Constrained and Unconstrained Optimization
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Lecture – 30
Degeneracy in TP and Overview of Assignment Problem

In this particular lecture, we are going to discuss two things; one is the degeneracy of transportation problem. And also quickly we will give an overview of the some other type of problem that is assignment problem. As I have told you earlier, the degeneracy in transportation problem occurs if the number of occupied cells after the finding initial basic feasible solution is not equals to m plus n minus 1, where m is the number of rows and n is the number of columns. So, if the number of occupied cells is not equals to m plus n minus 1 then only the degeneracy occurs in a transportation problem, we will see how we can solve the problem using one example directly.

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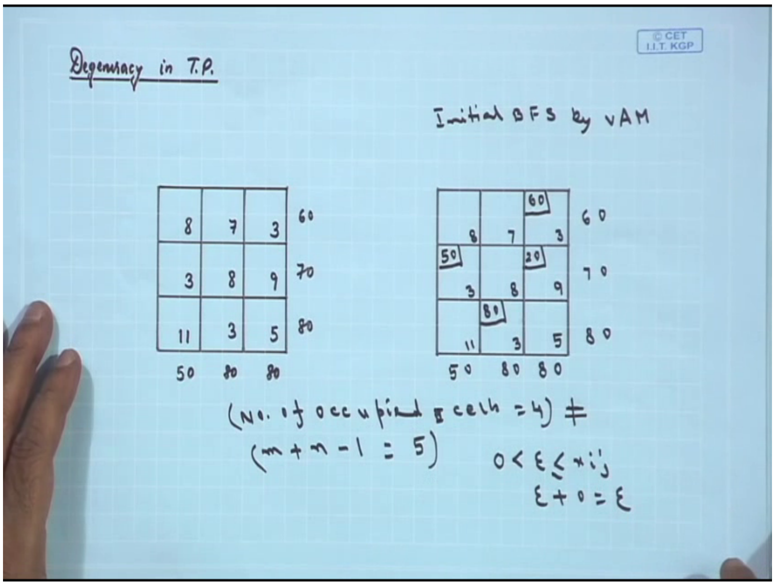

You see the problem I have a problem like this. So, here what we are doing at first I am finding the initial basic feasible solution by vAM method I am directly writing the solution here you have 8, 7, 3, 3, 8, 9, 11, 3, 5; and it is 60, 70, 80, 50, 80, 80. Since we have discussed the basic feasible solution by vAM method already, so I am directly writing the result of the vAM method. So, allocation would be like this here it will be 50, here it is 20, and 80 is coming over here. So, 60, 70, 80; here also 50, 80, and 60 plus 20 that is 80.

So, in this case what is happening your number of occupied cells number of occupied cells is equals to 4 - 1, 2, 3 and 4, number of occupied cells equals 4. And m plus n minus 1 m plus n minus 1 that is equals to 3 plus 3 plus 6, so minus 1 that is 5. So, what we are finding, this is not equals to number of occupied cells. And whenever this thing happens we say that the degeneracy occurs. So, whenever a degeneracy occurs in a particular problem, what you have to do, you have to assign a small quantity epsilon which is less than equals epsilon take certain values, such that epsilon plus 0 equals epsilon. And that means, a very small quantity is being allocated to a particular cell.

Suppose here; that means, what you are doing since the number of occupied cells equals 4 and m plus n minus 1 equals 5, so 5 minus 4 that is 1, one cell is unoccupied. Therefore, what we will do we will assign a very small quantity to a particular cell. Say we are assigning you can assign this epsilon to any unoccupied cells, suppose I will assign it at 1, 2 at this particular cell.

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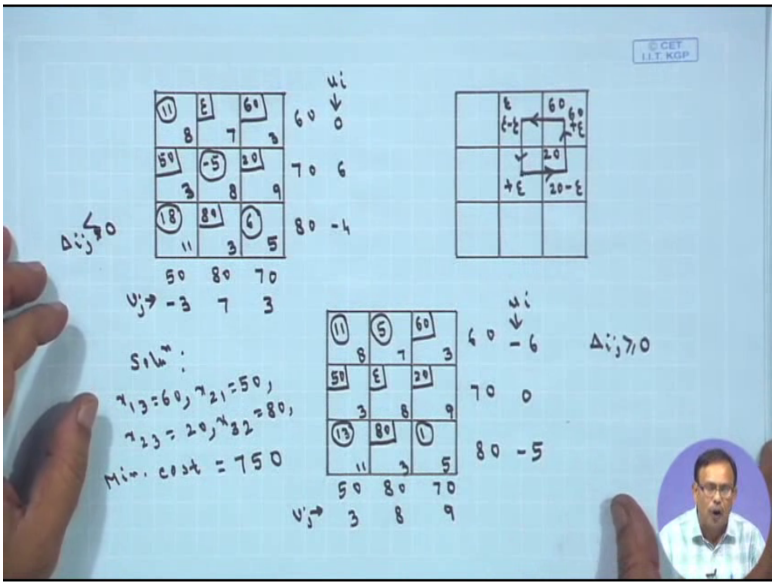

