Constrained and Unconstrained Optimization
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Lecture – 28
Vogel Approximation Method

So, in this class we are going to start the Vogel approximation method. In the last class, if you remember I have just told briefly about the Vogel approximation method for finding the initial basic feasible solution of the transportation problem. We absorbed other methods also for finding the initial basic feasible solution of the transportation problem.

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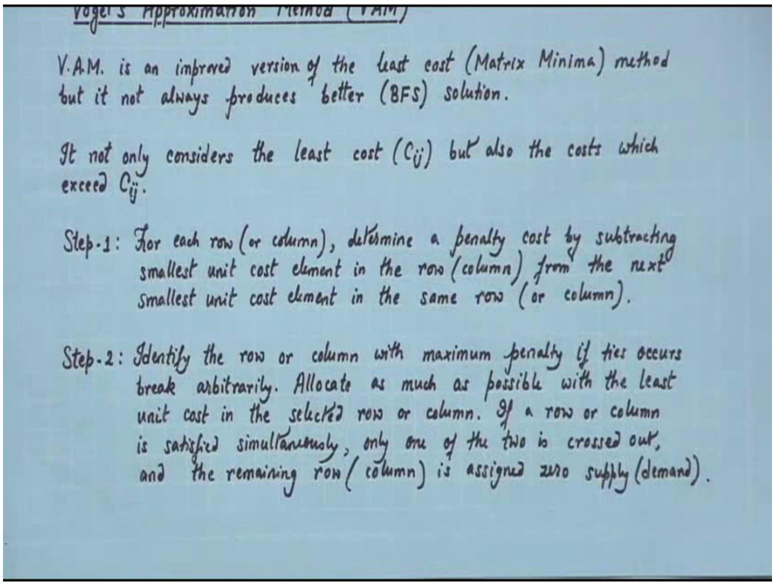

Let me just brush it up quickly. This is actually the improved version of the least cost method. So, in this particular case, what you are doing in step one at first for each row and each column, you are determining a penalty cost by subtracting the smallest unit cost element in the row or that column from the next smallest unit cost in the same row. Then we are identifying the row in step two, the row or column with maximum penalty if any tie occurs then break it arbitrarily that you can choose any one. Then allocate as much as possible with the least unit cost in the selected row or column. If a row or column is satisfied simultaneously, only either the row or column should be crossed out not both; and the remaining row or corresponding column is assigned with the zero supply or demand.

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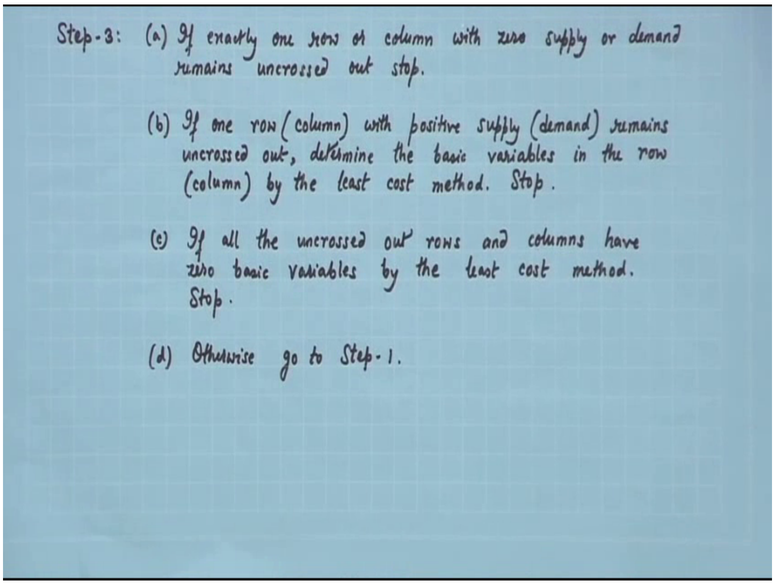

And basically this process will be repeated until on the all cross rows and columns are basic variables. So, let us explain it with example.

