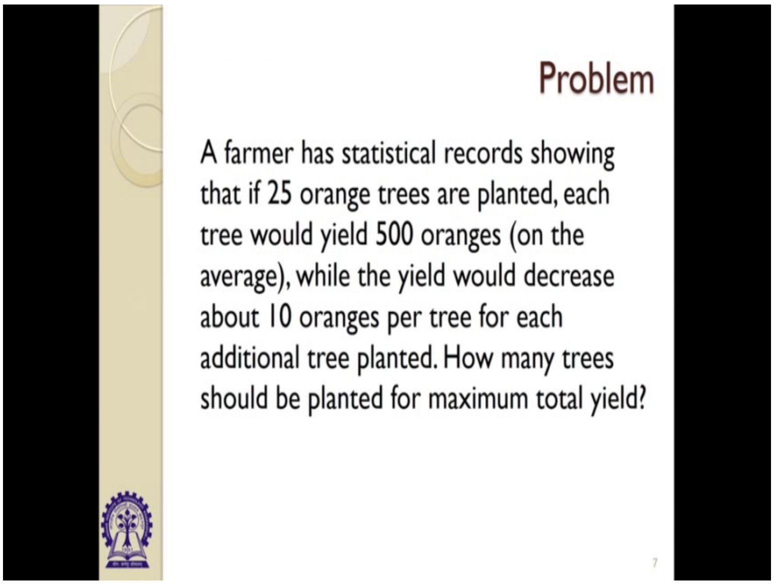

Now this is another problem just see. Now a farmer has statistical records showing that if 25 orange trees are planted each tree would yield 500 oranges, but there is one information that. If we plant only one tree extra, then 10 oranges per tree will reduce. Now the question is given to you how many tree should be planted for maximum total yield. For normal case we are getting 500 oranges from a tree. Now, but it the next information if we plant one tree extra from each tree train 10 oranges will be reduce; that means, we will get 500 minus 10 from each plant. Now the question is that how many tree should be planted. So, that maximum yield will come.

Now let us to construct is problem we have to just construct function for solving this. Now let me solve it manually.

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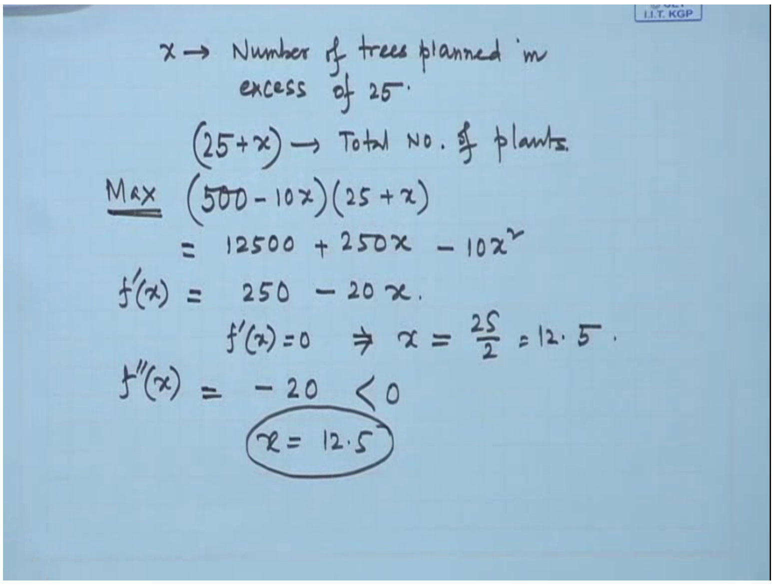

Let me considered the variable x as number of trees the farmer is planting in excess to 25, in excess of 25. Because the initially the information has been given that in the normal case is planting 25 plants. That is why in x is we are considering x plants are being planted. If x is equal to 1 then how many how much yield he will get he will get 500 minus 10 in to 25 because there are 25 plants. Now if there are 10 number of x number of plants 490 to 26 because 25 plus 1 26 you are right.

Now, if there are x number of plants, then what should be the number of for plants all together it would be 25 plus x total number of plants. Now how much yield I will get? If it is normal we will get 500 in to 25 plus x, but this is not normal there are x number of trees that is why I will get 500 into 500 minus 10 x. And total yield would be this in to 25 plus x. Now I have to maximize this this is my question, because I have to maximize the total number of yield all right.

