Constrained and Unconstrained Optimization

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Lecture – 31
Introduction to Nonlinear Programming

Ok. I will talk on introduction to non-linear programming problem. Now I will start non-linear programming problem all of you must have learned the linear programming problem, we have learned how to solve linear programming problem and basic concepts related to that, but non-linear programming problem is something different. In the sense that for the linear programming you must have seen if you know simplex algorithm and the other variations of the simplex algorithm you can solve any linear programming problem with those, but non-linear programming is not like that. Since the here the nature of the functions are non-linear, and there are different kinds of non-linear functions are there few functions are convex few are not. That is why there are there is a set of solution algorithm for solving non-linear programming problem of different kind.

Now, we have designed the course in such a way that we will discuss the varieties of the non-linear programming problems. And along with that we will discuss about the solution that is why I am start with the basic structure of a non-linear programming problem. And solution of that graphically. Then I will move to the classification of non-linear programming problem and one by one I will give you the solution procedures how to get the optimal solution for those.

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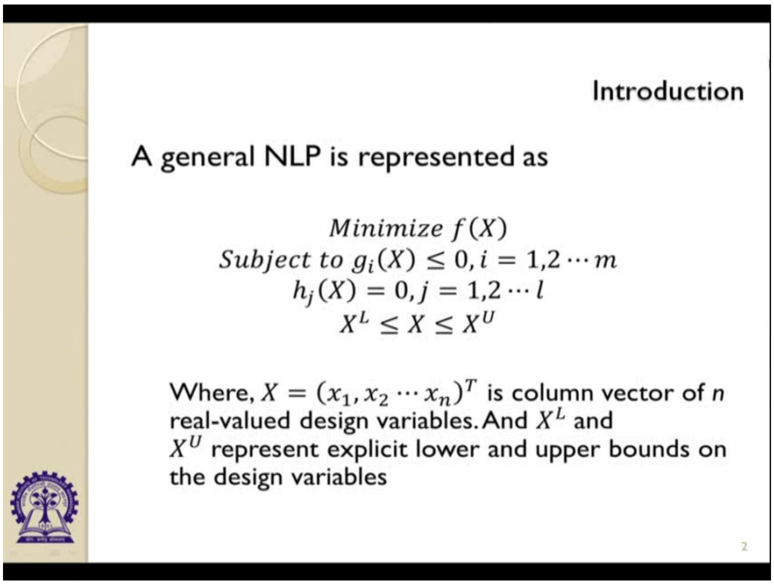

Now this is the general non-linear programming problem. Where you see the structure of the non-linear programming problem is that the objective function we have considered as fX. Minimization or maximization of fX subject to a set of constraints. Now here we have considered several functions you see the functions f is there that is the objective function and there is a m number of inequality constraint the functions are of gx giX rather and there is a set of linear constraints as well hj hjX, where j is running from 1, 2 l; that means, there are m number of inequality constraints and there is l number of equality constraints.

Along with that there is a bound of the decision variable. Now the decision variables may be restricted within a certain range that is why the bounds are given if it is not given. Generally, as we you know for the optimization problem we consider the this is non negativity constraint for the decision variable; that means, the decision variables can vary from 0 to infinity. But since we are considering the general form, here we have considered the lower bound as well as the upper bound of the decision variables. Now this is the general structure of the non-linear. If we look at the structure you see the functions f g h, any one of these can be non-linear in nature. This problem where any or at least one of the function is non-linear then we will say this is a non-linear programming problem and our simplex algorithm simply will not work. We have to learn you have to learn how to solve this problem.

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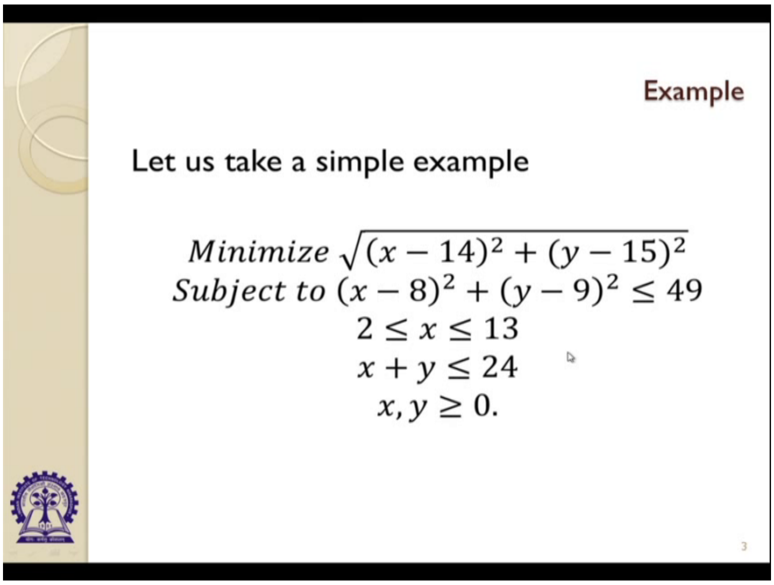