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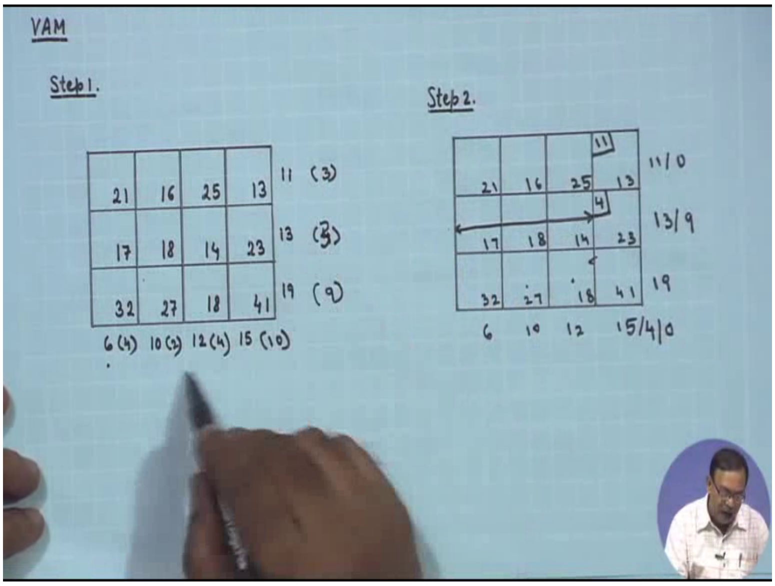

Let us see take one example over here. So, for this case, at first you have this problem, you have the origin, you have the destinations, you have the a values and you have the b values. So, for this particular case, what you do your step one was this one. For each row or column and column determined the penalty cost by subtracting the smallest unit cost element from the next smallest unit cost. So, if you see this row, the elements are 21, 16, 25 and 13; smallest cost is 13 and second smallest is 16, therefore, the difference of these two you write here within bracket, so 16 minus 13 it is 3. Similarly, for this row smallest is 14 second smallest is 18 difference is 4; 18 minus 14. Then for this case the smallest sorry smallest is 14 and second smallest is 17, so here it is 3, not 4; 17 minus 14. Then in the next one you are having the smallest is 18, second smallest is 27, so that the difference is 27 minus 8 that is 9.

Similarly, so for each row, what you are doing you are finding the smallest cost and second smallest cost and you are finding the difference, you are writing within bracket for each row like this. And same operation now I will do it for each column. So, for each column what happens, for the first column smallest is 17, second smallest is 21, so difference is here 4, I am just writing within bracket here 4. For the second one smallest 16 and second smallest 18, it is 2. For the next one third column, it is 14 and 18, so the difference is 4; and for the last one smallest and second smallest are 13 and 23, so the difference is 10.

So, now you see these differences, you are calculating then you see identify the row or column with maximum penalty. So, please note this one, identify the row or column with maximum penalty then allocate as much as possible with the least unit cost in the selected row or column. So, therefore, here if you see your maximum difference is 10 out of this 3, 3, 9 and 4, 2, 4, 10. So, basically what I have to do I have to allocate on this particular column this one. So, I am writing the next one that is 21, 16, 25, and 13, you are having the next row 17, 18, 14 and 23; your third row is 32, 27, 18 and 41.

The availability and demands are respectively 11, 39 and demand is 6, 10, 12 and 15. So, as you have seen on, this gives you the maximum difference. So, on this column, now I have to allocate. So, on this column, where I will allocate, I will choose that particular cell where which has the minimum cost entry, I will choose that particular cell which has minimum constantly, so minimum of 13, 23, and 41 is 13. So, therefore, I will allot on the cell one 4. If there was a tie that is if the cost is same for two cells in that case you can choose any one.