So, therefore, assigning epsilon in a particular cell, you will obtain a table like this that is you are having the initial solutions 60, 20 and 50; here it is 80 values are 8, 7, 3, 3, 8, 9, and 11, 3, and 5; these values are 60, 70, 80, and 50, 80 and 70. Now, you assign a small quantity epsilon in one occupied unoccupied cell. Suppose, I am assigning it at 1, 2.

Now, what I have to do, I have to calculate u i, v j and ultimately I have to calculate delta ij for the unoccupied cells to check whether the table after the occupancy whether that gives me the optimum result or not. So, these things again we have done earlier. So, I am directly writing the values you will check it afterwards of your own. So, u i values will be these things 0, 6 and minus 4; v j values will be minus 3, 7 and 3.

So, now you can calculate the values here for this case as you know c ij minus u i plus vj is the value for the occupied cell. So, for this one, it will be 8 minus 3 plus 0, so it is 11; likewise for all other occupied cells it will be this is minus 5, this will be 18, and this is 6 . So, here if you see 1 delta ij is less than 0, 1 delta ij is less than 0, therefore, the solution whatever you have obtained that is your solution is not optimal. And as you know I have to start now from here, I have to make this one as positive. So, I will start from this point at this point and I have to traverse around the occupied cells such that I will come back to this cell again.

So, your occupied cells I am writing here, first here it is you are having epsilon, here in occupied cell, you are having value 60. And in this occupied cell, you are having a value of 20. So, once I am doing this, so I can traverse like this from here to this occupied cell; from this occupied cell to these occupied cell; from here to this occupied cell and I can come back to this place again. So, I will come back to this place.

Now, here how much I will add, I will add the minimum of these occupied cells where there is a loop. So, in the loop you have epsilon 20 and 60 minimum obviously, is epsilon, because it is a very small number. So, therefore, I will add epsilon over here. So, once I am adding to make the balance, I will make it 20 minus epsilon, this will become 60 plus epsilon and this will become epsilon minus epsilon, so that now this occupied cell will become unoccupied. So, what is your result, here 60 you have given, here it is 20. Now, epsilon has been added here. So, this cell will be occupied by epsilon, here you are having already 50 was there, and you are having 80 over here, so that the values will remain the same.

So, here it is 60, 70, 80; and this is 50, 80, and 70. So, this is now becoming an unoccupied cell, and instead of these. And this since this epsilon is very small, so therefore, this 70 plus epsilon is equals to 70 only, let me write down the c ij values for all cells from the earlier one, so 8, 7, 3, 3, 8, 9, 11, 3 and 5. So, with this again check whether the this stable is optimum or not. So, for that you will calculate the u i value you will calculate the vj value; u i values will become minus 6, 0, minus 5; and your v j values will be 3, 8, 9.

Now, you calculate delta ij that is c ij minus u i plus v j for the unoccupied cells you will find that it is becoming this one in this case it is 13, here it is 1. So, please note that whenever there was one delta ij was negative, basically by assigning a very small quantity I am making this one as positive here if you see delta ij is greater than equals 0 for all the unoccupied cells. Therefore, the solution, which you have obtained, this is optimal solution and we have told earlier what is the basic theory.

So, what is your solution now optimum solution is x 1 3 equals 60, x 1 3 this is equals 60, then x 2 1 equals 50, this you will not assign anything because epsilon is a very small quantity approaching to 0. So, this will nothing will be occupied over here. The next one is x 2 3 will be equals to 20, and x 3 2 this is equals to 80. And if you calculate the minimum cost, your minimum cost will become 750 minimum cost you also know how to calculate.

So, just for repetition, whenever you have an situation like this, number of occupied cells is not equals to m plus n minus 1, in that case degeneracy occurs whenever degeneracy occurs in any of the unoccupied cells, you have to assign a very small quantity epsilon like I have chosen here the cell 1, 2. So, in the next table, I have assigned epsilon and then again I will check whether this solution is optimal or not by calculating u i v j and then delta ij. And I will repeat this process until I am obtaining the required optimum solution.

