Constrained and Unconstrained Optimization
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Lecture – 33
Types of NLP

Today I will talk on different types of optimization problem. We are dealing with a linear programming, we are dealing with a non-linear programming, but there are several types of optimization models available in literature. Today, I will just brief it in what condition what kind of model we are using. Solution procedure of linear and non-linear programming we are dealing in separate classes, but today I will just give you the brief idea of how many models are available in the literature.

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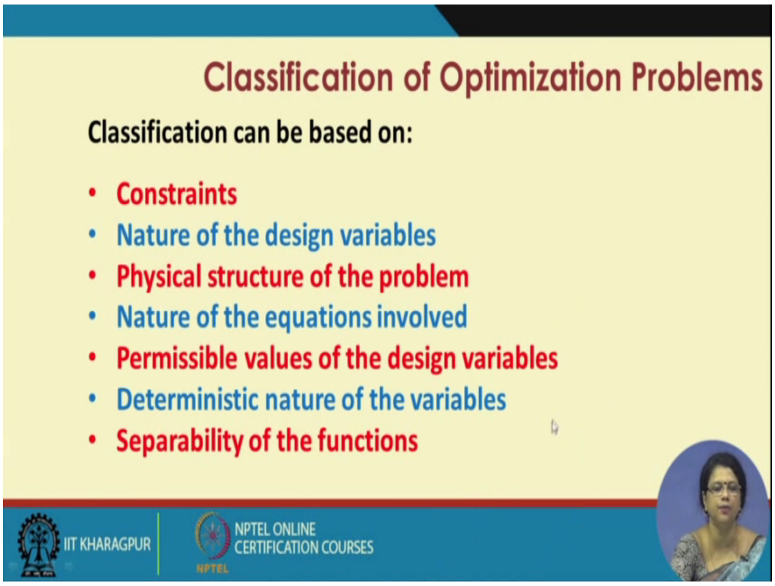

Now, the classification optimization models as you might have been seen in linear programming problem, any optimization model there are few things are involved, one is that we are maximizing or minimizing the objective function subject to a set of constraints that is the basic structure of any optimization model. But there are different building blocks of this model like decision variables, there are different kind of decision variables, they are working in different situation environments are different. And there are nature of the non-linear functions are different, in that way the classification has been done, and classification are mainly based on few that attributes.

One of that is that constraint the problem, the optimization model may have constraint may not have constraint. The second is that with the nature of the design variable, nature of the decision variable we are dealing with. Now, this nature can be of different kind it could be you must might have done the integer programming, where we are dealing with the decision variables, these are integer variables. The problem could be mixed of integer or non-integer variables this is one of the criteria.

The structure of the problem physical structure of the problem is one thing; and nature of the equations are involved this is one part; and the permissible values of the decision variables. And sometimes optimization model works on the stochastic variability, that is why stochastic nature are being captured the parameters involved stochasticity, we are considering the corresponding probability distributions of those, and we are preparing the optimization model for that.

Now, if we do the classification of different kind of optimization models different of depending on these criteria, let us see how really we are meaning that how it is different situations when constraints are different. And there is another part is also there that sometimes the objective functions and the constraints part whenever we are dealing that part. Sometimes the functions are inseparable in nature what does it mean and how really we are dealing with that I will explain you in the next part of my class.

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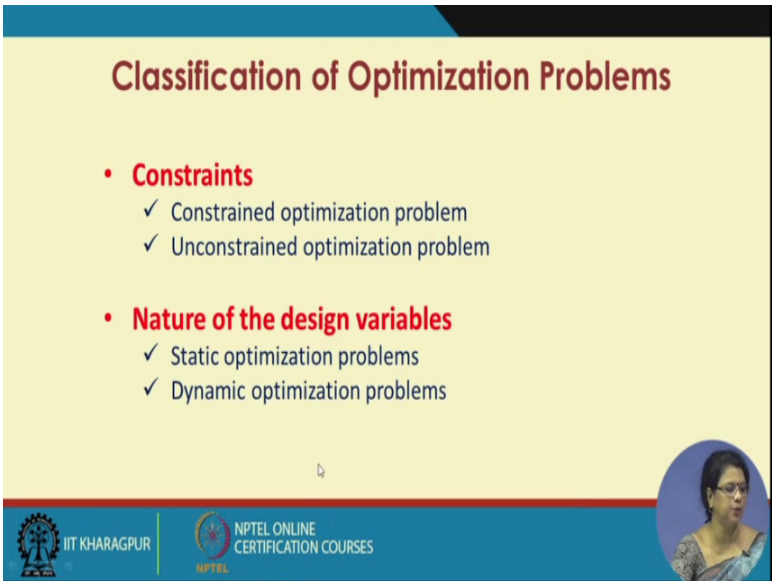

Now, if I deal with a constraint, there are two types of constraint variation we can expect in an optimization model; one is that constraint optimization model and another one is the unconstrained optimization model. Now, if we consider the design variable sometimes it may happen that the design situation is so big and the design situation is dependent on many parameters we see the decision variables or the design variables are sometimes the static and sometime it is dynamic in nature. That is why what we do whenever the system is dynamic enough we break the system in different stages and we handle that that is why we have the static optimization model, where there is no variation in different stages. And there is another type of model dynamic optimization model which are being changed in different stages in different steps in the situation that part how mathematically we can deal it I will tell you.

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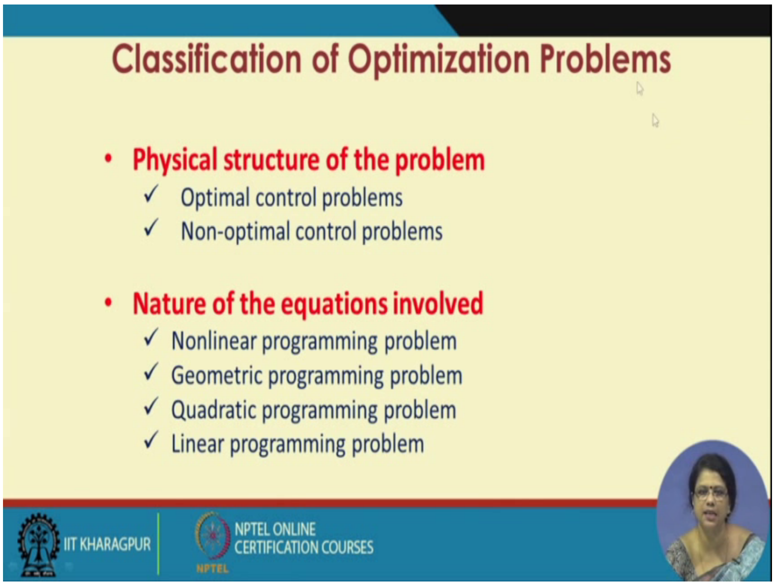

Now, if we just look at the physical structure of the model, one type is very much useful and is being used that is the optimal control problems. There are different control situations we are having. Now, how to optimize that depending on that controlling parameter where the control parameters depends on many other thing. For example, one car is moving on a hill road. Now, if we just see, if we want to minimize the traveling time, we will see there are different attributes, which are responsible to minimize the travel parameter. One of that is that that is the speed of the car, one is the fuel, one is the use of brake, all these things. But if we see that speed is dependent on the brakes condition and many other things and not only that that the hilly road that is why the road is not smooth enough that is why speed depends on many other dynamic parameter.