Now, you see we have decided on which we will form we will first allocate. In 13 you see the maximum availability here is 11 and here is 15. So, therefore, minimum of 11 and 15 that is 11, you can allot to this particular cell. So, effectively this becomes now 0 and here your remaining is 4. So, still on this column, these four demand four unit, you can allocate somewhere. How we will allocate again from the remaining choose which one is the lowest out of 23 and 14, the lowest is 23. So, on 23 here, it is 4 and here it is 13, so minimum of four and 13 that is four you can allocate. So, once you are allocating this thing.

So, basically in the second case, you are allocating on this one. So, therefore, you have allocated 15 here, and you have allocated 11 on this. So, now, from the next table, this becomes effectively 0. So, you have allocated the requirement in the first row, and in the fourth column. So, your remaining things will be how many only the second row and third row and three columns will be there.

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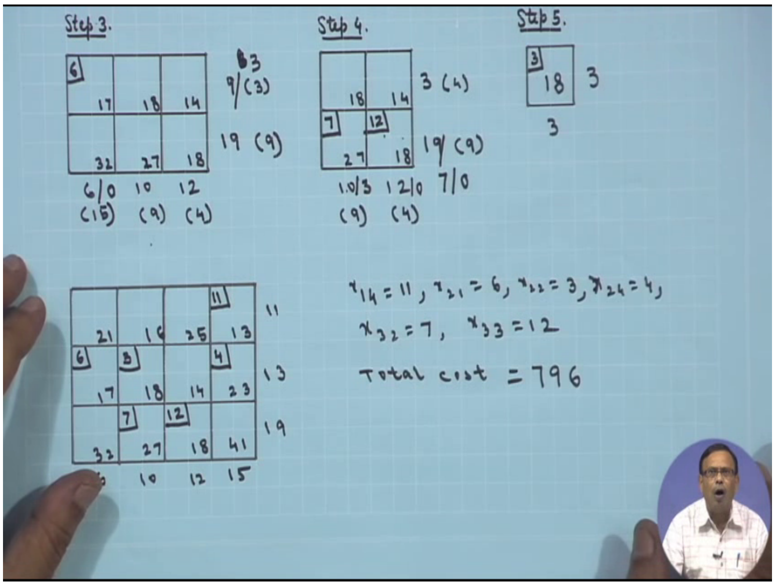

So, I am writing the next table. In the next table, what you are doing here at first from here you are writing the second row from here to here up to this that is second row and third row, you are writing and you will take only first column, second column and third column only, fourth column already you have allocated entirely. So, your second table becomes 17, 18, and 14, and then you are having 32, 10, 32 sorry 27, it is 27, and 18. Yet the cost will be since already you have allocated four on these. So, it has become nine from 13 it has reduced to 9. So, on this case, it will be 9 and 19, 9 and 19; and here the difference our costs are 6, 10 and 12.

Now, repeat the process whatever we have done on this table that means, again find the difference between the minimum and the second minimum, and choose whichever is the highest one. You have to repeat the process until you are allocating everywhere. So, for this case, here the difference is 17 minus 14. So, for this row, it is 3, for this it is 9, then for the columns 17 minus 32, so this is 15; here it is 9, and for this case it is 4. So, therefore, your maximum is coming on this column.

So, maximum you can maximum of the differences, differences are 3, 9, 15, 9 and 4. So, maximum is coming over here that is 15. So, here I can allocate how I will allocate I will use again the same process that is which has the minimum cost. So, minimum cost is occurring on this cell on the first cell. Now, what is availability and demand that is nine and 6, so minimum of them is 6, so you can allocate 6 over here, so that it becomes 0. So, it becomes 0 and this 9 will be reduced to 6. Since you have allocated fully on this column. So, in the next step I can remove this particular column now. Please note this one although sorry this will be 3 not 6, 9 minus 6, so it will be 3. Although, three units has to be delivered here, but I cannot do on these cells because the minimum was on this cell only on this column only. So, if there was any residue that has to allocate on the cells on that column only.