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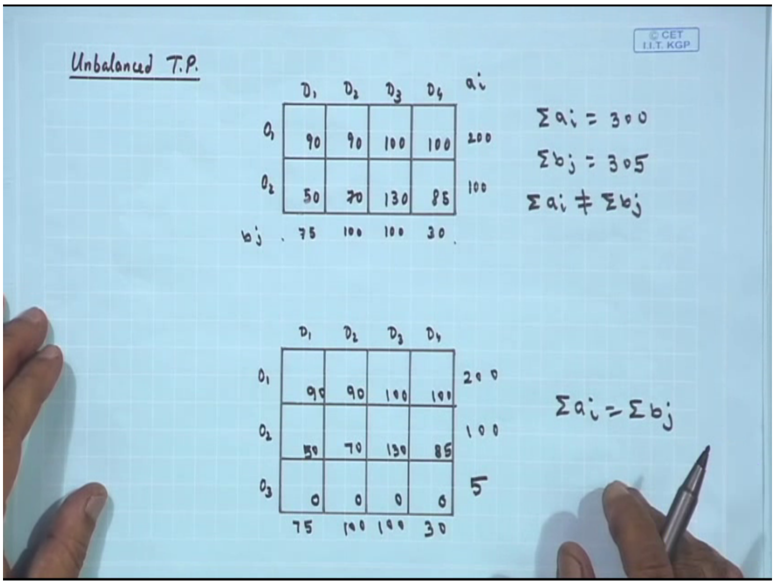

Next is the unbalanced transportation problem. Unbalanced transportation problem, when it occurs, here what is your summation of a i basically this gives you the a i's this gives you the b j's. Summation of a i is equals to 300 here. If you calculate summation of b j, this is to or 275 plus 30 that is 305. So, for this case if you see summation a i is not equals to summation b j. Whenever summation a i that is availability and requirement these two values are not same then we tell that transportation problem as unbalanced transportation problem. So, one unbalanced transportation problem occurs whenever summation of a i is not equals to summation of b j.

So, since there is a requirement available in a i that is a shortage, therefore, what I have to do, I have to add one row here like this. So, I will write down these two will remain same 90, 90, 100, 100, 50, 70, 130, 85. And for the new row the costs always will be 0, here it is your 75, 100, 100, 30 and here it is 200, 100. And what is the shortfall, shortfall was 305 minus 305. So, in the third row in the availability, I will make it 5, so that now summation of a i is equals to summation over b j.

So, once I am getting this, now I am transferring my unbalanced transportation problem into balanced transportation problem. So, by this way, whenever you are getting one unbalanced transportation problem, I can transform it into a balanced transportation problem either by adding one row or by adding one column. And then solve the problem as usual that is first using vAM or other method find the initial basic feasible solution and try to check the optimality if the table is not optimal, you make it optimal using the process whatever we discussed earlier.

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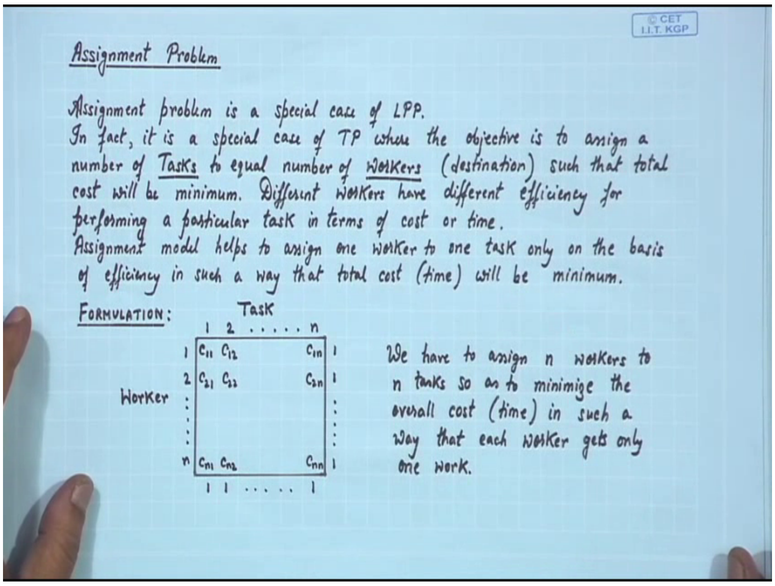

Now, let us go quickly to the other type of problem, which we call as assignment problem. In assignment problem, basically it is a special case of LPP and we can always tell that it is a special case of transportation problem, where our objective is to assign a number of tasks to equal number of workers or destination such that the total cost will be minimum. Now, it occurs at various places, different workers may have different efficiency for performing a particular task in terms of cost or in terms of money. And that means, one worker can do a job a little faster or the cost whatever wages are given that may vary from worker to worker.