How really we are dealing in optimal control problem, I will just show you. But there are certain non optimal control problem rather that there is the very straightforward models generally we are dealing those are all non optimal control problem I will show you few examples on that part. Now, if I just look at the structure of the optimization model, we will see that there are different kind of structure we are dealing. Structure in the sense that the functions which are involved in the optimization problem. Functions you must have been seen the linear functions in linear programming; non-linear functions in non-linear programming. Functions means objective functions, constraint functions etcetera. And depending on the nature of that non-linear functions for the linear programming, there is no such variation as such that is why the linear programming there is only one tool we do it that is the simplex algorithm, dual simplex, revised simplex these are all responsible to these are all the tools for handling the linear programming problem.

But there is no specific technique to handle the non-linear programming problem. Depending on the nature of the non-linear function, we are having different tools for geometric programming, for quadratic programming for, fractional programming all these things are available that is why how these are different, and what are the differences are really I will show you in the structure of the equations with examples in the next.

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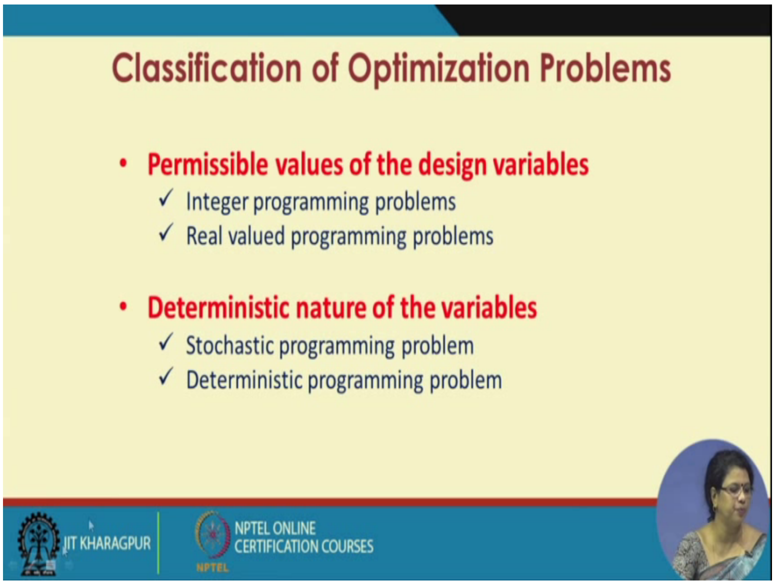

Now, other than this as I said that problem can be the decision variable can take the integer value, decision variable can take the real value as well. Now, that is why whether we are dealing with the integer programming, integer programming can be in linear programming, integer programming can be non-linear programming, integer programming when it is linear you know the methods that is the Bomiriya cutting plane method, branch and bound method these are all available for you.

But for the non-linear integer programming, we have to do it in a different way that is why you need to learn how really we are dealing with the optimization models when that is of non-linear type and the only the permissible values of design variables are in integer. It may happen that, we may have the optimization model where we have the combination of the integer and the real value that also you need to know. It may happen as well 0, 1 programming where the design variables are taking only the value 0 or 1 no other value is permissible for. Assigning the values of those design variables that is why you need to learn that techniques to handle the integer the mixed integer zero one programming non integer and everything that is why this is another kind of model for you.

Now, as I said we can have the stochastic programming problem, we can have the deterministic programming problem. Whatever model you have learned in your linear programming part, those are all the deterministic programming model, because there was no probabilistic nature was considered there, no parameter was uncertain in nature. Everywhere the parameter value was very much precise very much certain that is why that was the deterministic optimization model; other than, these you need to learn even how to handle the stochastic programming problem.

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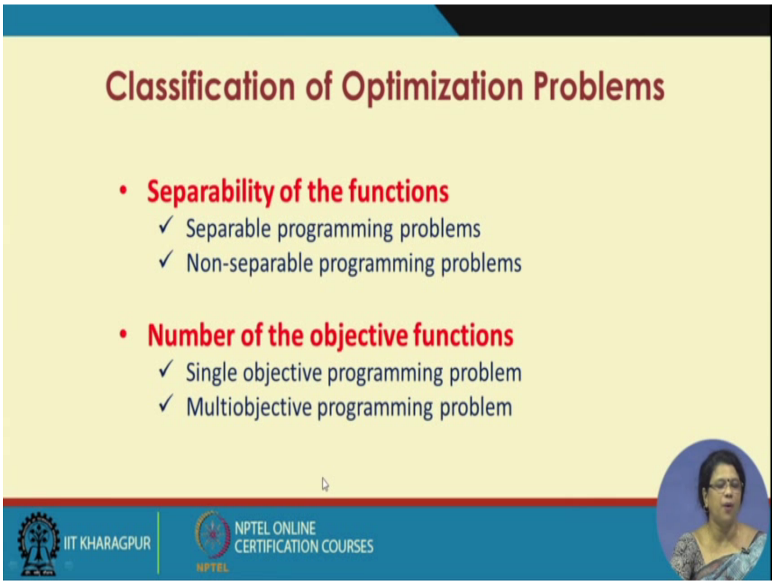

Now, other than these, we can have the separable programming, we can have the non-separable programming problem. And if we just looked at the constraint part as I said we may have constraint, you may not have constraint that is the constraint and unconstrained optimization problem. Similarly, if we just looked at if we just consider the number of objective function we can have single objective function we can have two objects functions; in general we can say we can have multi-objective linear programming problem, multi- objective non-linear programming problem. That is why all these are the types of models available in literature tilted I will show you one by one what I meant really.

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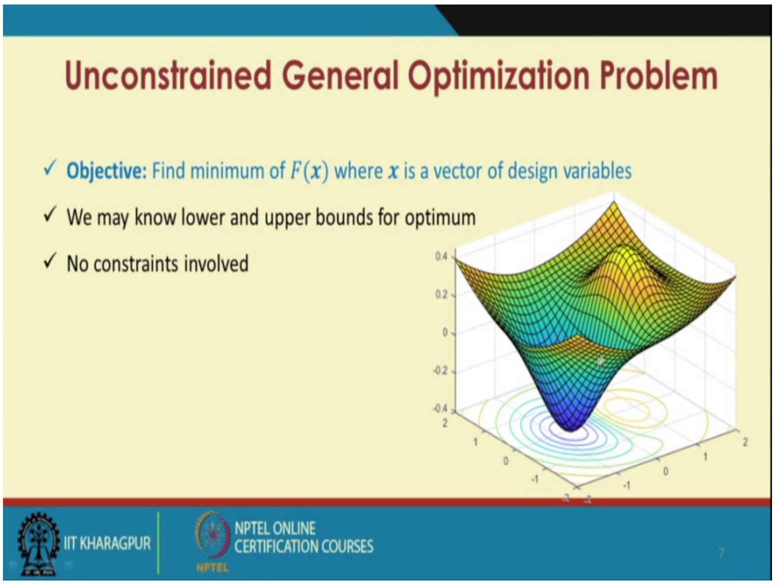

Now, this is a simple unconstraint optimization problem. Minimization of F x where x is a design variable x can move within us. Permissible range from a to b or x can move from minus infinity to infinity. Now, it may happen as well that this optimization model may have more than one variable, same two variables. Now, for example, if I consider minimization of x square plus y square where x is varying from minus infinity to plus infinity, y is varying from minus infinity to plus infinity. If we say so, it must be understanding you must be understanding that this is we are considering the circle centered at 0; and if we consider different level curves of the objective functions which means that we are just changing the diameter of the circle.