Now, if this is a problem for us then just look at the function this function is coming as 125 0 0 plus 500 minus 500 x minus 250 x. It is coming 250 x minus 10 x square all right we have to maximize this function. Now let us apply the classical optimization technique, what we are getting f prime x is equal to 250 minus 20 x. That is why the stationary points are if I just equate to 0, f prime x equal to 0 gives me that x must be is equal to 25 by 2 that is 12.5 let us find out f double prime x for this. Double prime x would be minus twenty; that means, always it is negative. That is why it is suggested that you plant 12.5 number of trees for getting maximum yield.

But you see the problem is such a problem here certainly, there is a restriction on the decision variable x that always x has to be positive. It cannot be negative number of trees cannot be negative. What else you are getting the restriction number of tree cannot be discrete. That is why this kind of optimization problem is being named as the discrete optimization problem, but you will see if we just find out feasible space the feasible space here we are considering the whole range from x is equal to 0 to infinity, which is totally continuous, but that should not be the always we will have the value for x is 1 2 3 4 5 etcetera integer numbers.

Now, we are getting x equal to 12.5. That is why this process is not correct process to judge that we are getting the result nice result for it. That is why that is the disadvantage of using the classical optimization technique. Now that is why there is a another series of techniques are available that is called the what is numerical optimization technique. There is range of variations of the methodologies. Now today I will just discuss one simple methodology for solving the through the numerical optimization. And the advantage is that even the space is discrete, we can not use those methodology and nicely we can get the solution of it the very well known method you must have been learned it that is the interval having process.

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1	12740
2	12960
3	13160
4	13340
5	13500
6	13640
7	13760
8	13860
9	13940
10	14000

11	14040
12	14060
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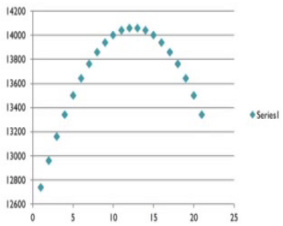

Now, before to that I get certain values I was calculating certain values for affects. And I just draw the picture of it. And certainly we are getting that at x equal to 12.5 we are getting the optimal solution.

That here if we see we are getting the points 0 1 2 3 4 5 like that up to 21 we would calculate. Now one thing is that one thing we can say that within the range of 0 to 21 function is unimodal. That is why whenever we are getting a function it is a better practice for us to just judge the property of the function, let us draw it for if I am having the tool. If I am having any softer for plotting the graph do it otherwise just use excel as I did here. Just simple excel will be sufficient for you would the value of x put the calculate the value of f x and draw it. That is why classical optimization suggested 12.5 it is very much correct.

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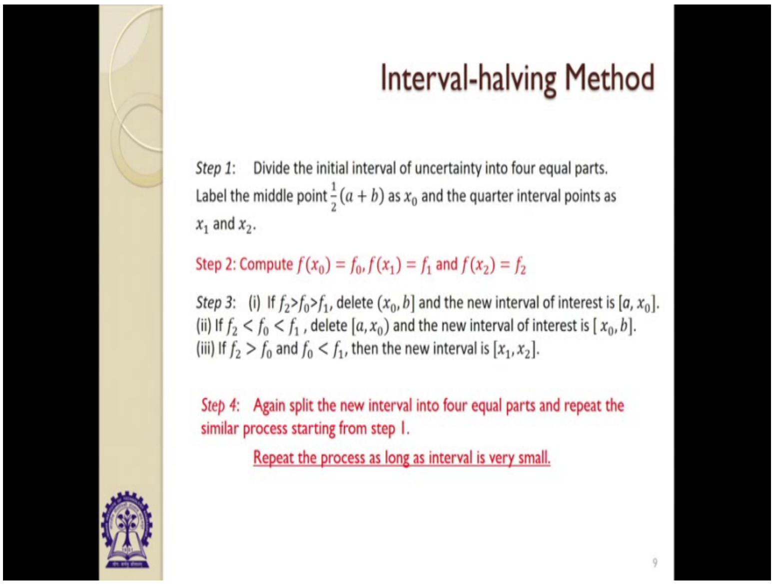

Now, this is not accepted. That is why we are going to the next level of technique optimization technique through which we will solve the problem this is the very well known methodology. All of you must have been done in numerical analysis interval halving process. There is another name to it that is called the by section method. We will solve the same problem and we will see how nicely we will get the solution for the same problem where the solution will be we can consider as integer all right.

