Constrained and Unconstrained Optimization

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Lecture – 34
One diamensional unconstrained optimization

Now, today I will talk on the non-linear programming problem, the simplest form of the non-linear programming that is a single variable unconstrained optimization. Now, there are several methods for solving non-linear programming unconstrained that is unconstrained single variable, but I will just first discussed what are the methods available then I will come to the process how to handle that kind of problem. Now, you see since I say the unconstrained optimization problem that is why you might be understanding that there is no constraint for this problem only the objective function is there that is non-linear in nature.

Now, since the non-linear function because of its complexity as I told you before that function can be kind of a concave, it can be convex and it can be neither follows any pattern of convexity or concavity that is why handling the problem would be a little bit difficult, when we have the function of the discontinuous nature. That is why whatever methodology is available till date, all the methodologies can be divided into two category one is the classical optimization technique.

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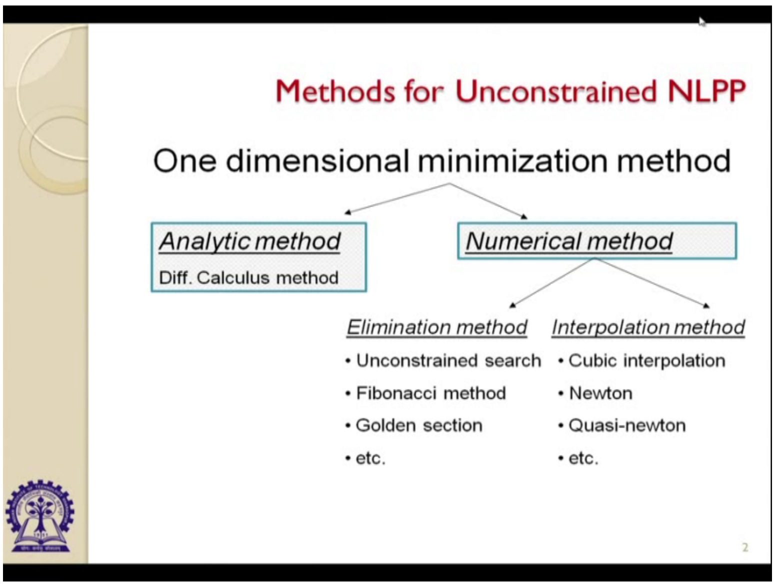

That is the analytical method and another method we call it as a numerical optimization, or the numerical method for solving unconstrained optimization problem. Now, the analytic method that is the classical optimization method is very much dependent on the fact that we consider the cases of the function with the non-linear functions are continuous in nature, continuous in the domain of the definition of the function. Since, it is continuous that is why we can assume that function is differentiable over the domain of the function that is why classical optimization technique or the analytic method is so much popular. But in real life situation in reality when we are dealing with the application, we will see that all the non-linear functions are not really continuous in the given domain that is why handling that difference classical optimize it through the classical optimization technique rather the differential calculus method is not all of is very much accepted.

That so we have another set of methodologies that is the numerical optimization. And in the next week, I will discuss about this and there are that also can be categorized into two parts that is a set of methodologies, these are the elimination technique, region elimination techniques and another set of methods are there that is the interpolation method. Interpolation method sometimes depending on the differential calculus method, but in general numerical optimization not always depending on the differential calculus technique. But you know though we have a range of differentia numerical optimization, but the basis of the numerical optimization is the classical optimization that is why classical optimization is very important to learn when we are dealing with a non-linear optimization problem.