So, in the next step, what I will do, I will remove the first column and I will write down the next two columns only, this column and this column. So, you are having 18, 14, 27, 18, and your costs are 3 and 19, here it is 0, 10 and 12. Again find the difference, so it is 4, here it will be 8, 27, minus 8, 9, for this case also, it is 9; and for this case it is 4, 18 minus 14 that is 4. So, here you see the maximum is 9 which occurs at this and this place. So, there is a tie, you can choose any one. Let me choose corresponding to this 9, corresponding to these how much we can do it.

So, out of these two, so if I choose this row, I would have chosen this column also. So, just by arbitrarily I am choosing this row. In this row, out of these two, 18 is the lowest. So, on this cell I have to allocate. Now, what is the minimum of 19 and 12 minimum of 19 and 12 is 12, so you are allocating 12, so that this becomes 0; your 19 now change to 19 minus 12 that is 7. So, on this row, still you can allocate on this table because seven you can allocate again here it is 10. So, therefore, 7 I am allocating to this one. So, once I am allocating these to 7, this becomes 10 and this is becoming 3. So, now, you are having this already you have completed this column total allocation you have made for this row, the allocation you have made. So, you can remove this row and this column, so that there will be only one.

What is the remaining here it is 3, here also it is 3. So, allocate three on this 18. So, that now your allocation is over. So, finally, if I have to say my allocation will be elements are 21, 16, 25, 13, 17, 18, 14; and 23, 32, 27, 18, and 41; here it is 11, 13, 19; and 6, 10, 12, and 15. So, from the original allocation that is I started with this one 11 then next step I allocated 6, like that way I have made the allocations which I am writing now. So, 6, after that you are having 3. On this row one more that is 4, and then you are having 7 and you are having 12, you are having 7 and here it is 12.

So, this is your final allocation that means, the solution of your problem will be x 1 4 equals 1 1, x 21 equals 6, x 2 2 equals 3, x 2 4 this is equals 4, x 2 4 equals 4, then x 3 2 equals 7, and the last one x 3 3 this is equals 12. This is the allocation. And total cost if you calculate as you know the cost in the occupied cells that is 11 into 13 plus 6 into 17 plus 3 into 18 plus 4 into 23 plus 7 into 27 plus 12 into 18, the value will be equals to, if you just check it will be 796. And it is reduced one the it is reduced compared to the earlier case. So, usually (Refer Time: 15:36) gives always better after total cost although the initial solution you are obtaining. So, like this way using either the north-west corner rule or row minimum rule or column minimum rule or row and column minimum rule or this Vogel approximation method, you can find out the initial basic feasible solution of the problem.

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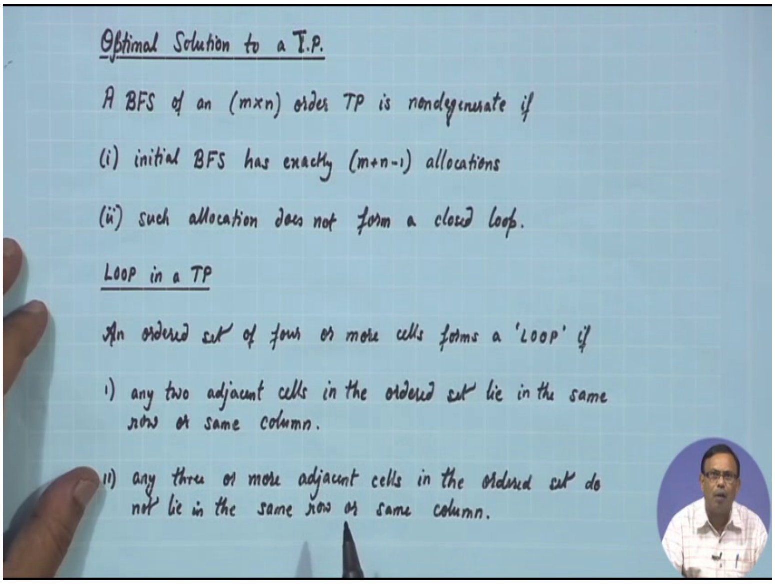

And once you are obtaining the initial basic feasible solution, then I have to check whether the optimal solution that is optimal solution or not for the corresponding TP - transportation problem. A basic feasible solution of an m cross n ordered transportation problem is non-degenerate. If initial basic feasible solution has exactly m plus n minus one allocation; such allocation does not form a closed loop. So, the solution will be non-degenerate, that means, optimum if initial allocation or initial basic feasible solution has exactly m plus n minus 1 allocation and the locations which we have made that does not form a closed loop.