Now, if we just change the radius of the circle, where x is varying from minus infinity to plus infinity, similarly for y, there is no end of it. We can have the circle which can be expanded up to infinity is it not. But if we just make the range within a specified limit, for example, x is from a to b, y is from c to d, then we can say that that could be the maximum value of the radius of the of the circle x square plus y square. That is y if we just maximize x square plus y square we will get a definite value. But in the other case, we will get the unbounded solution all right, but for the minimization of x square plus y square for both the cases, we will get the value for x and y as 0 that way. Now, that is why we may not have constraint in that case.

Now, if the lower bound and upper bound are known then its fine then just look at this non-linear function this is a function of two dimension; and in the third dimension, the z-value z is equal to f x, y is there. If you just look at the function, we have to optimize the function. Now, if we want to optimize this function just you see the structure of this function, there is a peak here, there is a just reverse peak here. And there are different functional value and different if I ask you what is the minimum value for F x in this picture that is very clear minimum is here. But if I ask you what is a maximum value of F x, we really do not know from the picture itself at least whether this value is high or this value high, where the height is more.

That is why there is a concept of local minimum, global minimum, local maximum, global maximum. This is locally one maximum, this is locally one maximum, but globally maximum if I have to find it out at least for this figure, there are five options for us, we have to check each and every one. This is the idea. All these ideas I will tell you in the respective classes regarding the unconstraint optimization problem for looking at the local optimality and how to get the global optimality for the problem where no constraints are involved in the model.

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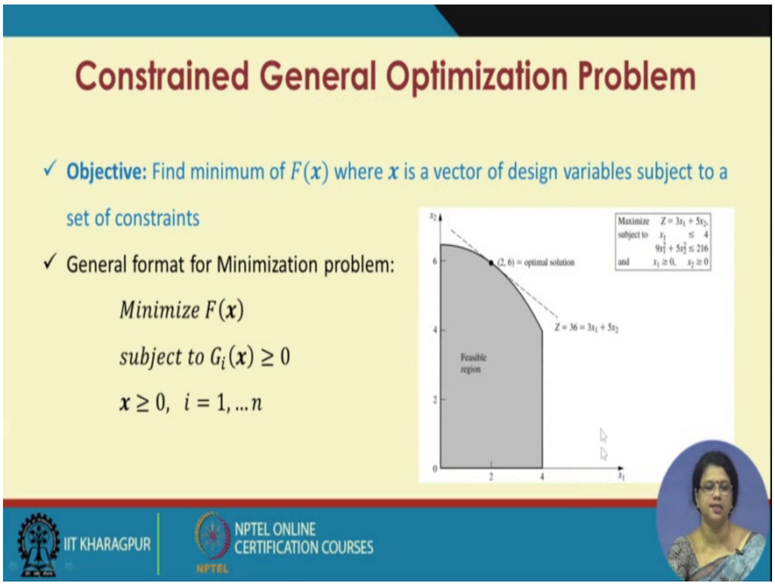

Now, if we just consider a constraint optimization model, now in the optimization model you see we will have a objective function and one constraint or a set of constraints which gives you the feasible space. Now, here if we consider F x and G x are linear, then this is a linear programming problem. And if we consider F and G are non-linear functions, then this is a non-linear programming problem.

This is a simple non-linear constraint optimization model, this is one of the example for that maximization of linear function subject to there is one non-linear function is there. Just you see if we just draw it graphically. And if we maximize the line then at z is equal to 36 that level curve will give you the maximum value that is why we may consider this model as constraint non-linear programming problem, because at least one of the function is non-linear in nature that is why your simplex algorithm will not work here. If all are linear then only we can apply simplex, but if one of that is non-linear we need to know the method to handle it.

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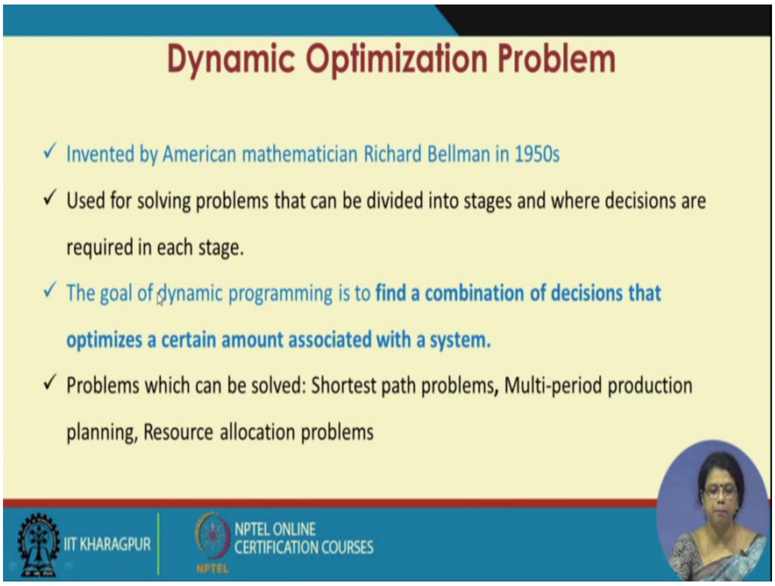

Now, as I said that the optimization model can work in the dynamic situation or it can be in the static situation. The previous model I showed you dynamic means in different time scale, the model is changing. And in the static model means at one point of time, we are getting one static model. The model I showed you before that was a static model at one point of time. Now, if the time changes it may happen the model will change, how the model will change, the structure of the function, objective function constraint may change the parameters may change the parameter values may change.

There may have uncertainty within that in that way how to capture the optimization model in different time scale or different stages that has been invented long back in 1950 by American mathematician Richard Bellman that is a dynamic programming problem where we are dealing the a big problem which is dynamic in nature. We are considering at each stage at each time scale we are considering as if that is the static model. And we are just dealing the dynamic model in that way I will explain more on this part. Just I wanted to tell you if we have a bigger problem, then in dynamic programming what we do we divide into different stages, and we are taking decision in different stages.

Now, the stages are connected in a sequence. The first stage, then the second stage then the third stage in that way that is why whatever output of the first stage it will go as an input of the second stage, whatever output of the second stage that will go as the input of the third stage, in this way it will go. That is why if you ask me what would be the final optimal solution then I will say whatever optimal solution you are getting at the nth stage, that is the optimal solution of the whole model. But in between as I said that the optimal solution of the first stage will be the input of this second stage; optimal solution of the second stage is the input of the first stage in that way that is why in dynamic programming we consider a combination of decisions and we optimize the whole system.

But in different breaking into different parts, and there is a principle that is called the principle that a Bellman's optimality principle. It has been said that whatever optimal decisions we are taking in different stages if we just combine it together in the sequence at the end that is the original optimal solution of the whole bigger problem that was the Bellman's optimality principle that is there. If you learn more on dynamic programming, then you will learn many things from there.