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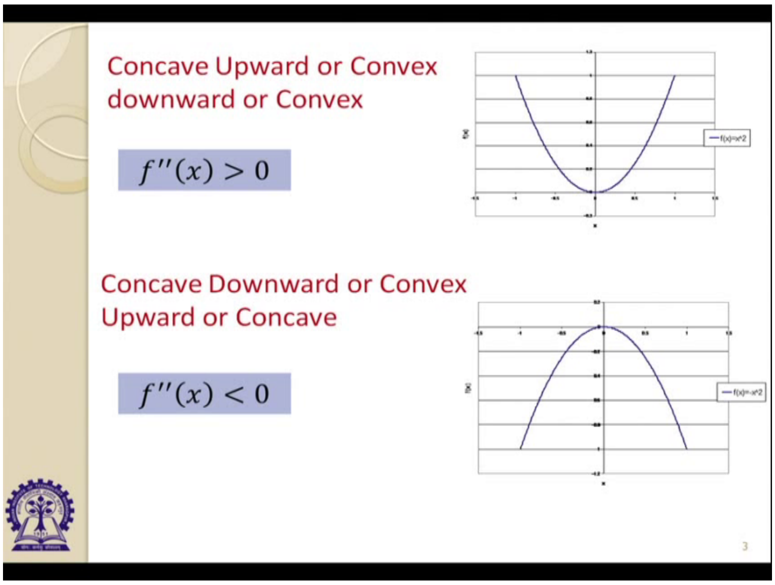

Since we are considering the non-linear function as we know that there are either the function can be convex or function can be concave or function is neither convex nor concave that is why due to the curvy nature of the function. I define in the last class that the convex is the how geometrically a convex function can be defined I already told you. Now, either we call it as a convex function, but it is a really we are dealing with the functions those are concave upward or convex downward or in general, we call it as a convex.

Now, how to remember that this is a convex function, there is a small trick about it, there is a very nice. Actually we are saying that whatever downward is there, convex we name it as a convex; concave downward we name it as a concave. But you see this is a concave upward that is the convex, concave upward to take the c from concave an upward cup then it coming as c u p cup that is why it is looking like a tea cup, and this is a convex function. You see I have given one definition has been that is the second order derivative is positive. And for the concave function, I have given that is the second order derivative is negative how really we can say this part, though we have defined concave function at the convex function geometrically, but algebraically how we can define the convex function and concave function.

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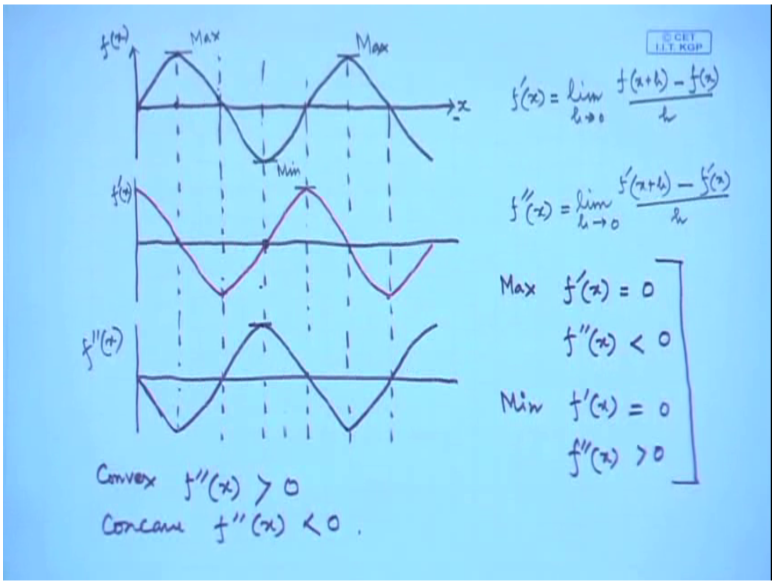

Now, I will just explain to you. Now, this is you look at this function. There is a curve; the nature of the function is combination of the concave and the convex part. This is the concave part; this is the convex part, again the concave part as if the function is moving like this all right. You see it is given that for the concave part second order derivative is lesser than 0. How really we are considering just look at this function. Now, if I move from this point to this point, now, if I consider the first order derivative rather the slope of the function at every point if we consider then we will see that the functions is a function is increasing, slope is positive from this point to this point.