So, what do we mean by a loop here? An ordered set of four or more cells form a loop. If any two adjacent cells in the order set, lie in the same row or same column. So, I will take an ordered set or cells of minimum four or more cells where if any two adjacent cells in the ordered set lie in the same row or same column that may form a loop. And any three or more adjacent cells in the ordered set do not lie in the same row or column. Please note this one. Any three or more adjacent cells in the ordered set do not lie in the same column.

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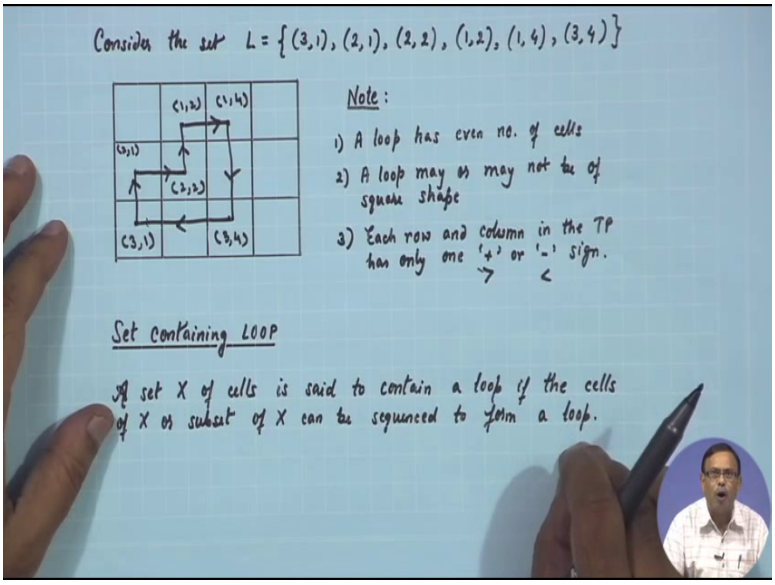

So, if you consider this one, if I consider sets like this 3, 1, 2, 1, 2, 2, 1, 2, 1, 4 and 3, 4. So, this is your you are starting say from here, this is your 1, 2; from this point you are having 1, 4. So, from here you are going up to this; from 1, 4, you can come to this which is 3, 4, you are coming to 3, 4; from 3, 4 you have 3, 1 also. So, I can go to 3, 1 then you may have this which may be 2, 1. So, I am following the shadow in this way the shadow after that I am going to 2, 1, 2, 2 is there. So, I am going to 2, 2, this is the point 2 comma 2; and again from 2 comma 2, I can go to 1, 2.

So, if I have these points 1, 1, 1, 2, 1, 4, 3, 4, 3, 1, 2, 1, 2, 2 and then again it is going at 3, 2. So, you see in each row you have only two adjacent cells or in each column you have adjacent cells. So, if I have started from here and whenever I am coming back this forms a loop. So, this we call at a loop. So, whenever you are allocated you have to check that if there is a loop or not. If there is a loop then your solution will be degenerate; that means, there will be no optimal solution.

So, therefore, you should remember note that a loop has even number of cells, a loop may or may not be of square shape, because if you see the shape is not square. So, it is not necessary that the loop may or may not be square; and each row and column in the transportation problem will have only one basically, it is this sign and one this sign means in each row or column there will be only one arrow sign which I am telling plus minus. At present we are considering it as a arrow sign there will be either, this greater than sign or minus sign in each row or column which has been formed. So, then we say that this forms a loop.

