Constrained and Unconstrained Optimization
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Lecture – 32
Graphical Solution of NLP

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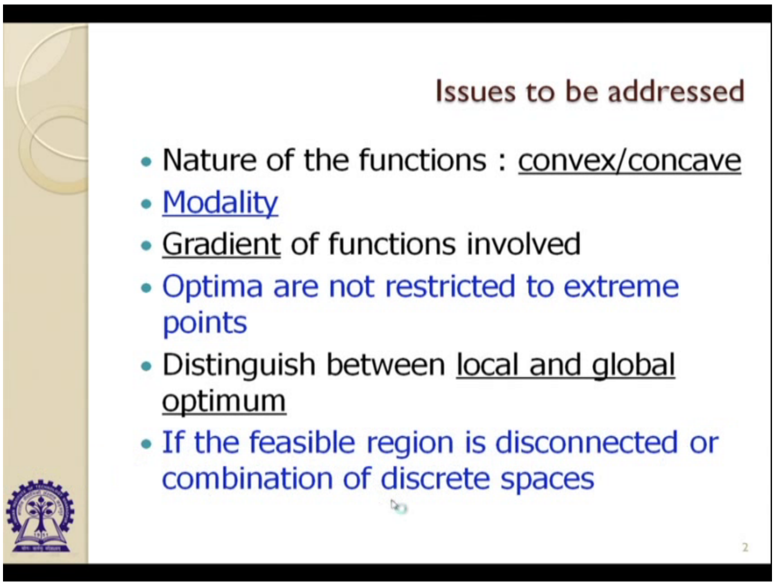

Now, last class was the introduction to non-linear programming problem for you, there I have introduced the non-linear programming problem and I have discussed several issues to be addressed for solving the non-linear programming problem, let me come to that point and let me elaborate few concepts which I needed for the non-linear programming problem one by one. Again coming to that point I was talking about in that in the last class that whether the functions involved that is in that means, the objective function as well as the constraints both can be non-linear in nature. If we consider in general all are non-linear.

Then we need to study few things, first of all whether the function is convex or concave that is why you need to know, what is the definition of convexity and what is the definition of concavity and further issues related to that and I told you what is the definition of unimodality that is why modality is the next concept you need to address for solving non-linear programming problem. And gradient of a function is another part to address. Now, today I will give a few graphical solution of the non-linear programming problem. And I will just show you how really the gradient and other things convexity, concavity, all the things are related to the non-linear program. Then you will know that yes these are the things we need to learn then only things will be better for learning. Now, we need to know the definition for the local optimality and the global optimality are formally that I will introduce.

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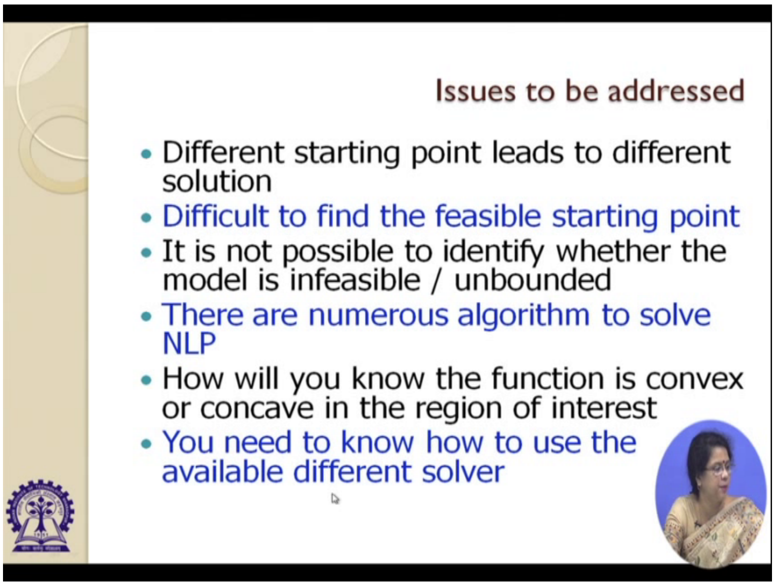

And as I was explaining in the last class that, if we adopt the searching algorithm for a non-linear programming problem, the initial starting point initial case point is very important for us. If the initial case point is somewhere which is far from the optimal solution, then number of iterations will be more and it will take more time to get the solution, and that is why you need to know about it. And if within the feasible space, starting point is feasible space itself is very complicated in nature that is why starting identifying starting point from there for infinite number of points which are within the feasible space for that is very important.

Now, you know in the linear programming you must have learned that there are certain symptoms in the simplest algorithm, where you could decide that function is not having at all a feasible solution. The solution space is unbounded in nature, but for the non-linear programming problem, it is very difficult to identify that whether the problem is having at all a feasible solution or not, whether the problem is having the unbounded feasible space or not that is very difficult to judge. Because that again the concept is that it is a very complicated in nature at the feasible space that is why within so much restrictions with so much complications only we need to find out the optimal solution.