Assignment model helps to assign one worker to one tasks only on the basis of efficiency in such a way that total cost or total time whatever you say will be the minimum that is we want to say that each worker will be assigned only one task, please note this one. And on what basis, so that total cost whatever was paid that will be minimum. So, you see the formulation over here. We have written worker in this side, we have written tasks in this side. We have the associated costs c 1, c 2, c 11, c 12, c 1n like this were c n1, c n2 c nn and we have n task and n workers. So, like transportation problem a may not be equals to n, but here m should be equals to n. So, please note this thing, and we are assigning 1, 1 to this. So, you have to assign n workers n tasks. So, as to minimize the overall cost in such a way that each worker gets only one work, please note this one each worker gets only one work.

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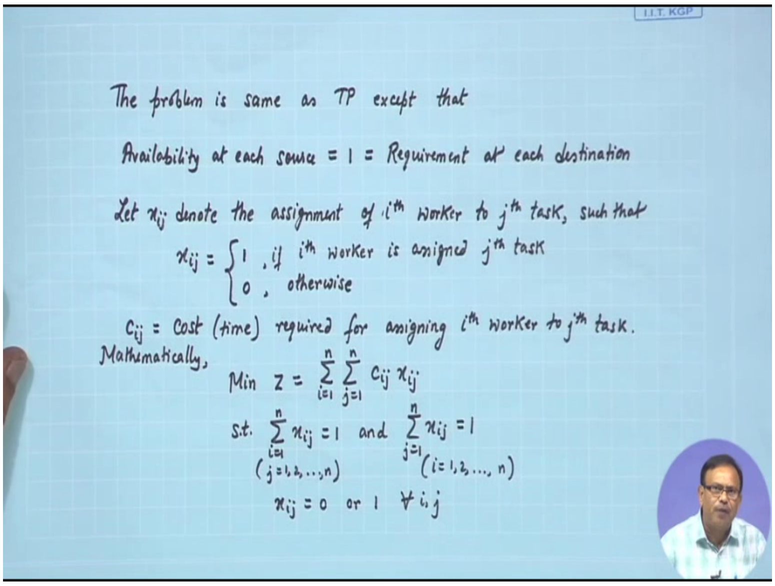

So, the problem is same as transportation problem except if you see the availability and requirement side the value is one. So, here availability at each source equals to 1 equals requirement at each destination. So, value of x ij will be equals to 1, if ith worker is assigned jth task otherwise it will be 0; and c ij equals cost or time required for assigning ith worker to jth task. So, mathematically if I have to formulate this particular problem, in that case, I can tell that minimize z equals summation i equals 1 to n summation j equals one to n c ij x ij subject to summation i equals 1 to n xij equals 1 and summation j equals 1 to n xij equals to 1. Both ai and bj will be equals to 1, and ij both will vary from 1 to n. And you have to note one thing here exerciser can take the value either 0 or 1, it cannot take any other value. So, in mathematically I can write it like this. So, I can solve it as a LPP, I can solve it as a transportation problem, but there are specific solution procedure or methods for this assignment problems also.

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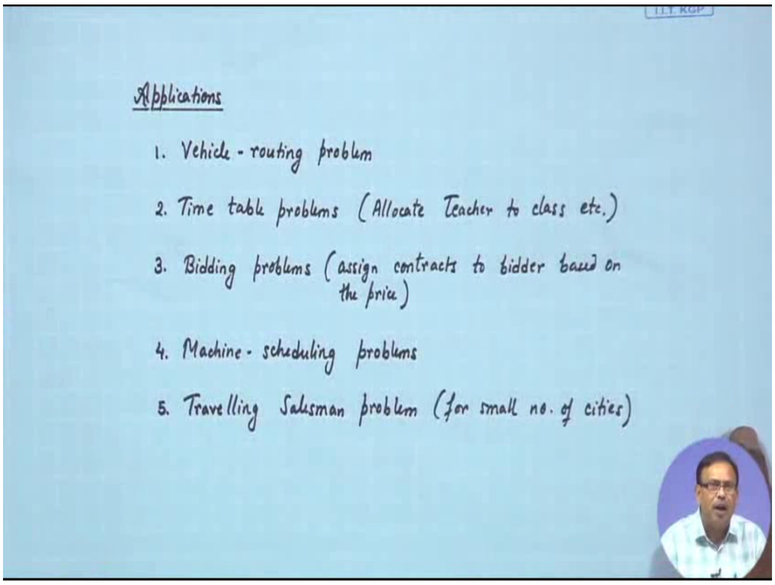