Now, how we are considering the definition of the first order derivative. All of you know that f prime x is equal to limit h tends to 0 f x plus h minus f x divided by h. It means that if we consider any point x over the real line x, then if I from the x point, if I just take a small very small increment x plus h, then limit h tends to 0 will give us the first order derivative value. Since, the function is increasing that means x is there and x plus h is situated in the right hand side of x that is why always f x plus h should be greater than f x that is why we can see that the first order the numerator is coming positive.

H is always positive that is why the first order derivative of f will be always positive for increasing function. That is why if I just move from here to here, from this point to this point, we will see that the first order derivative will be positive. Now, here the first order derivative is 0, because if I just draw a tangent that would be the parallel to the x-axis that is why the slope up at that curve would be 0.

Now, if I just move from this point to this point, then again we will see the function is decreasing, then first order derivative will be negative. That is why we are coming from positive to 0 then negative again may be negative from here to here, but again from here to here the function is increasing positive again further positive, again 0, then negative just moving like that. If it is draw it, just look at the curve. This is the curve for f prime x the first order derivative as I said at this point from positive to coming to 0, function is decreasing from this point to this point, that is why all are negative, but there is a change that at this point it is 0 coming negative. Here we are getting this point and here again 0 that is why it corresponds to this point, in this way the f prime x is moving.

Now, let us consider the next derivative f double prime x. What is that, this is limit h tends to 0 f prime x plus h minus f prime x divided by h that means we are considering the first order derivative of this function, that is the second order derivative of the previous function. Now, again you consider the same part here. If I consider from here to here function is decreasing that is why the first order derivative of f prime x would be negative; here it will be 0 increasing positive, it will move to positive, again it will be 0, again it will come to negative, in this way it will move.

If I even draw it just to see what is happening that if this is my f double prime x then the function is coming down all are negative 0 at this point, because again function is changing its behavior from negative to positive rather the decreasing to increasing that is why function is coming, this function is coming positive. Again coming down, here it is 0, because here the tangent at that point would be parallel to this axis – x-axis and here it is 0 again decreasing negative that way it will move.

What is the nice thing you are just noticing here, just to see if I consider the original function f x, this function is having the local and global minimum, local and global maximum because you know I have defined you know the definition of the local minimum and the local maximum etcetera. That is why I can say here is one local maximum, here is one local minimum, here is again a local maximum that way. What is the nice thing your noticing here, at the maximum point, first order derivative is 0 that is why we can easily say that at maximum point the first order derivative of f would be is equal to 0.

What else, you are seeing the second order derivative of the function at the maximum point is coming negative, you must have learned these in your school time that at the maximum point of a function always the first order derivative is coming negative. What about the minimization function? If I just see at the minimum point, again the first order derivative is coming 0 here at the minimum point, and second order derivative is coming positive. This is the fact we will use for non-linear programming solving unconstrained optimization solving.

What is the other part you are noticing here, this will look at the graph in the screen just you see in the convex part second order derivative is positive. Now, for this function where is the convex part the convex part is this part only, because this is the cup concave this is the concave upward that is the convex part. That is why this is coming positive here. But how many in concave parts are there for this function, this is one part, this is another part. In the concave part, you see we are having the negative of the second order derivative that is why we could establish from here again for the convex, and for the concave, now we will use this fact for solving the unconstrained optimization rather the unconstraint non-linear programming that is of single variable.