Not only that not only optimality, we have to say whether this is a local optimal or global optimal that is why many things to be considered together. And as I mentioned there are numerous algorithm for solving non-linear programming problem that is why which algorithm will be applied for which problem you need to know, then only you will get the better solution better means global and that is why let me come one by one.

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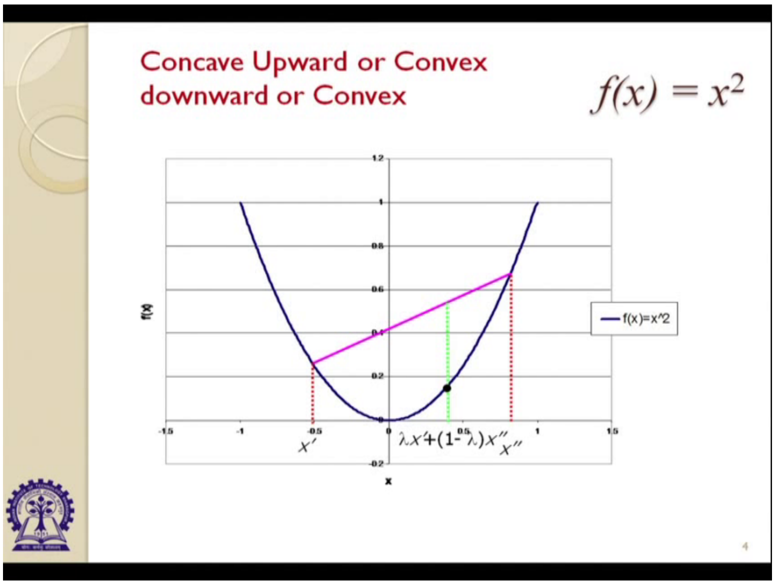

Now you see f x equal to x square, what kind of function this is, this is a convex function all of you know. How we can draw it, we can draw it. If you know the definition of convexity either we can say that this function is concave upward or convex downward or in general we say the function is convex. Now, what is the definition for this, how we can say this is a convex function? The definition is that if I take two points from the domain that is x prime and x double prime, and if I join the line and any value any for that point the functional value will be lesser than the any point joining these two points, any points lying on the line of joining two points all right that is the basic definition. How mathematically we can say I will tell you in the next. This is a convex function.

If I consider f x equal to minus x square certainly it would be the reverse that is why if I say that you must have learned in your linear programming as well minimization of objective function is same as the maximization of the negative of the objective function.

That is why if I want to get the minimum value for this object, if this is the objective function which is moving within a constraints space, then if I want to find out the minimum value of this objective function, then I can do the reverse, I will do the maximization of the minus of this function. That means, instead of minimizing convex function, I will maximize the concave function.

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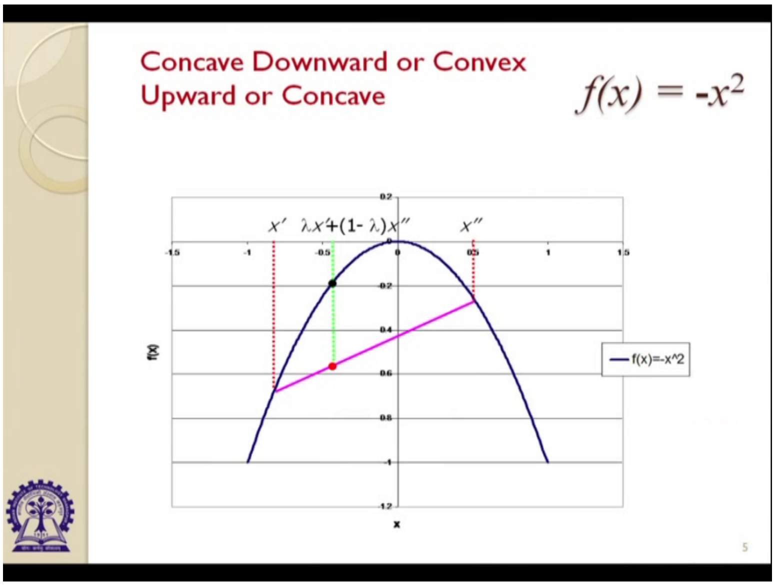

And this is the f x equal to minus x square this is type is this way. And how to define again we will take two points we will join these two points with a line, and we will see that any functional value of any point will be greater than of that point on the line value of the point on that line that is the definition. And this is the concave function either we say concave downward, that means, the cup downward, concave downward, convex support or we say this is the concave function that is the idea.