So, I hope what is a loop it is clear. So, once you have allocated and you have obtained the initial basic feasible solution after allocation, you have to check whether there exists any loop or not. If there does not exist any loop we say that the solution is optimal, but if there is any loop in that case I have to break the loop and I have to reallocate, we will check how we can do that one. Next one is the set containing loop a set x of cells is said to contain a loop if the cells of x or subset of x can be squared can be sequenced to form a loop. So, please note this one here also I have taken a set l and the way I have drawn it and I saw it that it is forming a loop therefore, the set l we call it as a it is a forming a loop.

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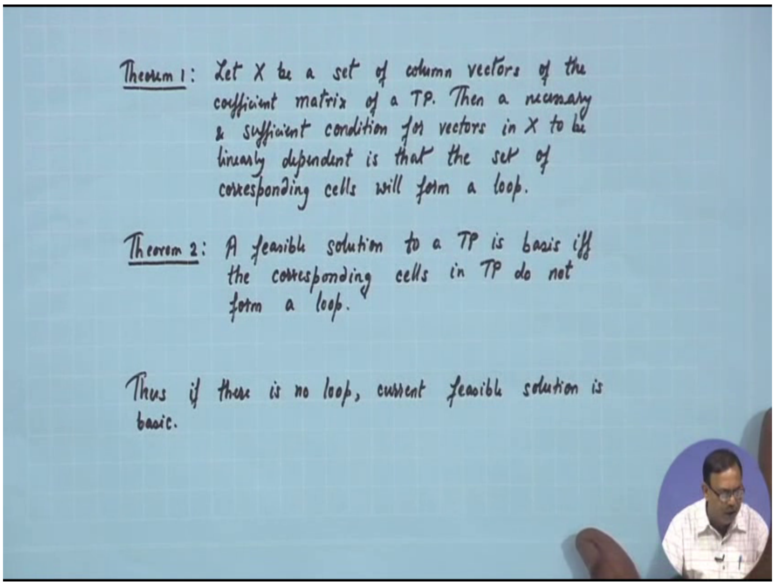

Two important theorems are there. One theorem states that let x be a set of column vectors of the coefficient matrix, x be a set of column vectors of the coefficient matrix of a transportation problem, then a necessary and sufficient condition for vector in x to be linearly dependent is that the set of corresponding cells will form a loop. Please note that it is a necessary and sufficient condition is that for the set of vectors to be linearly independent then is that the set of corresponding cells will form a loop. And we know whenever the vectors are linearly dependent; they cannot form an optimal solution, because linearly independent set only will form the optimal solution. So, therefore, if there is a loop it ensures that your solution vector is linearly dependent some of them.

Theorem two says that a feasible solution to a transportation problem is basis if and only if the corresponding cells in the transportation problem do not form a loop. So, it ensures that if there is no loop, then the solution whatever you have obtained by Vogel approximation method or other methods that will be optimal. Thus if there is no loop then the current feasible solution will be basic solution and it will be the optimal one also.

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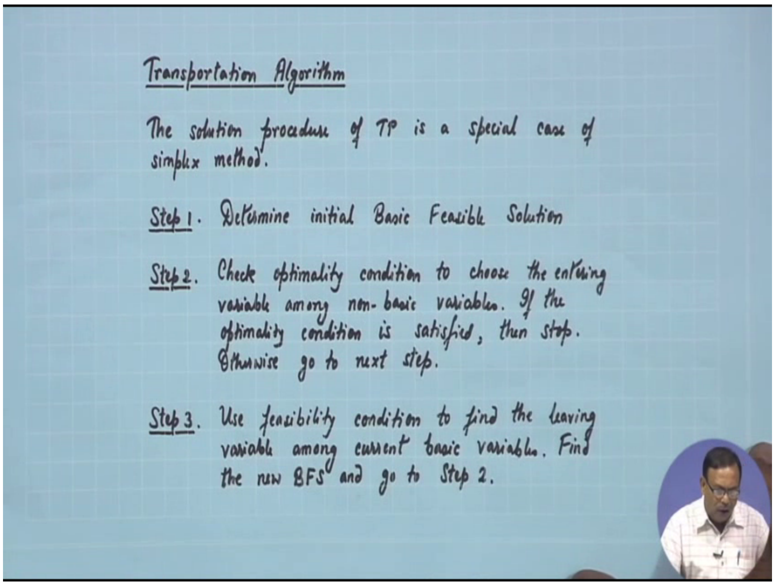