Now for that I have taken few examples with those examples I will try to explain you the concepts. Now here I have considered a problem of 2 variables.

Why I have considered 2 variables because the problem then I can draw it, I can tell you how graphically we will get the optimal solution for this. Now here you see the objective function we are having the objective function which is circular in nature, the circle we have conceded that is the centre 14 and 15. And there is a set of constraints one constraint ease of non-linear that is a another circle with the radius, 49 and the centre is 8 and 9 there is a bound for x. And there is a linear line x plus y less than equal to 24, but there is no bound for y, once that there is no bound for y generally we consider y is ranging from 0 to infinity.

Let us try to draw this problem.

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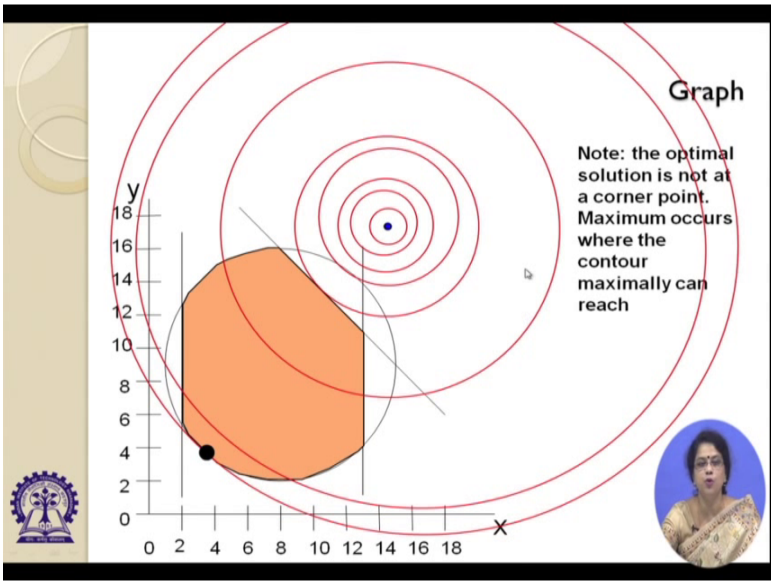

You see this is the axis for us, now we are first drawing the as you did for the linear programming same thing we will do and we will draw the constraints one by one. First we will draw a circle that is centred at 8 9 and with the radius 7. All right. Now this is the circle for you now there is a bound for x that is the lower bound of x is there and the upper bound of x is also there, and there is a line that is x plus y equal to 24. Now if this is show the space the points which is satisfying all the constraints that will be the feasible space you must have learnt that. So, from here we can declare that this is the feasible space for us. That is why the optimal solution must lie within this, but you must be seeing the feasible space is bounded by the non-linear functions by the non-linear curves and.

This is not the case like linear programming problem. And in the linear programming problem we have learned that all overs the optimal solutions are optimal solution is at the corner point. Or if there are infinite number of solutions at least that is at the edge of the feasible space. Now this is not the case for the non-linear programming problem, why this is. So, if I just explain you this problem graphically then you will understand. Now let us consider the objective function, objective function that is centred at 14, 15. That is why this is the centre of the objective function. And you see there are these are the contours that the red circles are the contours of the objective function; that means, the objective functions are moving forward. Then if I want to find out the minimization of the objective function, where I will get minimally it can once it is reaching to this point we can get the optimal solution because the objective function can go further.

Further and the at the minimum point, I will get at this one that is why we can have taken here that is the optimal solu minimum optimal solution for this objective function. Now let me explain this problem for the maximization type; that means, the objective function can move further what you have noticed here you have noticed here that the optimal solution is not coming at the extreme points or rather the corners of the feasible space rather it is somewhere at the edge of the feasible space. That is why graphically when the functions are very complicated in nature that is the first problem for us to draw the function. And next problem for us how to get the optimal solution for that, that is a another difficult part for us that. So, we need to learn graphically this is the simple function of circle that is why we could find it out otherwise very difficult for us. Now I am extending this problem for the maximization type.