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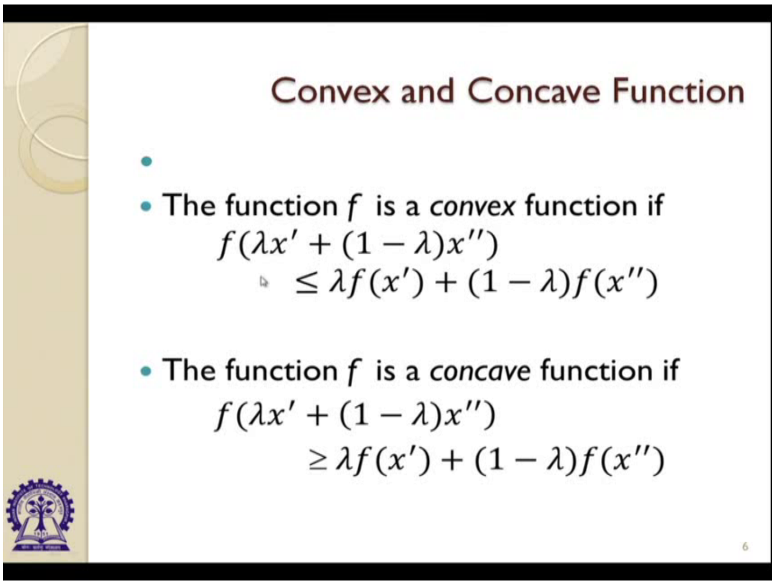

As I said that there are two points x prime and x double prime. If I take the linear combination of these two lambda x bar plus 1 minus lambda x double bar, where lambda is lying in between 0 to 1, then the functional value will be for the convex function, the functional value will be always the lesser than lambda f x prime plus 1 minus lambda f x double prime for any value of lambda between 0 to 1. And for the concave that this is the reverse case instead of less than time it will be greater than time as I showed you graphically all right.

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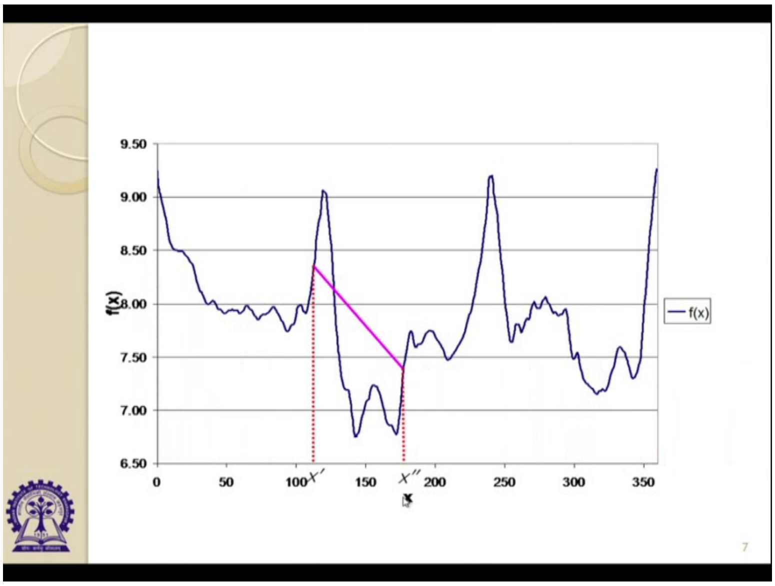

Now, look at this function, what you can find if I ask you whether the function is convex or concave, can you say not really because function is very complex in nature. There are several combination of convex function, concave function, and you know the point inflection point all the points are there that is why if I consider this range not really we can say that this is either convex function or concave function. That is why you must expect the non-linear programming problem where the objective function can be convex, whether the objective function concave and it may happen the function neither concave nor convex, even for the constraint function as well the same logic holds all right.

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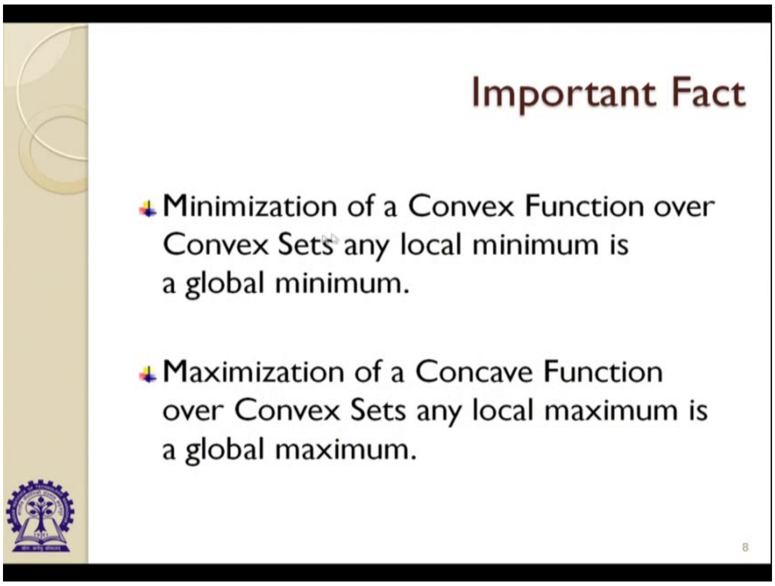