It has wide range of applications in vehicle routing problem; it is used that is in vehicle routing at various places, we use this one. In that timetable problem, that is allocated teacher to the classroom or allocate teacher to the subjects. Bidding problems where I will assign contractors to bidders based on price or other criteria. In machine scheduling problem, I can use this assignment problem. And in travelling salesmen problem, which is a very important problem which is NP hard problem in travelling salesmen problem also we use this assignment problem; obviously, for small cities not for bigger cities.

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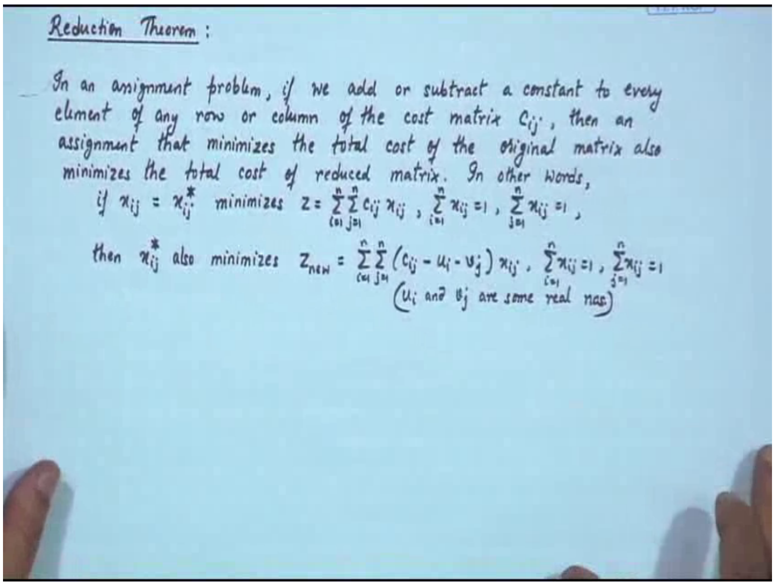

That is an important theorem. I am leaving each I am not going through the entire theory of this, which I am leaving for you, you can read the books and you can understand it but this important theorem which we call as a reduction theorem. If an assignment problem in an assignment problem, if we add or subtract a constant to every element of any row or column of the cost matrix c ij then an assignment that minimizes the total cost of the original matrix also minimizes the total cost of the reduced matrix. Or in other sense, whenever I am trying to obtain the solution, if I add or subtract a constant with a row or a column in that case the optimal solution of the original problem and optimal solution of the reduced problem will remain as it is.

Mathematically, I will say if x equals x ij star minimizes, z equal summation over i summation over j c ij xij, where summation over ix ij equals 1, summation over jx ij equals 1. Then xij star will also minimize another problem which I am writing z new equal summation over i summation over j c ij minus u i minus v j into x ij, where summation xij equals 1 and summation over jx ij equals 1. Or in other sense in both the problems in the second problem, the coefficient are reduced by u i minus v j where u i and vj are real number. Then for both the problems the solution will remain same which is xij star.

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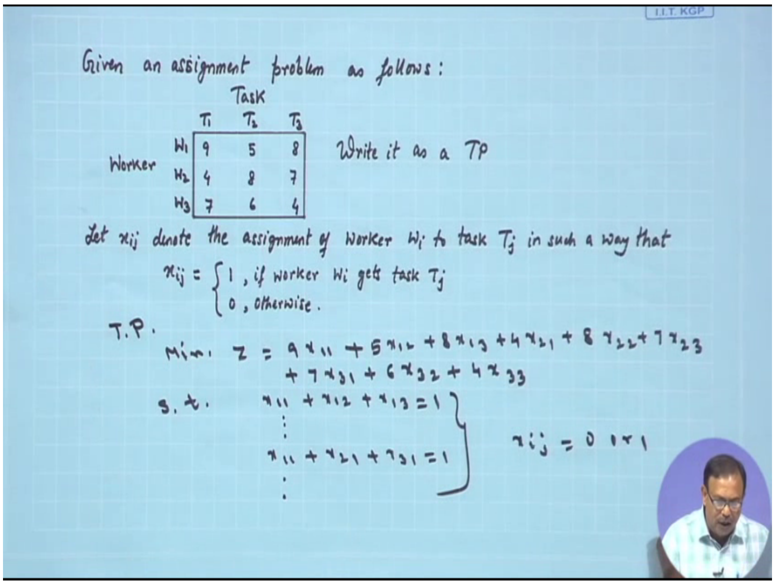