That's why this objective function is moving further and further. What did you see that once this objective function the contour is moving, once this objective function is a touching that is the last possible position for the objective function to reach to remained within the feasible space? If it is going further it will be beyond feasible space that is why, if I maximize the objective function certainly the optimal solution will come here. That is the therefore, we are getting the minimal minimum is this point and the maximum is the other point. That is the beauty of the non-linear programming. And there is a complex part as well you need to learn how to solve it. That is why may let us consider another set of problem now in the linear programming problem generally we are not considering the unconstraint problem in general.

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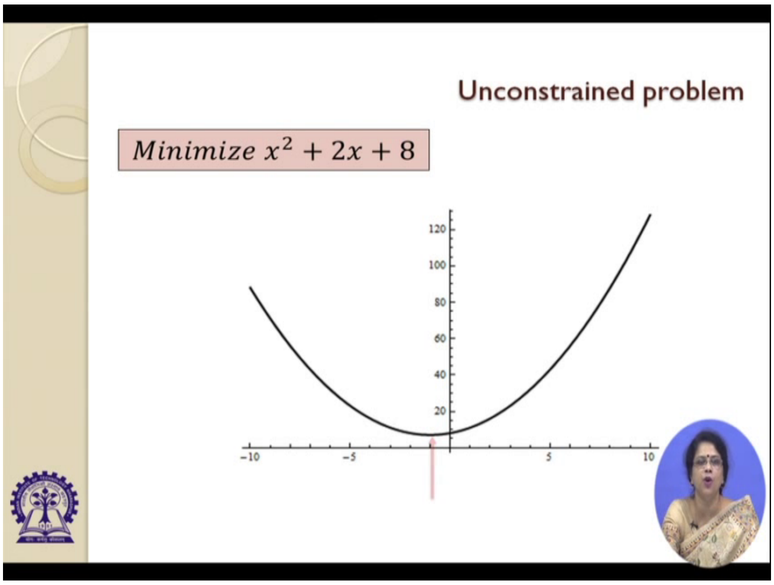

But you see for the non-linear programming problem, we may consider that are unconstraint because you see this is a function is only given to you. Now it has been said if x is in between certain range there is a domain for x, within this domain within this interval it is asking to you what is the meaning where the function has the minimum. You have done in your school time this thing for by finding out the differentiation of the function, but almost the same thing we will do you see this I just I have drawn the function here. Now where is the minimum this is the y that is a function fx and this is for you x. Now once you will get this point this point will be the minimum point, but if I ask you how did you get this point. Then you will say that the I have just done the differentiation of the function from there we got the critical point and we got the solution that way.

But this is not the case where the that we cannot do when the function itself discrete in nature all right. That methodology will not work. The differentiation can only be done if the functions are continuous, but if the function is not in continuous, we cannot do it that is why finding out the solution is not. So, easy for you need to learn how to find out the solution for it. In general, that is the part you need to learn for it. Now this is another function for you this is also non-linear in nature.

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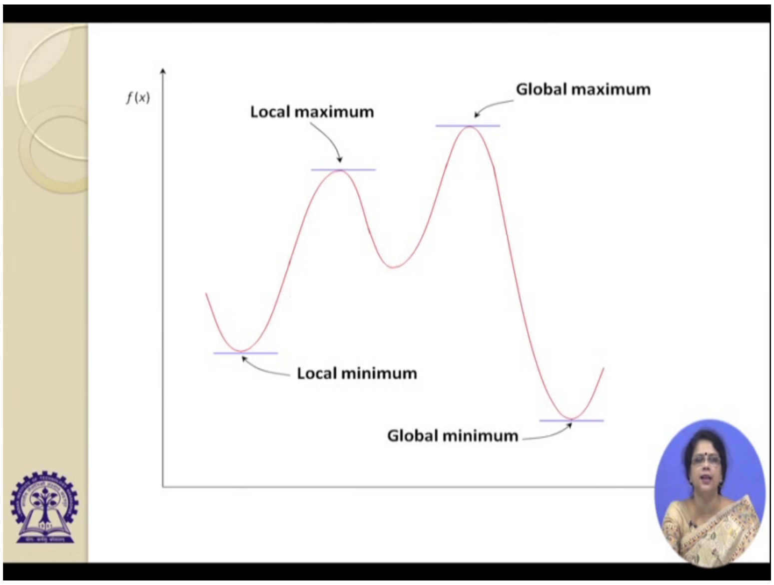

What do you see in this function if I ask you where is the minimum, you can say that there are there is a minimum point heres may be minimum here and maximum here because there are certain peaks over here this is the function of the single variable all right and you see there are certain peaks?