We cannot say anything that is why there are certain important facts you need to learn, this is you will ask me why we will learn the convexity, concavity, is it at all important for the optimization. It is really important because just you remember the fact we will prove it later on, I will show you how we are reaching to that point that is the very important fact in convex optimization is that. If we considered the objective function as convex and the constraint function as convex as well, and if we want to minimize the objective function, all of us we will get the global minimum.

And this is the reverse case for the maximum problem. If the objective function is concave constraint function is convex, if I want to maximize the objective function always we will get the global maximum that is the beauty of learning convexity concavity of non-linear programming problem. But as I said that if you have the other combinations that is; that means, objective function is concave or the constraint set is convex and we are minimizing the function then do not expect that you will get the global optimal solution out of it. It may happen the objective function is neither convex nor concave constraint function is neither convex nor concave do not expect that you will get the global optimality.

It may happen that you ha you will miss the global optimal solution in the search process you will you have to be satisfied with the local optimal solution that is why this is really important fact for non-linear programming problem. If you remember this fact then very easily you can say looking at the functions in the non-linear programming problem, you can declare at least for certain cases, you were achieving global optimal solution which is our ultimate target all right.

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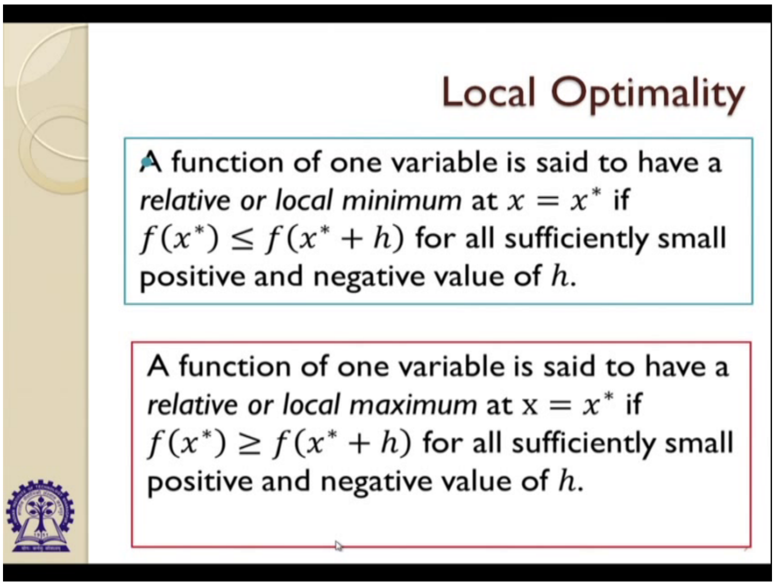

Now, you need to know what is the definition of the local optimality, global optimality etcetera because I mentioned many times, I uttered the term local optimal and global optimal that is so why let me tell you. Now, a point within the domain of the function the point is x star, I can say that this is a local minimum point if in the neighbourhood of that function we would not get any other value which is better than that. Better in the sense that we would not get any functional value which is lower than that is point then we can declare that that is the local optimal point that is why we are taking the point x star. And we are considering the h neighbourhood of that h can be positive, h can be negative. And you see the functional value is always greater than f x star that is why at that point there must be a local optimal point all right.

Similarly we can extend this idea for the local maximum the case is reverse. If I consider that x star is a local maximum point in the neighbourhood of that point we would not get any functional any point where the functional value is greater than that that means, the higher than that. Always all the functional values in the neighbourhood of that point will be lower than that, lower than the functional value at x star then only we can see x star is the local optimal.

But what about global optimality? If we extend this concept from the local to global then we can consider the whole domain of the function. If I say instead of taking the neighbourhood of that point if I consider in whole domain of the function, no point is giving the better value for the function f then that can be declared as the global optimal. Now, whether greater than or less than that depends whether we are considering the minimum or maximum that is why let me extend this concept for global optimality.