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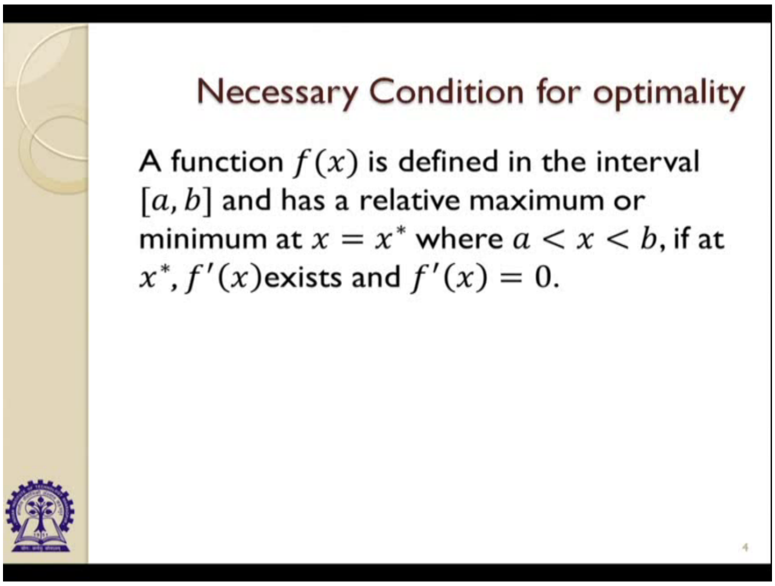

Now, coming to the part of it. Now, if I consider a function f x that is non-linear in nature, and the domain of function it is given the interval a to b, then we will get the relative maximum or minimum at point x star, if the first order derivative is 0, but it must exist then only. You know this fact, but I do not know whether you can prove it or not that is why I am coming to that part how to prove this. Why we are talking about this is a necessary condition, because with this condition we cannot really declare that. When at any point we see the first order derivative is 0 from the graph itself you must have seen it can be maximum, it can be minimum it can be saddle point as well.

What is the saddle point saddle point is that point where the function is changing it is nature from convex to concave or concave to convex. There is neither maximum nor minimum, we know that fact that is why this necessary condition is the first check to identify whether there is any local maximum or the relative maximum or local minimum or saddle point. Now, we will just go for the proof of it.

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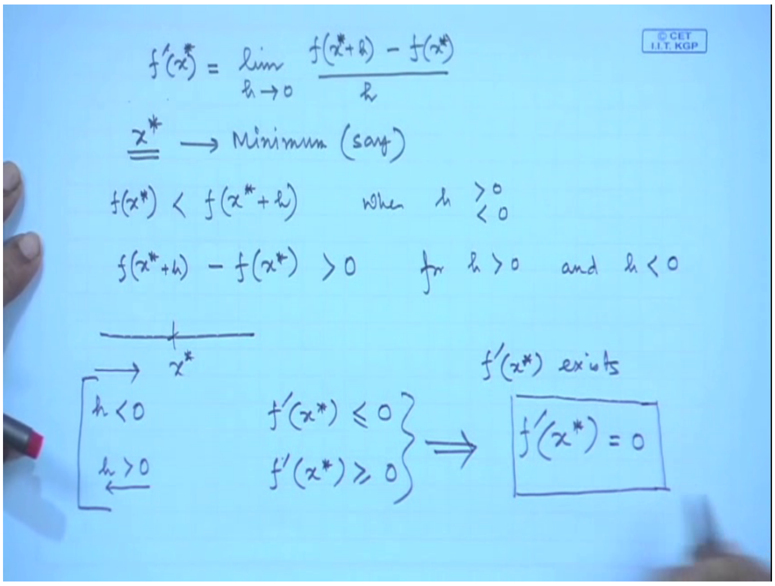

For proving our it, now I am coming to the basic definition of the differential of a function f. Just now I was mentioning limit h tends to 0 f x plus h minus f x divided by h. There are few things I have mentioned here just you see that function the first order derivative must exist.