That is why in non-linear programming problem there is a concept that is called the local optimal that is called the global optimal. If I ask you in the domain of the function where is the maximum you will say that this is the maximum point, and in the domain of function where is the minimum there is the minimum, but knowing only the minimum and maximum will not serve the purpose. Because I need to know even for you see once we are learning the optimization problem. This is a decision making problem and this is very much related to the real life situation. That is why in the real life situation always that is very much related to the management science rather the business environment when we need to take decision not only the maximum minimum along with that we need take the decision where the possible other peaks.

That's why in the non-linear programming problem, there is one concept that is called the local minimum local maximum. You see I can say this is a minimum local minimum point this is also local maximum, but this is not the global. This is locally here there is a maximum here, there is a minimum here there is a mac. That is the global maximum now here another global minimum, but here also we can have a another local minimum that way we have to find out not only the optimal solution, the maximum minimum we need to find as well as the which one is the local optimal what which one is the global optimal. That is why you must be understanding you need to learn the non-linear programming in such a way.

We need to get we need to have the solution, we need to get the optimal solution that is local as well as global, but it is sure that all of us we will try to get the solution that is global in nature, but all the solution at solution algorithm will not give you that level, in all the time you would not get the global solution because of the complicated nature of the function you have to be satisfy with the local optimal solution. That is why the global optimal solution and then local optimal solution both are important for us. Now if you look at the function what did you see other than these, you could see that that there is a conviction concave nature of the function. As well that is why if for learning non-linear programming problem once you are considering the non-linear function, not only the local maxima not only the global maxima or minima always we need to before to that we need to see what is the nature of the function.

How the function is moving within the domain, that is very important for us that is why study of convexity concavity is also very important for us. Now let us move to the function of 2 variables.

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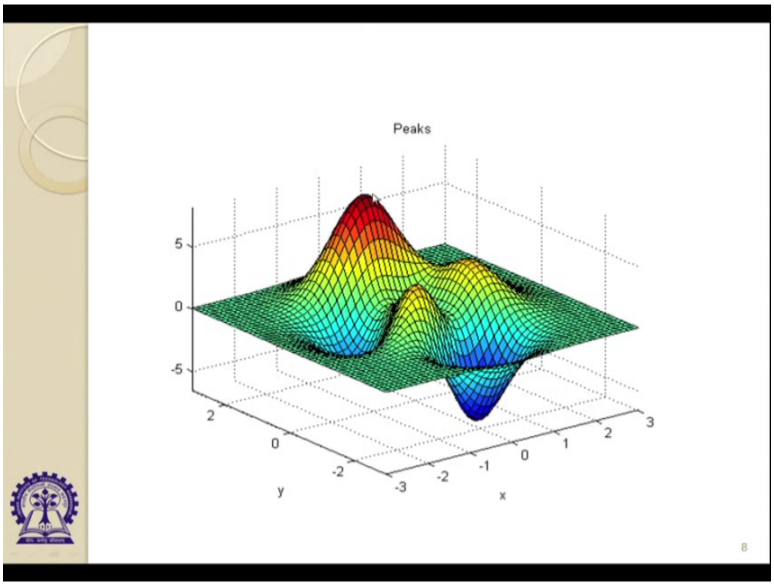

If I move to the function of 2 variables, what did you see the function. It is in the third dimension there are also certain peaks are there you can also define in the similar manner that they are a certain local optimal points at the end these are the local maximum points and this is the global maximum this is the local minima; this is the global minima if I just see the other part of the function. I could see even the local minima for this function that is why, once this is the function of single variable or 2 variables. We can visualize the local or global, but once the function is of in reality function is not of 2 variables you will get the functions.

Functions means I mean to say the functions which are involved in the objective as well as in the constraints these are function of several variables together. That is why you need to learn how to get the optimal solution for this.