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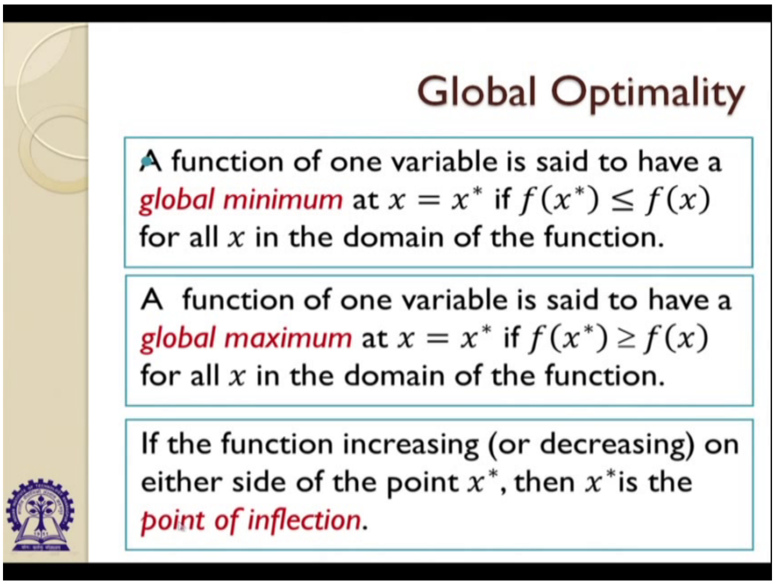

Just you see a function of one variable everything we are considering now I am just trying to explain you all the concepts with the function of single variable, so that very easily we can extend it for function of two variables all right. Now, you see the function is having the global minimum at the point x equal to x star, if f x star is lesser than equal to f x for all x in the domain of the function, that means, no point gives the better value for f in the minimum case all right. Similarly, for the global maximum, no point in the domain of the function you are having higher value of the function than f x star then that that is the global optimal. Then which one is the point of inflection all of you know, in the point of inflection in either side of the point the function is increasing either both increasing or both decreasing with the figures. I will consider I will show you at which point in the next class I will show you at which point we are getting the for function of one variable to variable as well when we are getting the global maximum, when we will get the local maximum, when the point of inflection is coming etcetera.

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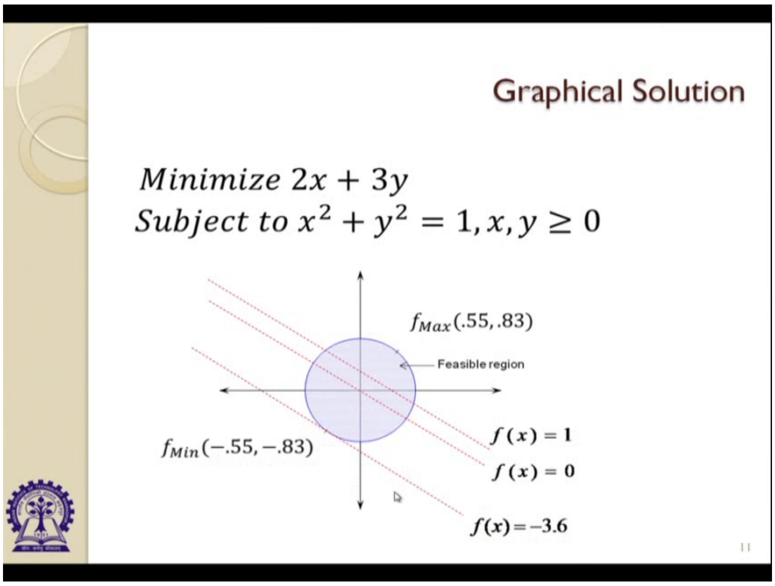

Now, we are moving to the graphical solution of the problem. Now, you see this is the objective function, objective function is linear in nature; minimization of 2 x plus 3 y subject to x square plus y square is equal to 1; and x and y all are moving within 0 to infinity. Then how if I draw the function, how it will look like, the constraint would be a circle again, and your objective function is a line. Let me draw it this is the constraint set, but you see I have considered only the x, y greater than equal to 0 that is why minimization let me update let me just modify the problem that x and y both are unrestricted in sign.

Let me consider that x and y both can move from minus infinity to plus infinity as you did for linear programming as well. You must have considered the set of variables which are unrestricted in sign and you know how to handle that in linear programming problem.

But in non-linear programming problem also we do the similar process, but for the time being for graphically explained to you let me relax this constraint that x y is greater than equal to 0, let me say x and y is unrestricted in sign. That means, both can take minus value and positive and negative value hold, then this is the constraint set for us.

Now we are trying to find out the minimization of 2 x plus 3 y. Now, my line is moving this is f x equal to 0 at point x equal to 0 y equal to 0 all right. Now, the function moves this way if I move this way functional value will increase and if I move this way the functional value will decrease all right. That you know from your mathematical knowledge that is why if I move this way f x equal to one functional value is increasing i will be further from the minimum point. That is why my direction must be the reverse one that is why if I move from here further and further then at this point at minus 0.55 and minus 0.83, we will get the minimum value of the function f.