Now let me tell you the steps of interval halving method first. Now the first step it saying that whenever interval halving method is tells you that, as we could see from the function fact and that always from maximum is there from 0 to 21. That is why it is suggested that let me considered 20. It is suggested that always from 0 to 20 you make the interval half of it. That is why we will get one part 0 to 10 and another part 10 to 20. After that each part you make you half it, that is why will get 0 to 5 5 to 10 10 to 15 10 to 20. That is why initially what you do you take the left point of the interval and the right point on the interval and in between in equally use space 3 points. So, that whole interval can be divided in to 4 parts. The process tells you that if a is the lower bound of the interval this interval is called is the interval of uncertainty, because we do not know that is totally uncertain to us where the optima lie. Only thing we have some information function is unimodal in between 0 to 20.

That is given information to us. That is why 0 is a and 20 is b we will have it and this name it with x 0. After that we will again from we will consider the interval points x 1 and x 2 in such with that x 1 will be in between 0 to x a to z naught. And x 2 would be from x naught to b that is why.

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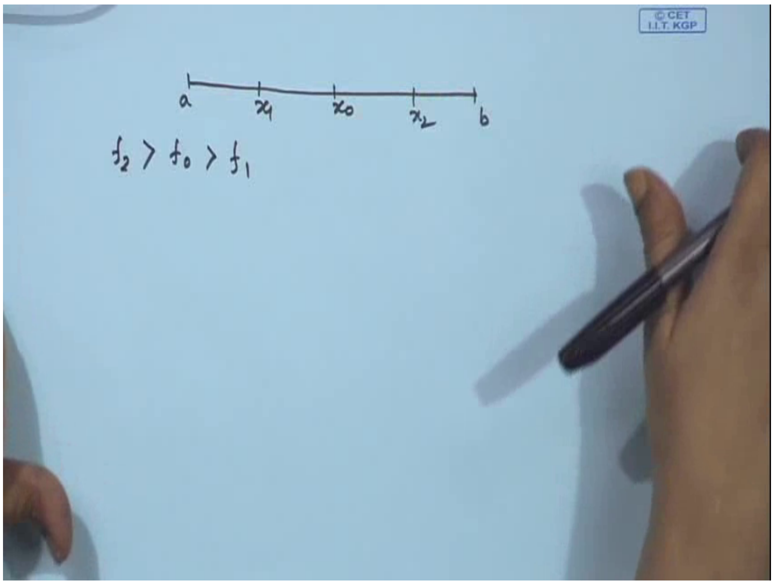

If I just draw the interval that interval will be a b x naught x 1 x 2. Now we could see here that we are getting the point x naught that is the middle point of ab x 1 is middle point of a and x naught and x 2 is the middle point of x naught to b. Just look at the methodology we have written here if f 2 is greater than f naught and greater than f1 and we are looking for the minimization minimum value of the function certainly we are function is unimodal. That is why there is only one more there only one minimum within the interval. That is the minimal cannot lie from within this region from x 2 to b because function is gradually increasing. That is why we will discard a part of the interval, which part we will discard we will discard the part from x naught to b. And we will get a new interval of uncertainty as a to x naught all right.

Let us consider the other case that f 2 is lesser than f naught and lesser than f 1; that means, in the left side function is increasing and then in the right side function is decreasing, but if I just minimize the function certainly, in the left side if in the left side function minimum cannot lie all right. That is why will discard that part that is why you see we have written the methodology deleted to x naught and you considered new interval of uncertainty as x naught to b.

Now, the other case, if we con if we see that f 2 is greater than f naught and f naught is less than f 1; that means, we are getting the around f naught we are having the minimum and f 1 one side this having the higher value f 2 one side higher value; that means, the minimum cannot lie bellow to x 1 and the beyond to x 2. That is why will get the new interval of uncertainty as x 1 to x 2. After what we will do once we will get the new interval of uncertainty, that one will consider at the initial interval of uncertainty we repeat the process. We will again find out 4 parts of the interval, we will find out new x naught new x 1 new x 2 we will repeat the process again and again and how long we will do in repeat the process as long as the interval of uncertainty is very small.

Or we have certain target that we want to get accuracy of 10 percent, accuracy of 5 percent. Then we can have the better pross this process can give us that kind of accuracy. In that way will we will get the solution. Now this is one of the method numerical optimization. What even numerical optimization method we will get we will see every where the challenge lies how to select x naught x 1 and x 2. Because these are only the guiding points for selecting the minimum value of the function. That is why from the next class I will give you few more nice methodologies. And after learning all the methodologies together we will just compare all the methodologies, that is all for today.

Thank you.