Now, if you ask what are the applications of dynamic programming, there are different applications shortest path problem, knapsack problem, resource allocation problem, multi-period production planning problem, multi-period inventory planning problem, scheduling problem, these are all the examples of the dynamic programming problem.

Knapsack problem there is we want to we have a knapsack of certain aid say the knapsack can have 10 kg, it can hold 10 kg material. Now, I am going for a trip. Now, I want to plan that what material shall I put there to get maximum benefit in the trip time. I have few materials in front of me as options, and the corresponding available side is also there, and the corresponding weight is also there. I have to schedule I have to plan the combination in such a way the maximum value must should be 10 kg, it can be lower than 10 kg, should not be more than 10 kg, but it should not be low like 2 kg. Then I would not get the maximum benefit at the trip that that is why it is being considered as a dynamic programming model why you know because this is not really dependent on the timescale.

We have the options, we have material one, material two, material three, material four, we want to put it in the knapsack to have the maximum benefit, the maximum load must be 10 kg. But we consider it as a bigger problem by just looking at the case like material one, first I will feel it then we will see how much space is there then we will go for the optimal decision for material two. Then we will combine together material one and material two, we will see how much really utility we are getting out of two materials then we will consider the material 3, then we will consider all together in this way we will club all together, we will get the optimal solution.

This problem can be considered as the dynamic problem, because we have the bigger problem. So, many materials I want to put it in the knapsack, but one material at a time; that means, I am considering one stage at a time. I am taking decisions only for one material at one point of time. After that I will consider the next in that way I will proceed. And I will show you an example I have prepared for you that is for the shortest path problem how really it is being done, I will show you how dynamic programming model is considered there.

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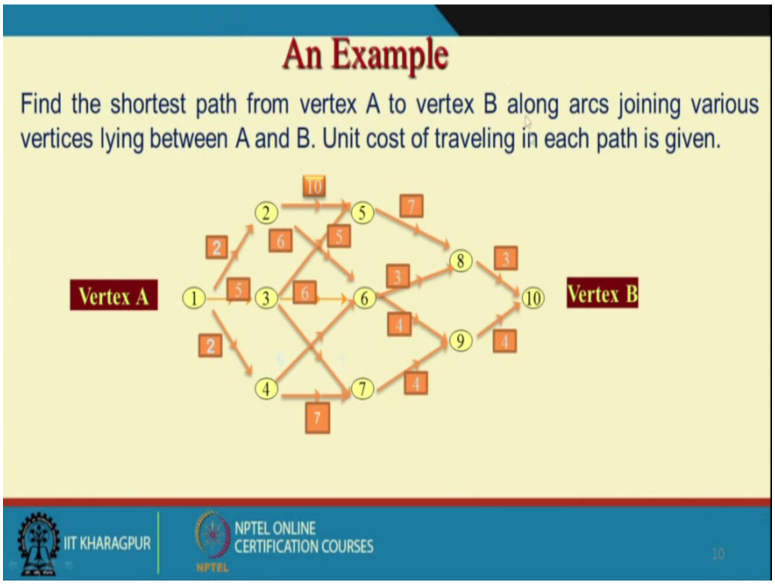

Now, this is one of the shortest path network problems. Now, here it has been given I have to move from vertex A to vertex B. There are different paths the nodes are given this is a starting node 1 the ending node is 10. And the if I just move from one node to the other node corresponding traveling cost or traveling time whatever you may consider. Some measure has been given all together I want to reach from 1 to 10 which path I do not know, but I want to minimize total travel time or total travel cost this problem can be considered as a dynamic programming model.

How, the problem the dynamic programming model generally is being considered through backward recursive formula or forward recursive formula. What does it mean? It means that I will just plan in this way. If I could have planned from 1 to 10, what should be the path this is the forward recursive approach, I will consider, why it is recursive, I will tell you. And if I do from 10 to 1, what exactly the path I will follow. If both are matching then fine that would be the optimal solution. There should be the approach for dynamic programming then dynamic programming adopts this forward recursive formula and backward recursive formula.

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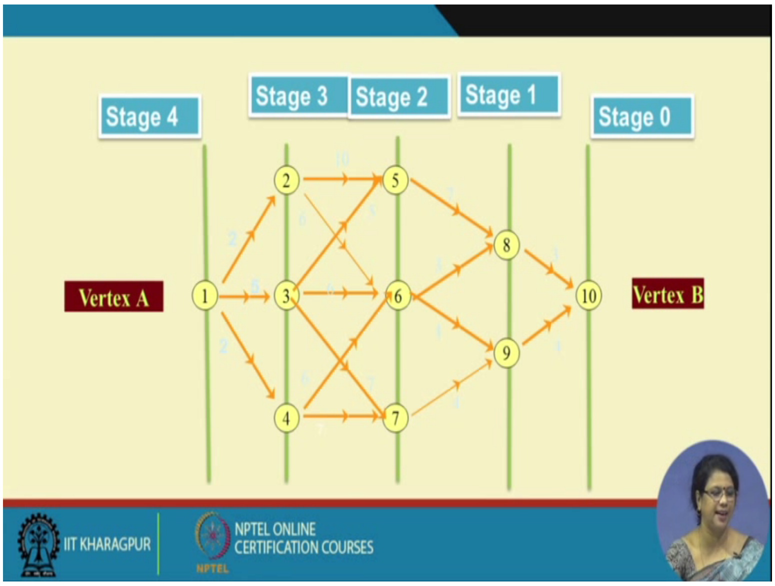

Why the recursive what is coming again and again, because this model is being braked in two different stage. Just you see we are breaking the model in different stage, we start from stage 0, we are moving backward then we are going to stage 1, we are taking decision if I move from first stage to second stage, shall I move from 10 to 8 or 10 to 9.

We have the corresponding cost figure with us we know what is the minimum value. From stage 1 to stage 2, then I will move I will forget about the stage 0. In the memory, I would not put any picture about stage 0, I will only consider the current stage and the next stage, I will find out the possible options I will find out the minimum.

Then once the optimal solution we are getting from stage 2 to stage 3, I will go without remembering the situation I had come across in stage 1 and stage 0 that is that nice thing we will do so that the whole problem is so bigger problem. Every time we are finding out the minimum of this thing, minimum of this thing that is why the same calculation, we are doing again and again in different stages that is why we are calling it as a recursive process. The same recursive formula works in different stages only the objects are different that is why I can do forward, I can do backward everything I can do and I will get the solution accordingly.