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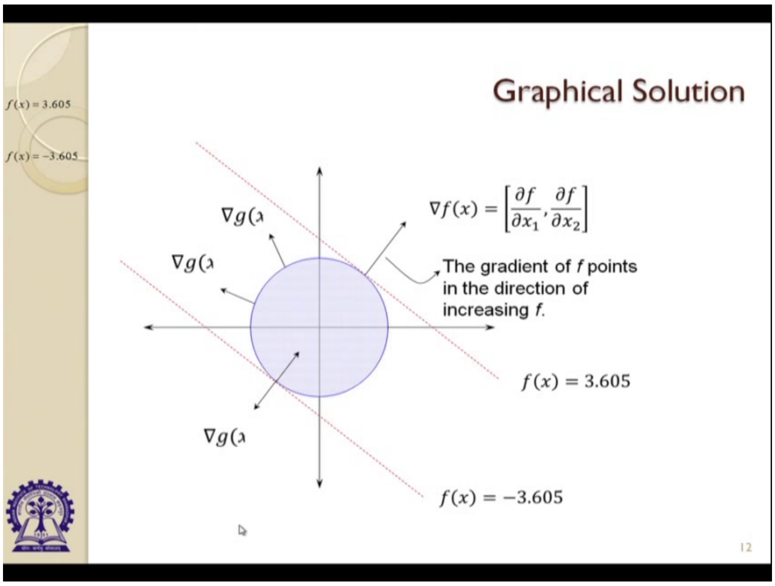

And you see I will explain a few things here with the same example as I said that the gradient is very important for getting the max optimality shall optimal point maximum point or minimum point that part I will just explain you with this example. Now this is my constraint set I am considering the previous example as well. Now, what about the gradient of the constraint set, you know always the gradient would be this way, because if I at any point the gradient would be directed to the outside. Because, if I consider the tangent then normal to that tangent would be the gradient as you know that is why these are the gradient at each and every point on the constrained space, these are all x all right.

Now, this is the gradient of the objective function as I said that if I move this way objective functional value will increase; and if I move this with the objective functional value will decrease all right that is why the gradient of the objective function will be this way. And you know how to find out the gradient of a function, if I consider the function of two variables at any point the gradient of the function would be the partial derivatives of the function at that point, that is why I will consider del f by del x del f by del y. And at that point whatever value will come that would be the gradient of that function that is why this is the gradient of the objective function all right. And this is the gradient of the constraint.

Now, you see how you are getting the relationship between the gradient of the objective function and the gradient on the constraint at the optimal level. Now, you have achieve the minimal point here all right. Now, what did you see this is the gradient of the constraint and this is the gradient of the objective function at this point. What did you see at the minimum point the gradient of the objective function and the gradient of the constraint all are moving in the reverse direction. We will see when we will see the non-linear programming problem, we will just give you further conditions in a further theory then we will establish this fact that always at the minimum point gradient of the constraint and gradient of the objective function would be in the reverse direction.

But if I consider the maximum of that function that means, function is moving, and that should be the maximum point because if I go further with a function then I will be beyond the objective function space, that is why this is the maximally I can move at that point. Then this point must be the maximum point and the coordinate of this point would be the reverse one the 0.55 and point 0.83 so like that as a minimum we got. And what did you see here gradient of the objective function and the gradient of the constraint all are in the same direction that is the important fact again for the non-linear programming problem that gradients are rivers in the minimum point and gradients are the same direction in the maximum point. This is very important in non-linear programming problem; you just remember it, we will establish further.

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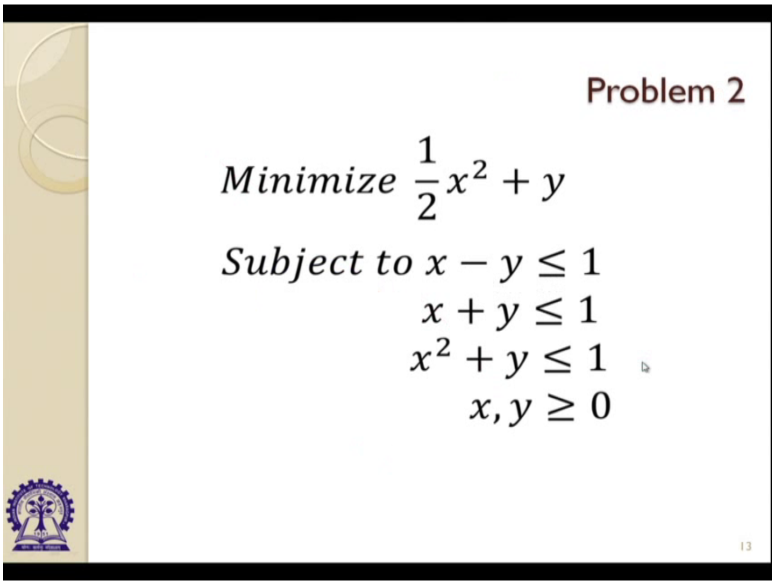

Now, let me the thing with other example. The same thing I will explain to you, but the problem is little bit complicated because here we have considered both the constraint set as well as the objective function both are non-linear in nature, but few constraint I have considered the linear in nature all right.