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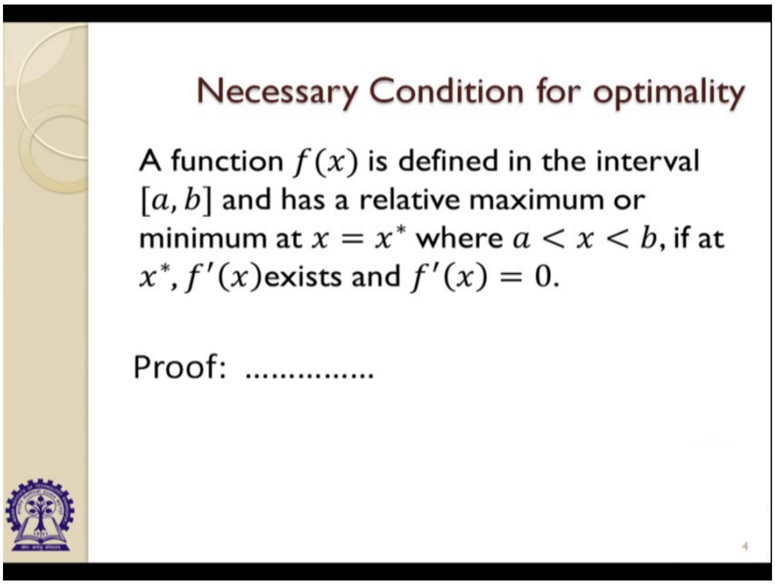

That means the limit value must exist. What is the meaning of it? Limit value exists only when le left hand limit right hand limit and the value of the function at that point all are equal then only limit value exists and then only the first order derivative exists that is the consideration. Now, we are saying that there is a point that is x star, and we are declaring that this is either minimum or maximum.

Let us consider that x star is minimum; we are assuming. We will prove it again for if x star is maximum. If x star is minimum then I can redefine this definition f prime x star is equal to f x star h f x star this one all right. Now, what we have said that f x star is minimum that is why always f x star would be lesser than f x star plus h because you have learnt that if there is a local minimum, then in the neighborhood of that point, we would not get any better value rather any other minim lesser value for that function other than that point. That is why we can say that f x star is always less than f x star plus h when x is greater than 0 or less than 0 then only we can consider the neighborhood we are considering the function of single variable. That is why we can consider that is there is a neighborhood in the real line either h in the left hand side or h in the right hand side of x star.

If this is so look at the numerator from the numerator what we can say that f x star plus h minus f x star always will be positive for h greater than 0 and h less than 0 even. Now, you see h is tending to 0 that is the point is there x star h is tending to 0 either from the right hand side or from the left hand side. If h is negative, what does it mean, it means then approaching from the left hand side to x star all right. If this is so then we can say if h is negative then the numerator is positive; h is negative. What we are concluding f x star must be less than equal to 0, because the denominator is coming negative. Now, if we consider h greater than 0 that means, we are approaching to x star through the right hand side of that. If I approach to that what is happening we are coming from the right to x star then we are getting the numerator is positive, h is positive that means, this is equal to greater than equal to 0.

And you see both cannot happen together at the point x star, because we have assumed that f prime x star exists then what is the only possibility from these two combination. We are getting only one option we can have that f star f prime x star must be is equal to 0 that is why that is the condition to check that at x star function is having either minimum or function is having either maximum or at that point there could be a saddle point. Because at the saddle point function is changing his pattern from convex to concave that is why the first order derivative 0 at the point where it is changing the pattern all right.

But for the maximization, when x star is maximum only this part will reverse, but the whole logic will remain just reverse to it, we can prove it very nicely. But this condition is not really sufficient to say that there is a maximum or minima, because there are certain drawbacks to it, that part I will discuss it. Now, what is the drawback? For example, function is having maximum, where is the function has the maximum certainly at this point, function is having the maximum value.

But you see at this point you would not get the first order derivative, because there is a discontinuity, you would not get the limit value of the function that is why we cannot apply this definition that is why this definition is not really full proofed to check. Now, what is the other disadvantage, disadvantage is that for that we cannot declare whether it is maximum or minimum or saddle point, that is why there is another set of conditions are there that is call the sufficient condition.