So, please note this thing. Therefore, if you have a problem like this, suppose I have given an assignment problem like this workers corresponding to task and I am denoting x ij this one. I can write down it as a transportation problem also. Minimize z equals 9 x 1 1 plus that means, you are just writing 5 x 1 2 plus 8 x 1 3 plus 4 x 2 1 plus 8 x 2 2 plus 7 x 2 3 plus third row now 7 x 3 1 plus 6 x 3 2 plus 4 x 3 3. Subject to what you are doing the now all the availability and requirements are equal that is x 1 1 plus x 1 2 plus x 1 3 equals to 1, then x 2 1, x 2 2, x 2 3 equals to 1 like this way three constants. And then again column wise x 1 1 plus x 2 1 plus x 3 1 equals to 1, and again two more equations and xij equals to 0 or 1. So, this I can formulate as a transportation problem this I can formulate solve as an assignment problem. So, whenever this structure is given, I can always solve it using transportation problem also or even if LPP also.

For assignment this is the basic flow chart you are starting the setup cost or table for the problem if it is a maximization problem then convert it to the minimization problem, if no then we will check it is a balance problem or not. Like we have done it for the case of assignment problem, if it is not I am adding row or column to this one after that I am obtaining the reduced cost table here.

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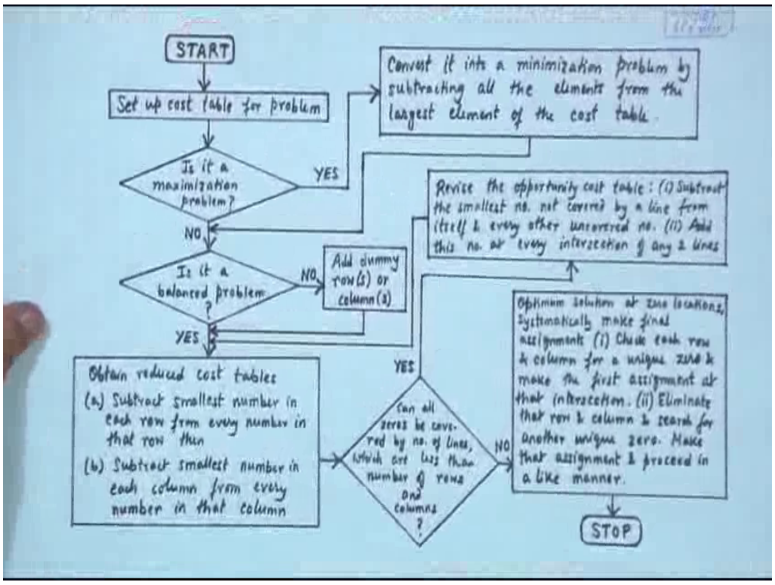

You see this is important subtract smallest number in each row from every number in the row to get the reduced cost table and subtract smallest number in each column from every number in that column. So, here I am reducing it first, I will explain it by one example. Then we are checking can all 0s be covered by number of lines, which are less than number of 0s or rows. If no, then optimum solution at 0 location we have assign and we are stopping; otherwise, we have to revise the opportunity cost table as we will discuss in the following example.

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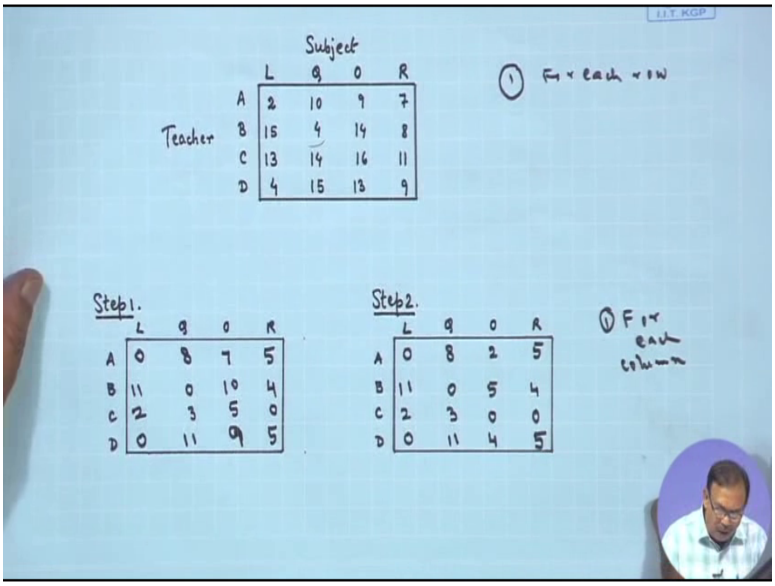