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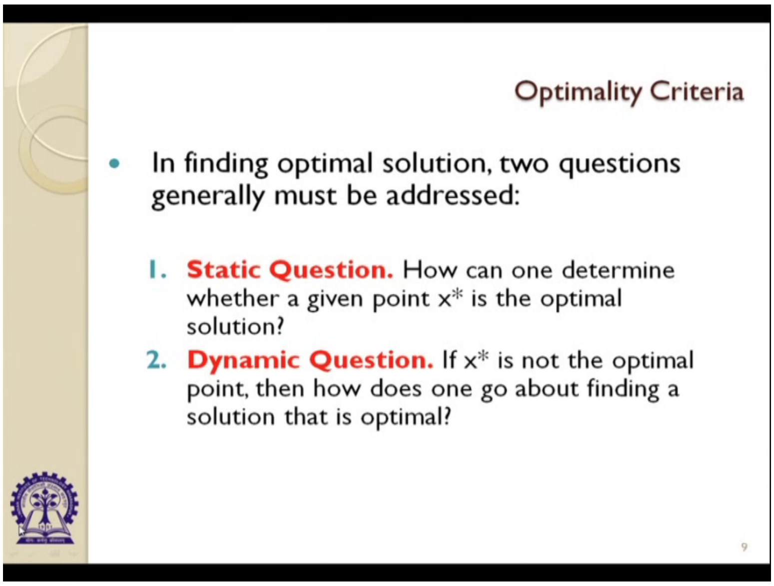

Now then what is our objective our objective is to find out the optimal solution. Now how to get it if you just use your common sense then very easily we can say that from the domain of the function, I will pick up any point I will check whether this this point is optimal or not. If the function is at that point function is optimal then we can declare this as the optimal. That is why optimality criteria checking is the first criteria for us how to check the optimality, this you need to learn all right. And if the point is not optimal then we have to move further, we have to search for the next and next and further and that is why we need to move that is why you need to learn that how to move to the optimal solution further. And how to design the optimality criteria for the non-linear programming problem. As I was told you that non-linear programming problem will involve involves different types of non-linear functions, that is why optimal criteria also differs, but there is a general guideline that I will talk about it.

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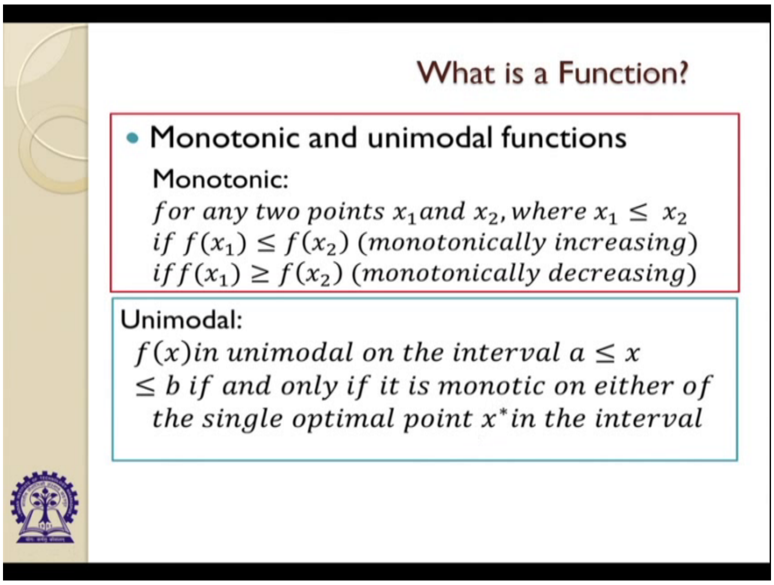

Now, this is another very important part of the objective function. Before learning the convexity and concavity you see there are certain peaks of the objective functions that is means then function is having the curvy nature, if the function is not having the curvy nature that this function is linear, but if the function is having the curvy nature then the next question is coming whether the function is the increasing function or the function is decreasing function if the function is increasing whether it is a monotonically increasing or there is an ups and downs within the function.

Function or there is any value of the function within the domain. Not only that we need to see how many peaks are there within that function and how many negative peaks are there within that of that function within the domain all these things we need to learn. That is why there is a very important property in non-linear programming problem that is called the unimodality property. You need to learn what is unimodal function why I am talking about the unimodality, because I will talk in detail in my next week classes, but now I will just mention the definition of the unimodality and I will tell you. If I refer to my previous example you must have seen that we dealt with a function of single variable.