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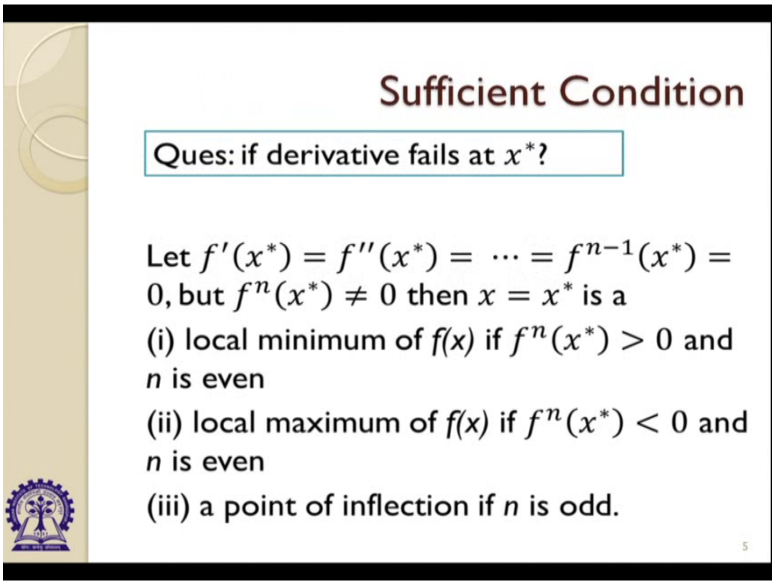

And in the sufficient condition, we are saying that in the first I have mentioned that if the derivative fails at x star, how to do it, that is why the sufficient condition is coming. And in the sufficient condition, we are saying that if up to n minus 1 if derivative if we see that function derivative is coming at that point 0, but at the nth derivative of the function at that point x star is coming nonzero. Then we can say x star is minimum local minimum rather if f n x star greater than 0, when n is even. If n even f n x star must be less than 0; and if n is odd then that x term must be point of inflection this is the sufficient condition for optimality. Now, we are going to prove it.

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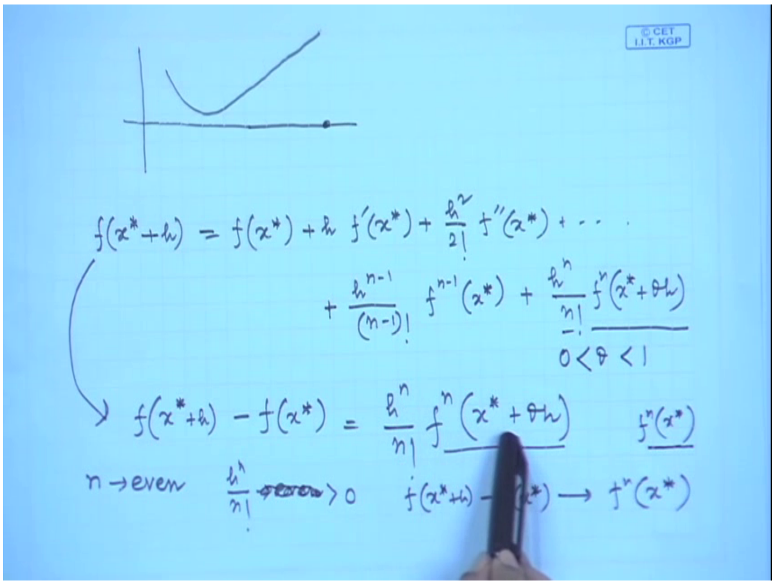

For proving this one, I am using the Taylor's expansion. We know f x star plus h how to expand it, f x star plus h into f prime x star plus h square by factorial 2 f double prime x star and we are considering the Taylor's theorem, where there is a remainder term after the nth term. That is why we will move up to nth term like this n minus 1, this is the nth term. And there is a remainder at the nth term f n x star plus theta h, where theta is in between 0 to 1, this is my remainder term all right.

Now, in the sufficient condition what we have considered we have considered that f prime x is 0 f double prime x is 0 and after f n minus one in x is 0. That means, from here what we are getting we are getting f x star plus h minus f x star is equal to h n because all other term will vanish f n x star plus theta h all right. Now, theta is running in between 0 to 1. Now, what it has been given it has been given only we know the fact that that fn x star is nonzero.