Now, quickly let me talk about the transportation algorithm. The solution procedure of a transportation problem is a special case of simplex method as we have mentioned earlier. What you have to do you have to determine the initial basic feasible solution which we have done by the not by simplex normal way, but using the VAM or other methods which we have discussed earlier.

In step two we are checking the optimality condition, to choose the entering variable among non basic variables. If the optimality condition is satisfied, we will stop; otherwise, we will go to the next step. And use the feasibility condition to find the leaving variable among current basic variables. Then find the new basic feasible solution and go to step two. This we will follow if we are trying to solve it by the simplex method. But sometimes it may happen that sometimes we do not want to use the simplex algorithm, in that case I have to use the other technique.

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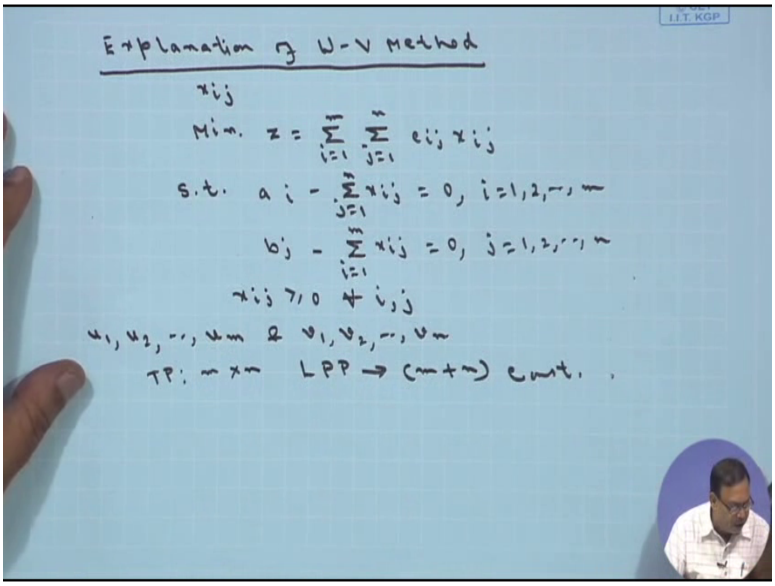

The technique we call it as U-V method the explanation we call it as U-V method or sometimes we call it as method of multiplier also. So, basically what you want I want to determine x j which minimizes z equal summation i equals 1 to m, summation j equals one to n c ij x ij subject to a i minus summation j equals 1 to n x ij this is equals 0, where i will vary from 1 to m and b j minus summation i equals 1 to m x ij, this is equals 0 j equals 1, 2 n. This already we have discussed the requirements. So, your x ij should be greater than equals 0 for all i comma j.

Now, here you have basically you have transportation problem of order m cross n, order m cross n, so that means, here you have m variables m equations and n equations. So, total m plus n equations would be there in the LPP. Suppose, I want to convert it into the corresponding dual, so I am assuming u 1, u 2, and v 1, v 2, v n be the dual variables which are associated with the source and requirement constants. So, since I have told that the transportation problem TP is of order m cross n is an LPP with if it is of order m cross n, then it will be an LPP with m plus n constraints, it will be with m plus n constraints respectively. So, hence it is dual will have the opposite one, m n constraints if I take the dual of these, it will have m n constraints m into n constraints and you have m plus n dual variables, so that you are reducing the size.