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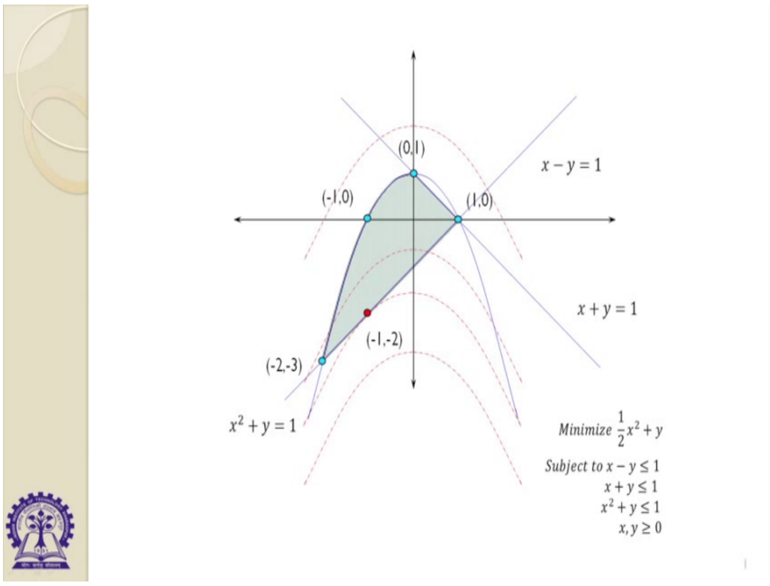

Now, let me draw it. This is my problem. Now, let me draw the function. Now, if I draw it then this is the line x minus y is equal to 1, this is another line x minus x plus y is equal to 1. Surprisingly, this is the negative image has come, this is the function x square plus y is equal to 1 all right. Now, this is the objective function because forget about this, if I draw it all of you are understanding, these are the reverse images somehow we got it. Now, this is the constraint set and the points are the feasible space is combination of those points, which are being satisfied with all the constraints, that is why I can say this is my feasible space all right.

Again I am relaxing x and y from negative and positive otherwise we would not get this feasible space, I want to show you the bigger picture that is why I am relaxing this fact all right. Now, what is my objective function, objective function is again another parabola for us. These are the points the intersecting points for us. Now, this is the objective function is moving that is why what is that function problem is minimization problem that is why if I move this way then I will get the better value of the objective function. If I move in the lower side, I will get the lower value of the objective function that is my gradient this way. And since this is the minimization problem, I will just move reverse to the gradient direction, because that will be satisfy me, I am searching for the minimum value.

Now you see if the function is coming down and down, if I come this point, I am out of the feasible space all right. I do not want to be that is why certainly that must be the optimal solution for this problem. If I relaxed the non-negativity constraint then only and my optimal solution is minus 1, minus 2. Here also if you just say this is the task of yours that just to check whether at the minimum point you are getting the gradient direction for the objective function and the constraint both are in the reverse direction or not that you check.

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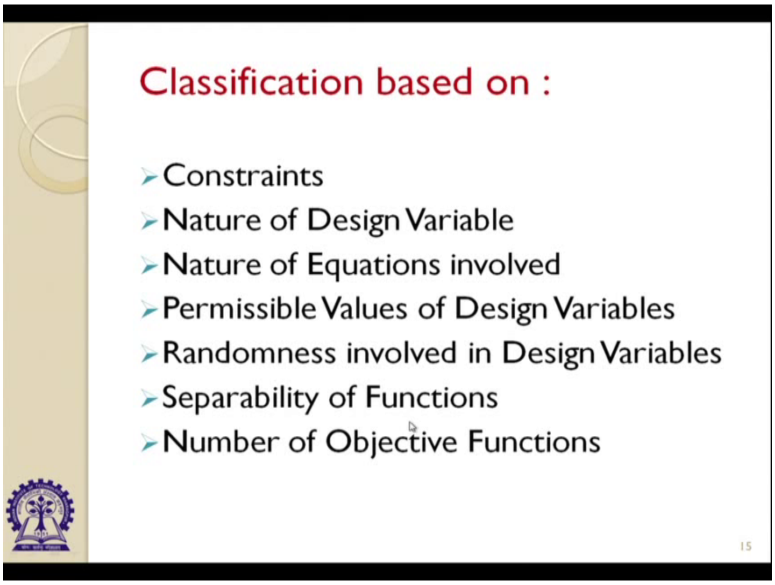

Now, whatever we have seen till now, what are the things we have seen there are few things to be remembered. That you see there are we need to consider the constraints constraint can be linear, constraints can be non-linear, constraints can give you the discrete feasible space, constraint can form the continuous feasible space that is very important for us. Now, the next thing we need to considered the nature of the design variable. In sometime it may happen design variable means the decision variable. Sometimes it may happen that the decision variables are all discrete in nature that means, the decision variable can take either 0 value or one value, you must have done the integer programming problem in linear programming the course that this decision variable can take the discrete value.