One thing you see here whatever sign of f n x star is there, it will be maintained the same sign on f n x star plus theta here theta is in between 0 to 1 that is why I can say that whatever sign we have for the same sign we are having for this. Now, you are see if n is even, then what we are getting h n by n factorial will be even n is even then this would be rather greater than 0 not really even this would be greater than 0. That means, what we have saying that fn x star plus h minus f x star the sign of it must be same as the sign of f n x star all right.

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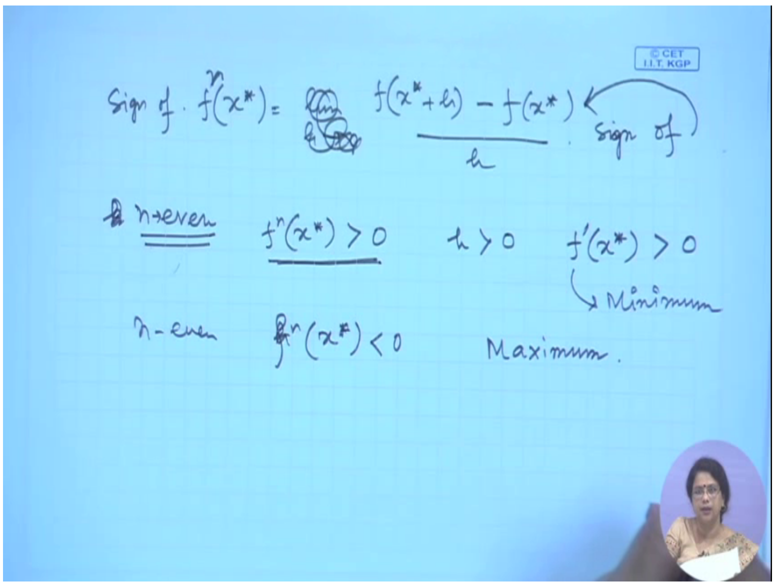

That means, what we are getting from this fact that, we know the definition for this that f prime x star is equal to limit s tends to 0 f x star plus h minus f x star divided by h, we are saying that n is even for this condition we are now first considering. What we are seeing that n is even then if f n x star is positive. If h is positive then what we are getting fn prime x star is positive all right. That means, we are getting this fact that if n is even we are getting a minimum of the function, because from the necessary in a condition we have seen that f prime x star must be 0 that is why from that condition we are checking that this must be having we are having the minimum. If we have not really it is not f prime, we really we would not consider f prime f n, f n x star, the sign of this , sign of f n x star will depend on the sign of this one all right. Now it is clear.

Now, if n is even, f n x star is positive then we are getting from here f prime x star greater than 0 all right; that means, we are saying that at minimum value if n is even we are having f n x star is positive. Similarly, for the n is even what we are saying, we are getting if it is say f n x star negative, then we are having that we are getting the maximum value is it clear? But when n is odd look at this function if n is odd we cannot say anything from here that is why we are not able to see, now you see what we are getting from the Taylor series that f x star placed h minus f x star is equal to h to the power n by n factorial f n theta x star plus theta h.

Now, it is given that f and x star is nonzero. And whatever sign of f n x star we are having it will have the same sign with f n x star plus theta h because theta is very small from 0 to 1, and h is also very small that is why we are having the same sign of these two. That is why if we look at the sign of f n x star from there we can we can conclude whether at x star we are having the minimum or not. Now, from here if n is even, h and n by n factorial must be positive.

Now, if f n x star is positive then this is also it conclude that this is also positive. If this is positive, what we can conclude that is greater than f x star that means, we are having the minimum at x star because all other point other than x star we are getting the higher value of f. That is why from here if I just look at this the first condition of the sufficient condition, we are having the local minimum of f x, if n is even and f n x prime is greater than 0 all right.