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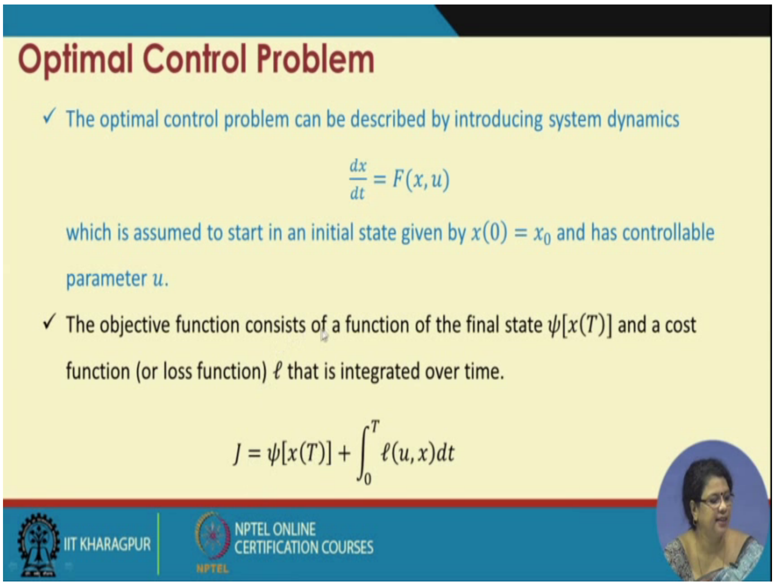

Now, the optimal control problem is another kind of problem, where really we are having you see we want to optimize the function capital F, where the capital X is connected with differential equation, and u is a parameter which is dependent on t. And l is another cost function has been given this is the objective function it involves the final stage which is depending on time. You could see it just behaves like integral equation that is why that is the combination of differential equation integral equation, but we are going to optimize the model that is why this kind of model is being named as the optimal control model.

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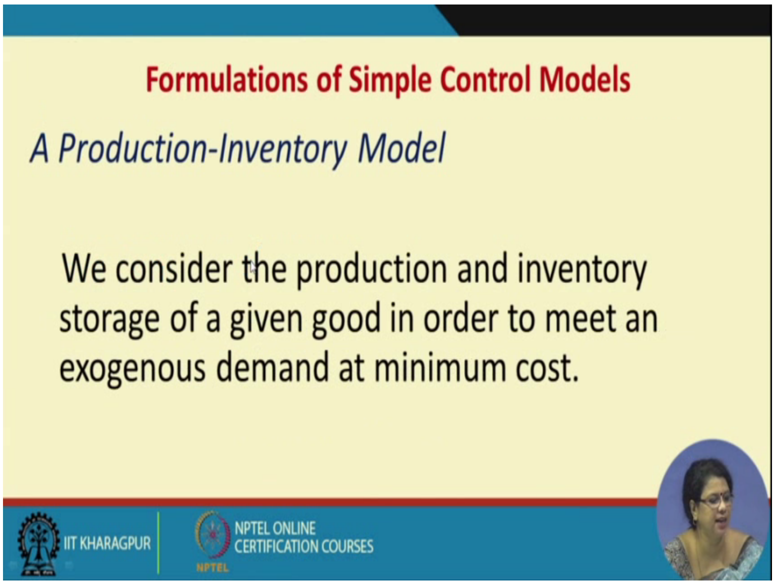

One of the production-inventory model, we can refer for it. Just you see we are having different state variable control variable objective function, which is dependent on the state constraint, control constraint.

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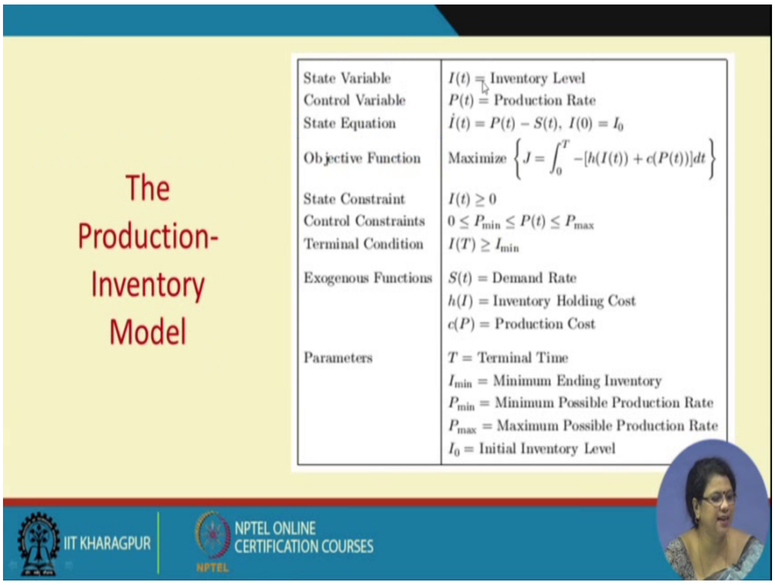

We are having the demand rate, we are having the holding cost pattern, we are having the production cost pattern there are different functions for it. We are all considering together at different point of time, we are looking at the pattern of it and we are optimizing as a whole the objective function or we can go for the minimization of the total cost as well which will look like this, one of the example of the optimal control problem.

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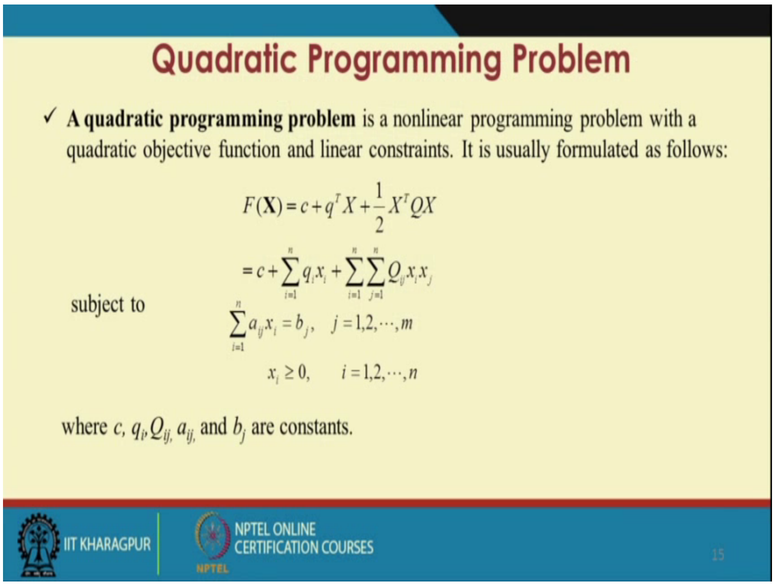

Similarly, we can have the quadratic programming problem that is another non-linear programming problem we are considering, where the objective functions is quadratic in nature. I can give one example for it. Just you look at the function. Here, the function is quadratic in nature X T Q X; you know that is a quadratic function.

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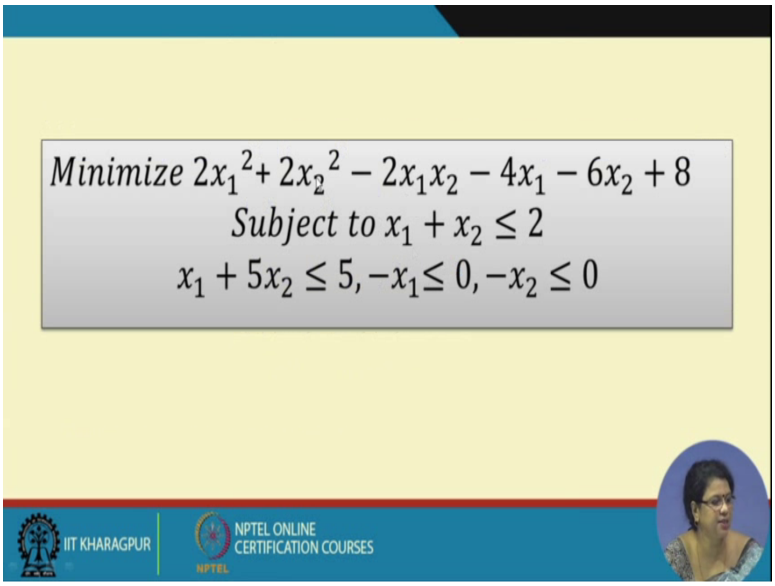

I am giving you one example for it just you see. I can write down in the form of c that is 8 plus q t that is a that is a simple matrix.