For example, my the problem is till now I did not consider any real life problem, but you must have come across in the linear programming problem, where we need to decide for certain situation. How many chairs are I have to produce in my factory that means, that the decision variable must be integer in that case; that means, if this is the case that is very important that is why even in the non-linear programming problem, if my design variables are discrete in nature, how to handle it that is very much important.

Now, what about the nature of the equations, what are the equations are involved that is very important for us, because you know for the non-linear programming problem, non-linear function can be of different type. It may happen the non-linear function is in the fractional type.

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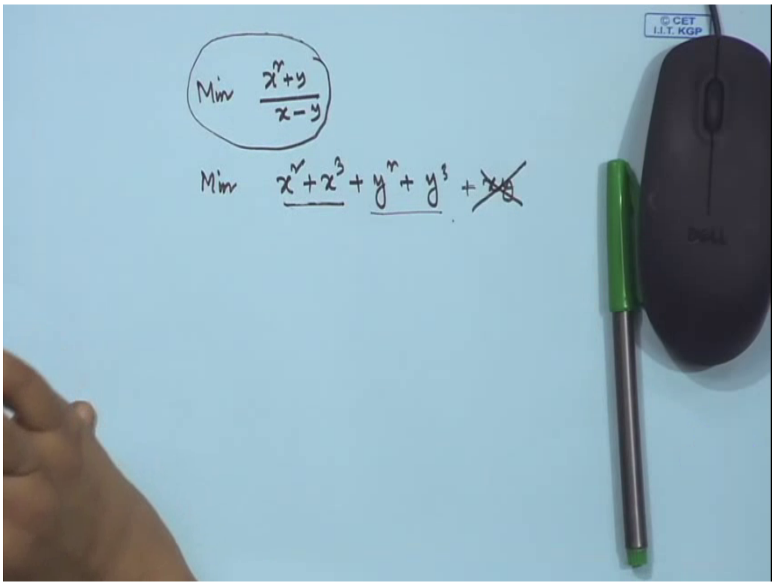

For example, if I consider the function can be this way function of two variables. If I consider x square plus y divided by x minus y minimization of these or maximization of these. You see the function is a fractional type that is why how to handle this one. Now, it may happen the function is separable in nature for example, x square plus x cube plus y square plus y cube, there is no term like x y that means, I can separate x in one side, I can separate the function of y, then how to handle this.

if x y is coming to this then it is not a separable non-linear programming problem, how to handle that kind of problem that is why in the next class I will talk on the classification of the non-linear programming problem. Depending on the nature of the constraint, nature of the design variable, nature of the functions involved, nature of the permissible design variable. Another thing is very important the decision variable which are involved in the function that may be affect that may have the stochasticity that means, in it may follow certain probability distribution. In that case, how to handle that there is a separate branch in non-linear programming problem that is called the stochastic non-linear programming problem that is why it depending on the stochasticity, it depending on the deterministic nature of the design variable, how we can categorize the non-linear programming problem that also I will talk in the next class.

Now, what about the separability as I talked about it, there is a separate branch of non-linear programming problem that is called the separable non-linear programming problem. Sometimes the non-linear programming problem is called as a geometric non-linear programming problem. And the beauty of the non-linear programming problem is that as I was mentioning different kind of non-linear programming problems all are having different algorithms to solve that is why you need there is no one solution process like your simplex in linear programming that you will apply that one on any kind of non-linear programming problem you will get the solution that is why you need to learn all kind of algorithms separately. I will talk about that.

And the last point that is very important also in real life situation. It may happen we may not have one objective function like whenever we are considering the minimization of objective function; generally we are considering the cost function. Whenever we are considering the maximization of objective function, we are considering the profit of objective function or we are considering the goodwill of the firm that is the objective function that we are trying to maximize.

It may happen both the objective functions are together; that means, we are trying to minimize the cost of a certain product as well as we are trying to maximize the profit. Is it at all position possible, shall we get any optimal solution from that complicated situation? There is a separate branch on that that is called the multi objective non-linear programming problem that not really is within the syllabus of yours, I will deal with the single objective non-linear programming problem in the respective classes from the next.

Thank you for today.