So, let us directly take the example over here; from the example it will be very clear to you. So, suppose you have some teachers and you have some subjects, subjects name I have given as say linear programming problem, king theory, then operating system real analysis something like that and the costs associated costs are there . So, at first what you do the your first step, step one is for each row you take for each row what you do you subtract the lowest element of the row from all other elements. So, you see the first row first row contains 2, 10 9 and 7. So, subtract here the minimum of this row is two therefore, you subtract two from all the elements of the first row. So, if you subtract 2, then it will become 0, 8, 7 and 5.

Similarly, follow the same process for the second row that is minimum element is 4 here for the second row, so subtract 4 from all other elements. So, you will obtain 11, 0, 10 and 4. For the third row, the minimum of the third row is 11. So, subtract it from all other elements, so here it will become 2, 3, 5 and 0. Then in the 4th row you are having the minimum element is 4, so subtract 4 from all elements, you will obtain 0, 11, 9 and 5. So, please note that in step one, what you are doing you are subtracting the lowest element of each row from all are from the elements of that row like this way.

Next, in step two, what you will do you will perform the similar operation for each column. Similar operation for each column that is for each column find the lowest element, and from the lowest element you subtract all other elements. So, here it is basically 0 is there. So, this column will remain as it is then you are having the next one here also you have a 0, so 8, 0, 3 and 11. Next one, no zeros are there, 5 is minimum therefore, you will obtain 2, 5, 0 and 4. The next one is one zero is there, so it will remain unaltered, so it will be 5, 4, 0 and 5. So, in the first step for each row you are subtracting the lowest element from all the elements of the row you are repeating the same process in step two for each column of the element.

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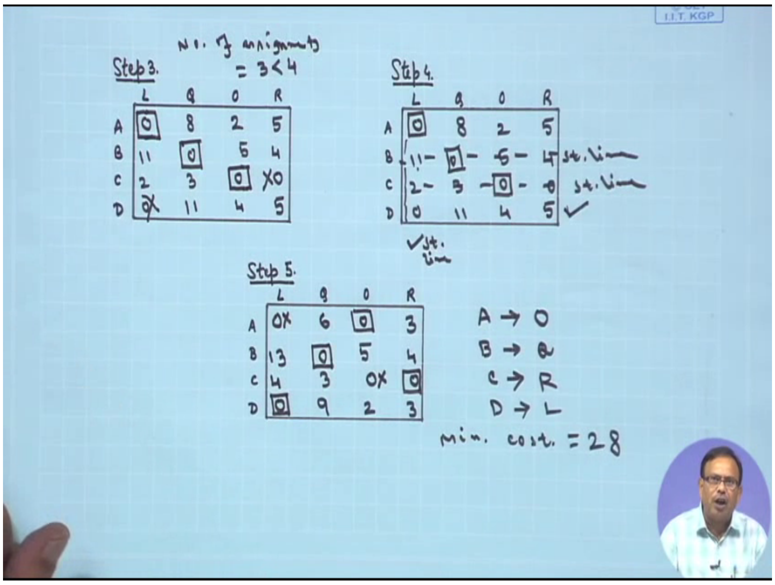

Now, in the next step even step 3, what you will do I will first write down the table earlier table that is whatever I got in step two. It is your 0, 8, 2 and 5; 11, 0, 5 and 4; 2, 3, 0 and 0; 0, 11, 4 and 5. Now, examine in step 4, examine each row carefully first. In each row you check whether you have only one zero or not. So, in the first row, you have only one zero, then write it inside a square like this; that means, I am assigning. But please note that if you have only one zero in one row, then only you can do this thing. Once you have done this one then corresponding to that row in that element on that particular column that is in the first column itself if any 0 is there, then cross out that 0 that means, this 0 I am crossing out. So, what is the policy, you examine each row individually starting from the first row. If there is only one zero in the row, you put it in the square and check the corresponding column if any 0 is there crossing that is afterwards you will not use this one, repeat it the process for each row and for each column.