How many modes it had how many peaks, it had it had for maximum there were 2 3 peaks were there all right now once; that means, if I have to find out the optimal solution now we need to find out at which point the mode is there, it may happen that I will get one mode and I will declare this is the optimal solution, but it may happen there is a local for you that is why what in general what we do we break the whole domain of the function in to their several intervals. So, that in each interval function is unimodal. So, that in a better way we can analyse the function that is why we will in different intervals we break the function, and we consider function must be unimodal means there must be only one mode, if there is one mode if that is the convex or concave in nature then certainly we can find out very easily.

That's why mathematically how to define the unimodality. Let me it is first tell it talk it talk about it now this is the monotonicity as you know a function f is monotonically increasing, if I consider 2 points x 1 and x 2. Graphically I will tell you when I will just show you in the next. And so, that the x 1 is the lesser than the point lower in the line in the real line than x 2 then function fx 1 must be f less than equal to fx 2 then we can say the function is monotonically increasing. And if this is the other case x 1 is lesser than x 2, but fx one is greater than equal to f fx 2. Then the function is monotonically decreasing function. Now what about the unimodality as I say there will be only one mode. Now see within the interval a b if I see that function is monotonically increasing in one side and monotonically decreasing in other side or the reverse monotonically decreasing in one side of a point x star and monotonically increasing.

In the other side other side of the x 1, then we can say the function is a unimodal once again I am repeating now the function fx is a unimodal within the interval a to b, the formal definition is that if and only if it is monotonically monotonic on other side of the single optimal point x star. I should not say optimal point in general let me just say any point x 1, if I see in one side it is monotonically decreasing and the other side monotonically increasing or the reverse one then we can say the function is unimodal. I will show you that example of that and unimodality is very important property as I said in the non-linear optimization, all right.

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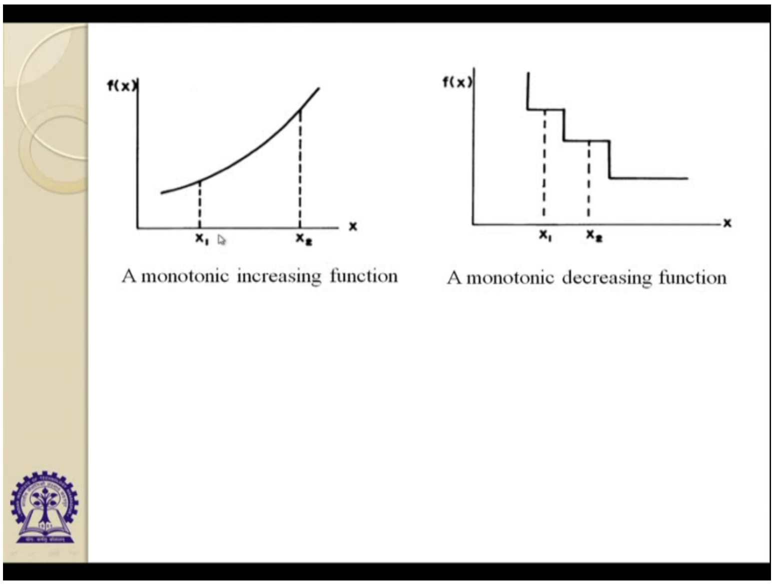

That is the definition now you must be seeing here the first example is a monotonically increasing function. If x 1 is lesser than x 2 if x 1 is lesser than fx 2 decreasing if x 1 is less than x 2 if x 1 is greater than fx 2 decreasing. And this is the unimodal function; that means, you see there is only one mode within this interval. In one side is it is decreasing and the other side it is increasing all right. Unimodality is very important property for finding out the optimal solution.