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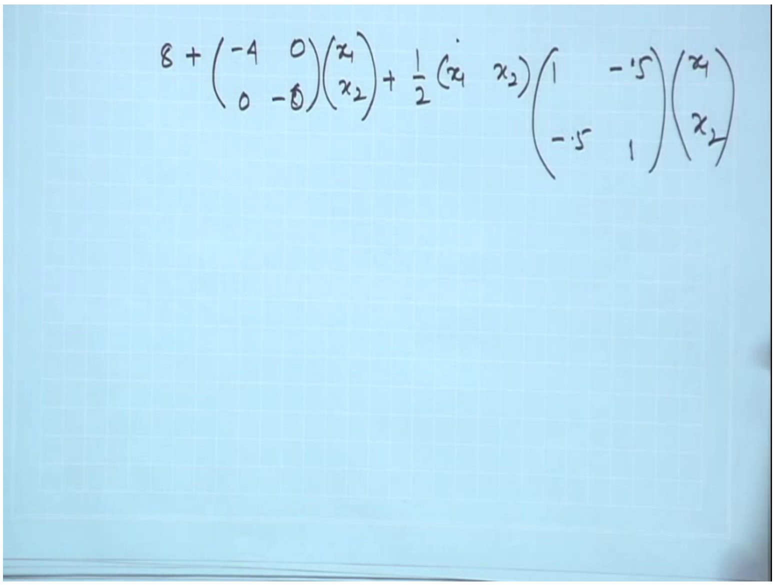

We can consider like minus 4 0 minus 6 0 x 1 x 2 all right plus half x 1 x 2. And the matrix we can consider 1 1 minus 0.5 minus 0.5 x 1 x 2. See this is a quadratic function, I can put in the form of as I showed you in the previous slide c plus q T X plus half X T Q X, but the constraint is linear I have considered. Now, this kind of model is being handled by there are different methods Vols method, Vills method we handle the quadratic programming that way.

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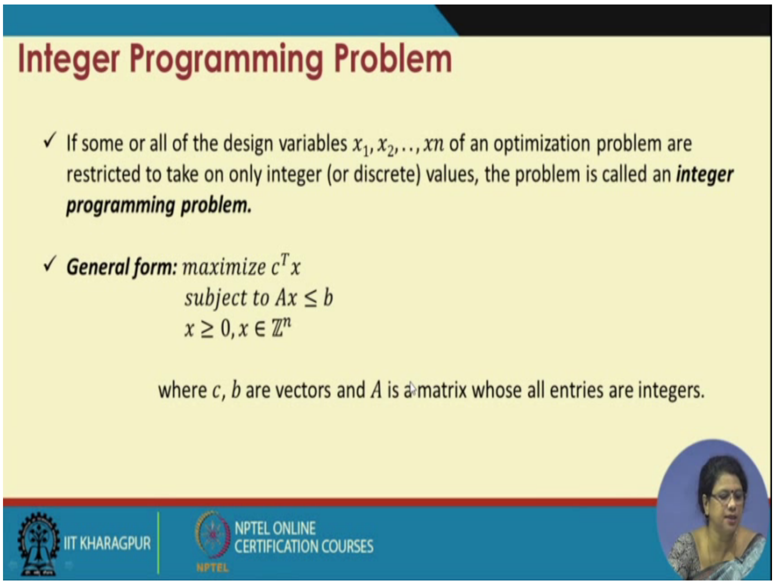

Now, we can have the integer programming, I need not to say the example for integer programming because you have already done the integer programming and linear programming, but we can have the non-linear programming as well.

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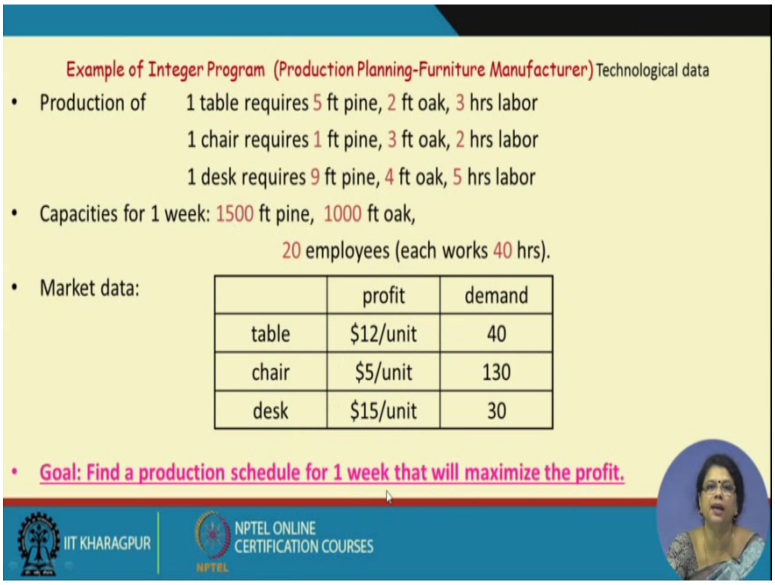

This is one of the data set. We are going for production of table chair and the desks, and we need to find out how many table, how many chairs, how many desks, we will plan for one week to maximize the profit. Here you see table cannot be 1.5, chair cannot be 2.3

that is say always it will take the integer value this is one of the example of integer programming.

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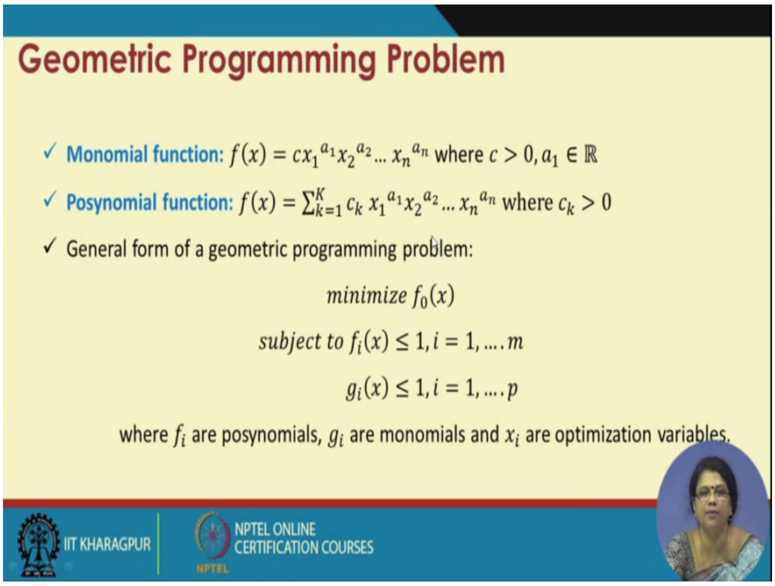

Similarly, there is another very nice thing we have in non-linear programming that is called the geometric programming. Geometric programming is a problem where we are considering the monomial functions or the posynomial functions how really it looks like that just you see the structure. There are n number of variables and in the n number of variable at the we are considering the exponents like a 1, a 2, a n these are all the real numbers these can be positive these can be negative as well. If you see we never consider in any other non-linear programming problem where we consider the exponent as minus 1.5 minus 1.2 these are all being considered in the geometric programming problem.