So, let us see the second row again second row is having only one zero. So, put it on the square and there is no 0 in the corresponding column, so I cannot do anything. Now, in the third row you see you have two 0s. So, you cannot do anything you have to proceed next please note this one. In the third row, since you have more than one zero you cannot do anything. In the 4th row, you do not have any 0, so you cannot do anything.

Now, go to the columns in columns if you find first column already one zero, you have a sign. Second column already one zero. You have a sign in the third column you have only one zero and once you have only one zero, you put it under square. And now do the opposite that is corresponding to this if in the row if there is any 0 then cross it that means, you are crossing this one. Now, you see if you check no 0s are left behind either that we have put it under square or we have crossed it, but here you see number of assignments for this case, number of assignments is equals to 1 to 3 which is less than 4 because I have to make 4 assignments over here.

Since I have to make 4 assignments over here, but number of assignments is not 3. So, what I will do now, this is not optimum. So, let me write down in step 4 the earlier one zero, 2 8, 5; 11, 0, 5, 4 and 2, 3, 0, 0; 0, 11, 4, 5. You first tick mark the unoccupied rows and column tick mark the unoccupied row and the corresponding column in the unoccupied row where you have 0. So, you give a tick in the unoccupied row, your unoccupied row from step 3, we are observing it is this. So, I am putting a tick on this. And corresponding to this row, if there is a 0, if there is a 0, in that case this also that particular column also you tick mark. So, this is the first step.

So, your occupancy was on this three. Now, you draw a straight line through all unmarked rows and the marked columns. Please note this one, draw straight line through unmarked rows and marked columns, your marked column is this one. So, I am drawing a straight line, this is unmarked row, I am doing putting this. Now, this is also these. So, I am giving an straight line. So, you are drawing one straight line here, you are drawing one straight line here and one straight line here.

So, for unoccupied unticked rows, you are putting a straight line and correspondingly for ticked column also you are putting it since already here. So, now, the remaining elements are these which are not covered by the straight line. You find out the minimum of them. Your minimum here minimum of them this value, this value, and in the third case this value this will remain as it is. For the unoccupied cells find the minimum unoccupied cells the elements are 8, 2, 5, 11, 4 5. So, minimum is 2. So, now, what you do you subtract for them for the unoccupied cells that is 2, you subtract two from these two elements.

So, the first the first row it will be 6, 0 and 3. For this case, it is remaining, for this one I am coming to this let me write down 11, 4, 5, so it will become 9, 2 and 3 this is becoming 9, 2 and 3. And then you put wherever there is a crossing of straight lines you add that minimum value at those points, so that crossing one is coming here that is second row first column and third row first column. So, this value 11 and 2 will be incremented by two, since two we have done this thing. So, it will be 13, it will be 4, all other values will remain as it is. So, it is 0, 5, 4, 3 and 0. So, this is the process and then you reassign or reallocate again using the process whatever we have told over here.

So, for this case, initially first row we cannot give because two 0s are there; in the second row I have only one zero, so I am allocating. In the third row I have only in the third row what is happening again two zero, so I cannot do anything in the 4th one only one. So, I am putting this and corresponding column I have to check this one, then go to the columns first and second column already I have assigned third column has two zeros, I cannot assign. Fourth column as one this one, so these 0 in the corresponding row again it will be this I will cut it, so only one zero is remaining and I am assigning this.

So, now you have if you see the number of assignments are 1, 2, 3 and 4. So, I obtained the optimal solution. So, what will be the assignment the teacher A will get O, teacher B will get Q, and teacher C will get R, and teacher D will get L. So, this is the assignment and what is the minimum cost minimum cost from the original problem you can find out, what are the corresponding values in these assignments that is it will be 9, 4 plus 11 plus this one 4. So, that minimum cost will be 9 plus 4 plus 11 plus 4 because these were the assignments you made in the last column, so that if the sum is minimum cost will be 28.

So, by this way you can find out the solution of an assignment problem. And if the assignment problem is not balanced assignment problem in that case you can make it balanced by adding either a row or a column suitably. So, this is an very easy process compared to if you see the procedure whatever we have done for solving a transportation problem or solving a linear programming problem. And for this reason, assignment problems, we solve it by this method which is known as Hungarian method. For transportation problems, we have given some other kind of solution process; and for linear programming problem, we have provided the simplex algorithm method for finding the solutions.