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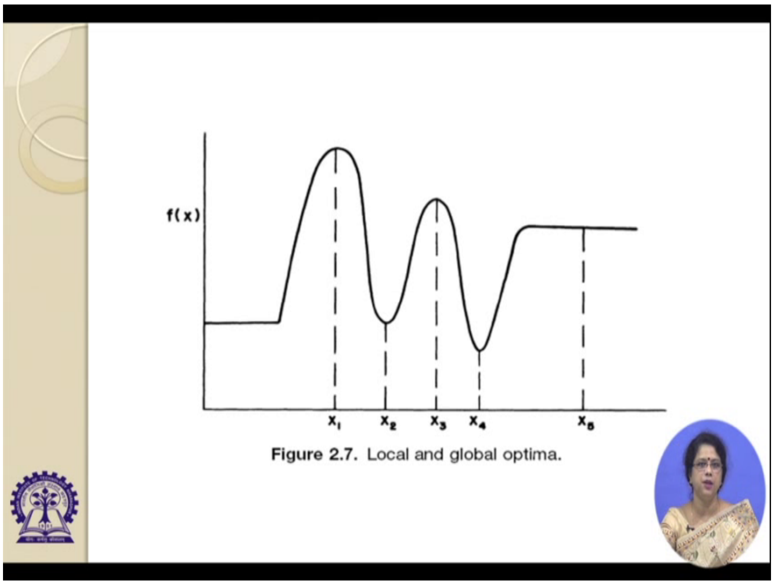

Once again I am repeating. Now you see this is the another function of single variable how many modes are there in x 1 there is a mode x 2 there is a mode for the minimum x 3 is maximum x 4 is the minimum. There are several modes are there. If I say if I ask you to find out the optimal solution for this, then the if you use your common sense then what should be the case it will start from here you will search whether the function is monotonically increasing or not.

If I move from here further we will see function is increasing, further I will show you one example on that you see I am starting from here all right. Now in this point I am here now I want to move on either side of this point, I can move this way I can move this other way as well. In which way I will move if I move right side then I will be beyond the optimal points is it not I will search and search I would not get the optimal solution at all because function is constant further.

That is why my search would be such that the procedure should not guide me to move in the right hand side. That is why I will move in the left hand side that should be the right process for me I am moving this way. Now the next question is coming if I even move in the left side how long shall I jump. So, that I will get the optimal solution I do not know where is the optimal solution at all.

Because this is a simple graph is given to me if I say you jump from here to here you will get one optimal, but this is not the case in general that is why I can jump from here to here even I can jump further. Then what will happen I will miss the optimal point that is why you need to decide 2 things from where I will start to get the optimal point in the real line not only that how long shall I move. So, that I will I own miss the optimal point I will get the solution exactly, even it is local I sometimes I have to be satisfied with the local optimality. Because as I told you that local optimality is something that in many cases is serve the purpose. That is why instead of searching global optimality means I have to do more number of jumps I have to do more number of iterations. In lieu of that better let me we satisfy satisfied with the local optimal all right. That is why say that there are several things to be considered all right that is just I wanted to.

Now, it may happen I will just jump up to these in this way I will jump to the next. So, that I can reach to x 4 for finding out the local minimum point that is the case.

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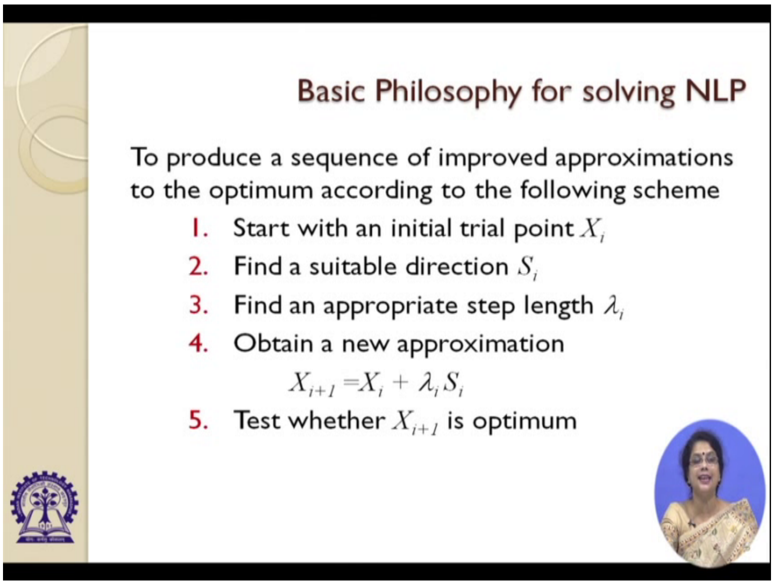

Now that is why as I told you just mathematically, I have just written the steps just to see the steps, I will start from initial trial point. I will find a suitable direction I will find out the appropriate step length I will move to the next point how by considering the first point plus the step length as well as the direction together all right. If I am moving to the left direction will be negative if I am moving to the right direction will be positive. Now lambda is the second that is the step length sorry, then I will get the next point xi will move to the xi plus 1, I will do the same process xi plus 1 will move to the xi plus 2.