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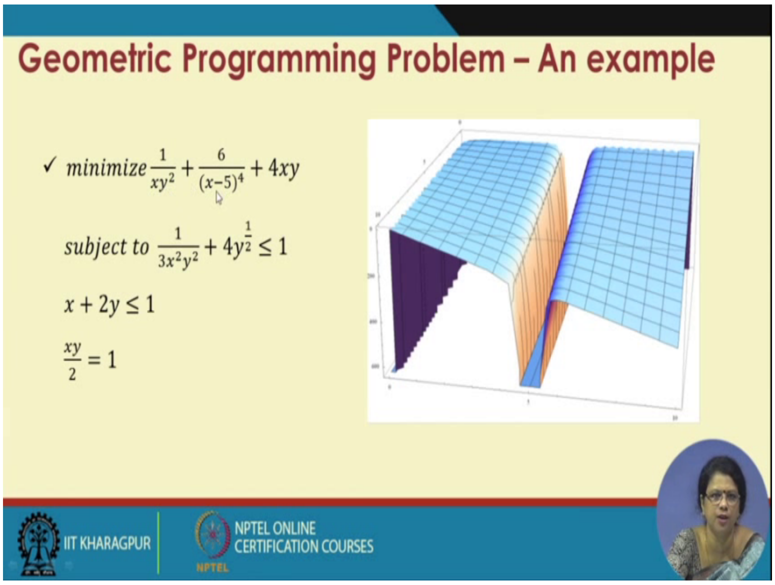

But the beauty of geometric programming is that if I just show you one example just you see here the power is y power of y is minus 2, power of x minus 5 is minus 4, power of here y positive, here negative this is the combination we are getting one optimization model. Just this is the function we are consider the objective function look at the objective function there are certain singularities. If we just see function is discontinuous in at different point. Just look at the objective function where the function is discontinuous at 0; at y is equal to 0; at x equal to 0; at x equal to 5 everywhere function is discontinuous that is why whatever classical technique we have the differentiation technique etcetera will not work here.

For geometric programming, there is a series of problems are there lot many applications are there where the geometric programming the consideration is we are considering the posynomial we do not call it as a polynomial, we call it as a posynomial. And this posynomial geometric programming problem is being handled differently using the dual of the primal that way we considered you know the duality theory.

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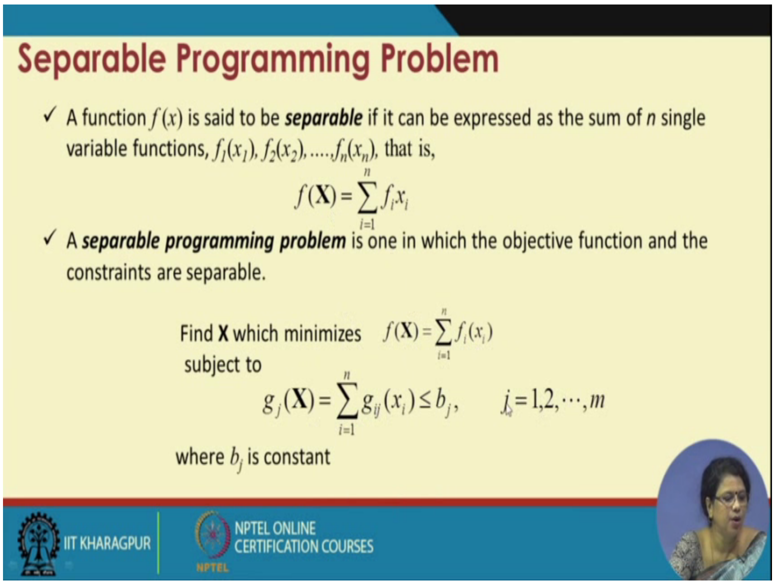

Now, this is one of the application of duality theory in the non-linear programming problem that is the beauty of geometric programming problem. Now, as I say the separable programming problem where the function can be separated with respect to the variable I am giving you one example for it here you see the objective function is f X equal to summation f i x constraint is summation gi x.

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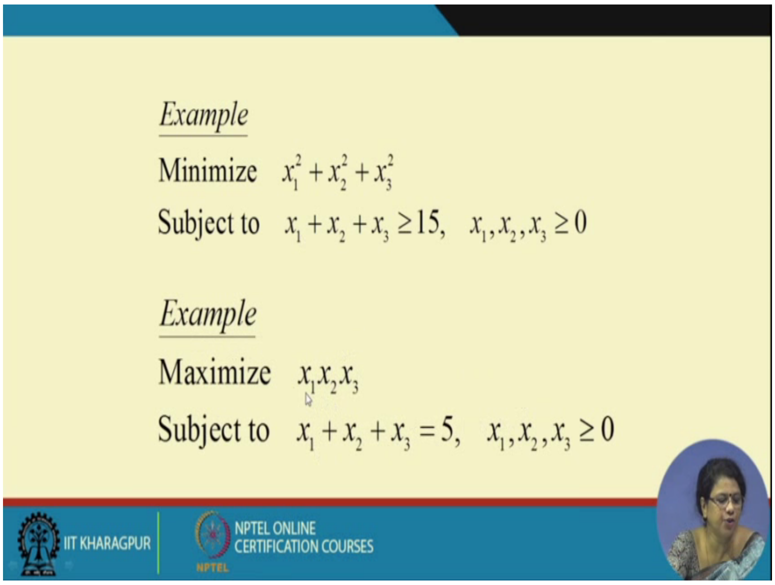

One example I am showing you. Just you see if we consider this x one square one function x 2 square one function, x 3 square one function; x 1, x 2, x 3 these are the another group of functions. Now, this separable programming we can handle in non-linear programming problem and this separable programming problem can also be considered by using the dynamic programming technique we can solve it. But if you just look at another example, this is not separable because we cannot write this function as a summation of f i x, but constraint is separable function, but not the objective function that is why this is not a separable programming problem.

And another thing we can do in the separable programming problem, we do a non-linear function we approximate with linear approximations. And we handle it as if that is the combination of linear programming problem, there is a different technique to handle separable programming problem that you need to learn also in non-linear programming. These are all the varieties available in front of us. And a stochastic programming problem this is another variability, where the objective function may have object.

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This is by definition, it has been given wrongly only the thing is that stochastic programming problem, the objective function and the constraint part may have the decision variable, which are probabilistic in nature.