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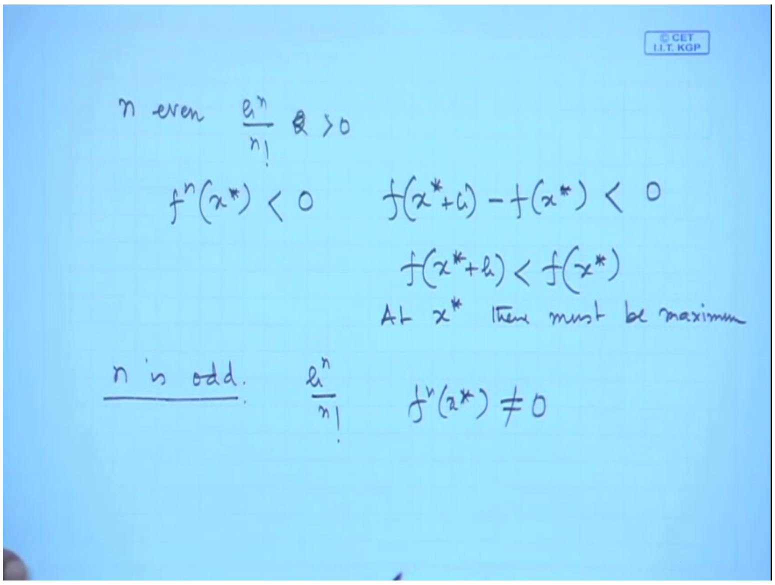

Just let us see the reverse case. When n is even, we are having h n by n factorial is if this is also positive n is even. Then if f n x star is negative then certainly x star plus h minus f x star must be negative. What we are getting that for any other value other than x star, we are having the lesser value functional value. From there we can conclude that at x star, there must be maximum all right, this way we can see that is why we are reaching to the second condition of the sufficient condition. We are having the local maps maximum of f x if f and x star is lesser than 0 and n is even.

Now what will happen if n is odd? If n is odd, you see we are having h n by n factorial depending on the n value odd value it is depending on the x value depending on the h value, this value will change its sign positive and negative, positive and negative that is why we cannot conclude. Whether even the f n x star is having positive or negative anything we cannot conclude that function is having minimum or maximum that way we can say that is why we are reaching to the sufficient condition.

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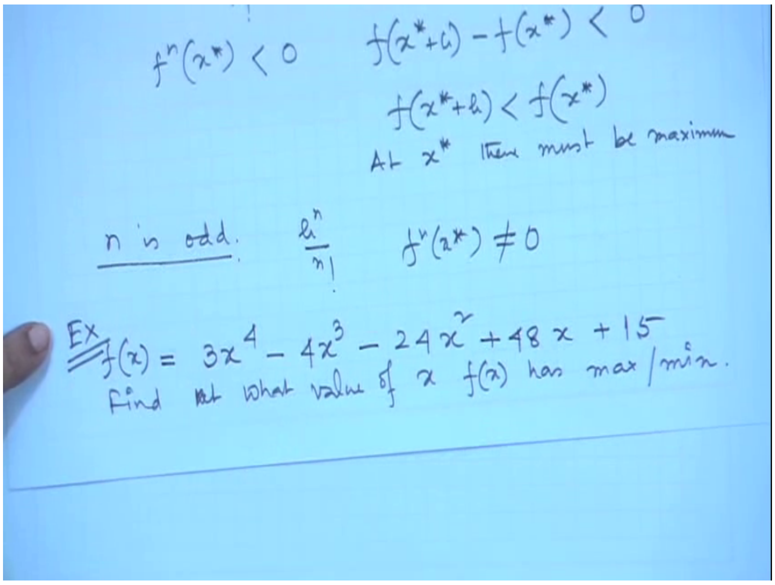

Now, this is the task of yours for the next class, find out the minimum or maximum whatever is there for this function otherwise I will discuss it in the class. This is a non-linear programming with single variable; find at what value of x f x has maximum or minimum. Use this necessary condition and sufficient condition and with that try to conclude.

Thank you for today.