In that way sometime, I will get the optimal solution, but again the question comes one by one. The question will come how to select the initial trial point if the initial trial point is far from the optimal solution, then the number of iterations will be more. Then I own get the solution that is why if I am how can I use my common sense, but in science there is no space for the common sense always. There is it has been said that there is a guideline which one to be selected as the initial trial point all right. That is why you need to learn this one the next you need to learn in which direction shall I move how to select the direction now for the single variable case if I say the plus would plus minus moving to the right moving to the left all right.

But if we consider function of 2 variables there are infinite number of direction from a point all right. That is why with the plus and minus it will not work it should tell me that in which direction shall I move all right. I need to find out some direction you have learned in mathematics the gradient of a function that part I will considered here and I will define the direction and as well as the step length. And in this way we will move this is a very basic idea how to solve a non-linear programming problem, but this is not the case this is not an efficient process to get just since this is the first class of yours to have a feel how really we are a getting the solution, just I have given you the feel of that that is all that is why you see.

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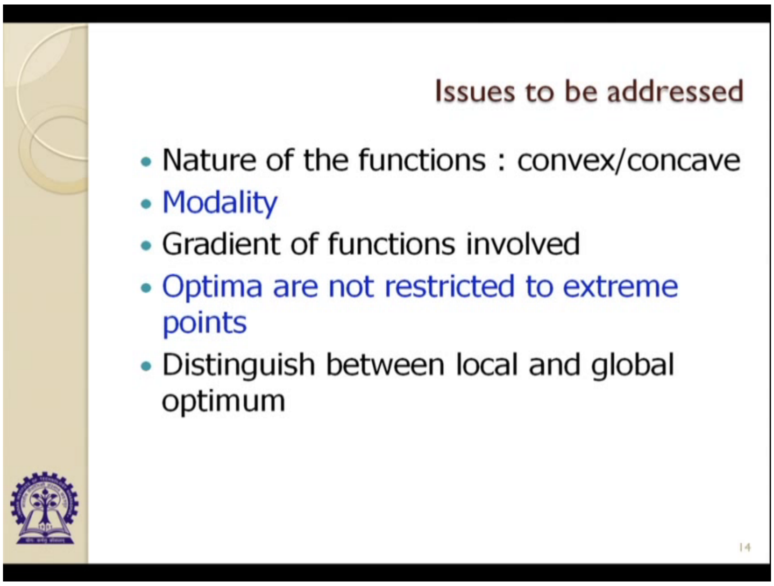

Few things we need to consider here up to this from a lecture you must have realized that first part of the first part of non-linear programming is that we have to judge.

Whether the function is convex or concave, if the function is depending on the nature of the concavity or convexity, we need not to even judge the function whether we are getting the local optimal or global optimal we can declare there are certain kind of conditions are there. Because there is a branch of optimization that is called the convex optimization there is a branch of optimization problem convex optimization. And in that convex optimization very easily we can say that always this methodology for this kind of problem will give you the global optimal solution that is why story of convexity and concavity is very important for us in the next class. I will talk on it talk about it all right modality whether function is having unimodal or not that is the another part for you. Next thing you need to learn the gradient of the function. As I said in which direction you will move gradient will guide you to get it. Now since optimal are not restricted to the extreme points that is why that is very difficult for us to get the optimal solution.

Infinite number of points at the edge of the feasible space. You know the extreme points if finite number of points, but here we are having the options that is infinite number. That is why which one will be the optimal solution that is very important for us we need we need to know we need to distinguish between the local optimal and global optimal that is why you need to know the definition of these mathematically.

In the last next class, I will discuss about it and it may happen the feasible space is disconnected feasible space is discrete in nature you are getting the feasible space that is piecewise continuous in one place continuous in other place it is continuous I will show you some picture of that. For example, if I just tell you the feasible space is like this just you look at this, this is only the feasible space where I am giving the shade, this is not the feasible space.

This is the feasible space. Then getting optimal solution local global everything studies of unimodality study of convexity concavity all these things you need to know together for non-linear programming problem and your differentiation will not work you cannot find out the gradient, that is a very difficult for you that is why you need to know how to find out even for the disconnected space. Now these are the issues I will consider in the next class and I will start my class with that.

Thank you very much.