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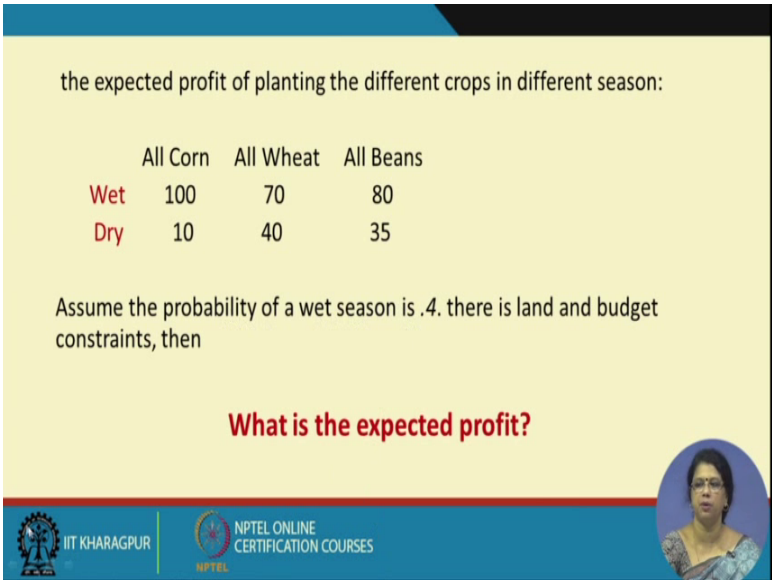

Just you see there are certain this is one of the example these are the expected profit figure for corn, wheat and beans in wet and dry weather. Now, you see the productions are different, the expected profits are different, but in the wet weather and dry weather the production of corn, wheat and beans are different also, because for every agricultural thing, we need to depend on the season. There is another thing is also there, there is a probability of old season as well. Now, if we have a land constraint if we have the budget constraint then if we ask you what is the expected profit, you will see it may happen that the instead of having the probability value we may have the probability distribution that is the way how to handle it. Then you need to do in stochastic programming problem all right.

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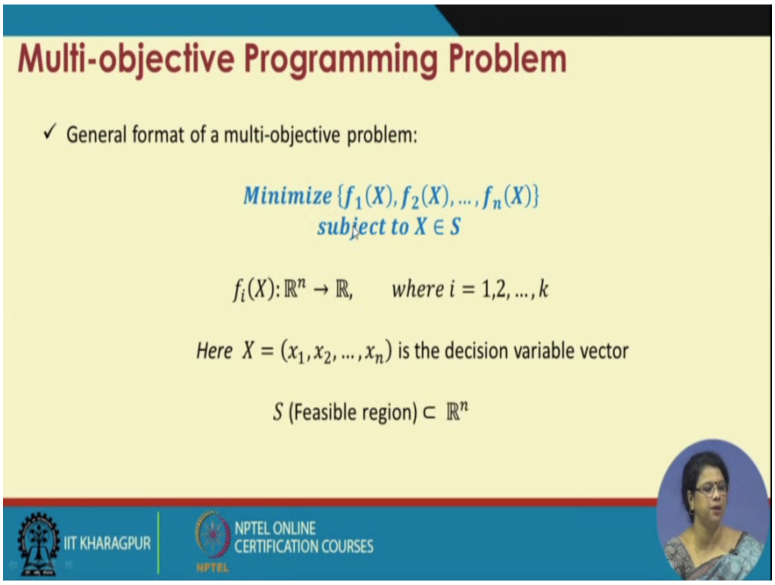

Now, the next is that as I said another criteria to handle the optimization model instead of having one objective function we can have several objective functions together.

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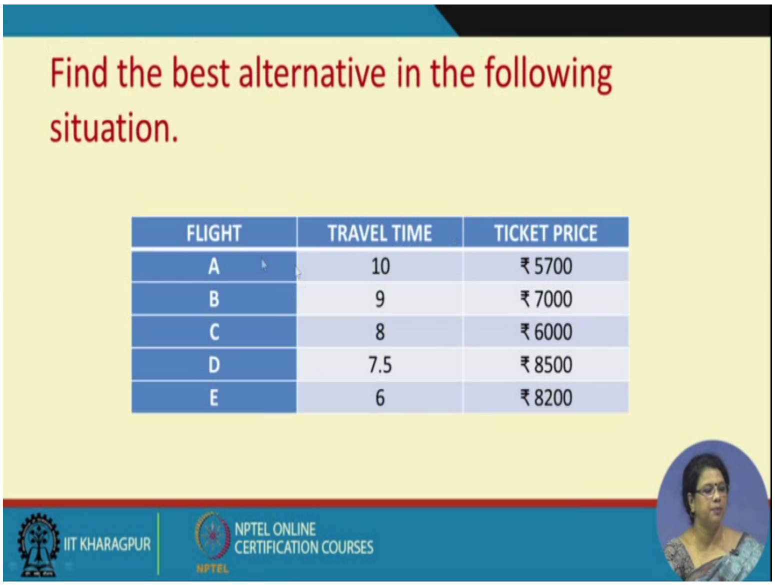

Now, you see minimization of all objective function simultaneously for example, I am giving one example we have different flight options with us, different travel time and different ticket prices. I want to minimize the travel time, I want to minimize the ticket price, but if you look at the figure will you get a single option from it which where you will get the minimum travel time as well as minimum ticket price. It is really difficult that is why getting optimal solution for this multi-objective problem is much more complicated than any optimization model where we are dealing with the single objective.

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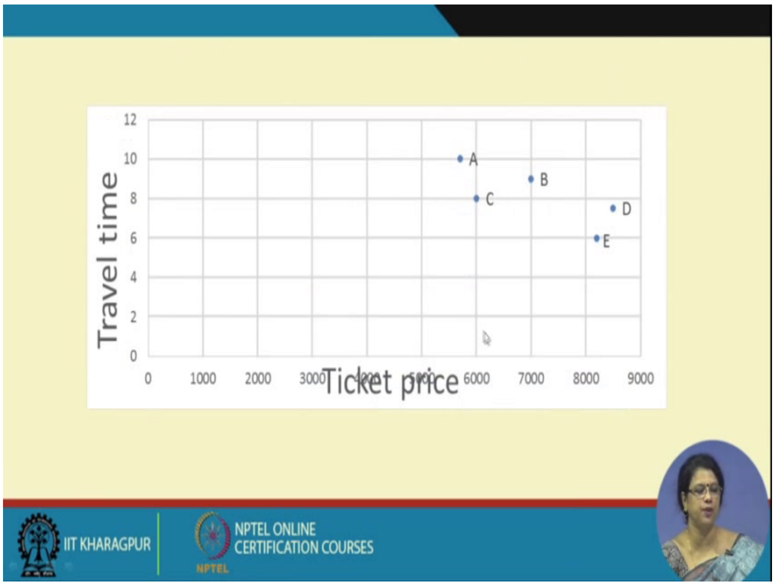

Because in multi-objective model if it is draw this figure in the two dimension where in one side that ticket price, another side is travel time is there we will see that we need to depend on the partial ordering of the options. Generally, for single objective, we are having the total ordering we know the objective functional value at different point of time we can select the minimum. But here getting minimum for both the cases is really difficult that is why we need to learn how really we can handle the multi-objective problem. Now, with that I am ending today's class, concluding today's class. And these are the types available there are more other special types available like fractional programming and other this thing, and you should know from the later units.

Thank you very much for today.