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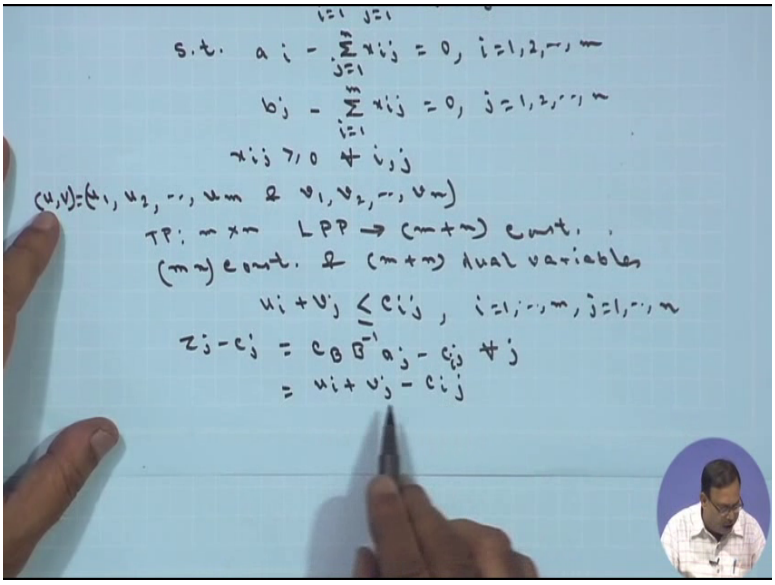

So, whenever I will take the dual in that case it will have m n constraints and m plus in dual variables. Now, the new constraints whatever I will take this is dual variables which I have denoted by u 1 to and v 1 to v n. The new constraint in this case will become this one that u i plus v j less than equals c ij. So, new constraint basically u i plus v j equals c ij where i is from 1 to m j is from 1 to n; and u i and vj are unrestricted in sign we do not know what is the sign of this from the duality theory itself we can tell.

So, your z j is what z j minus c j this I can write down C B B inverse a j minus c j for all j. For any transportation problem your u, v is this one whatever we have writ10 uv is this one. So, this portion C B B inverse B can be replaced by this one, so therefore, this z j minus c j this equals you can write down u i plus v j minus c ij sorry this will be c ij c j sorry c j for all j. So, this I can write down z j, z j means z ij minus c ij this I can write down again in the new form.

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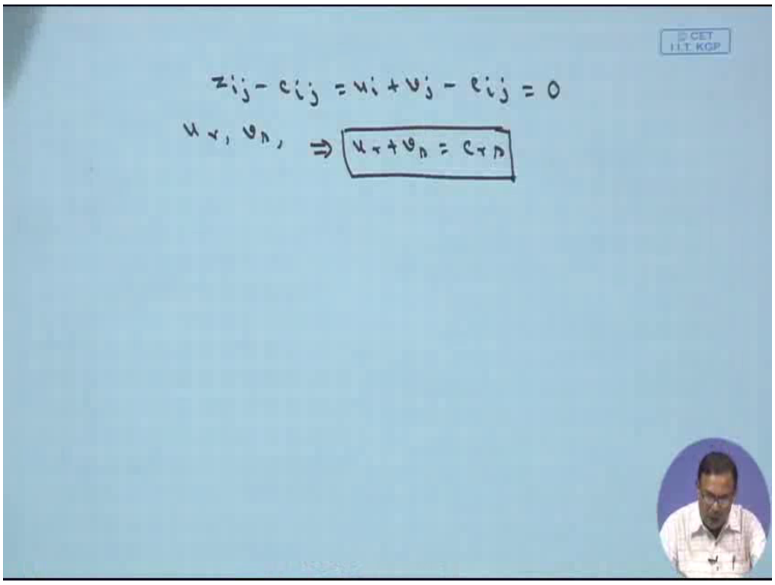

Z ij minus c ij this is equals u i plus v j minus c ij. And we know it that for basics if a for a particular case if u r and v s has to be basic then this value should be 0 or which implies u r plus v s, this is equal c rs. So, basically we will use this concept c r s equals u r plus v s where u r and v s are nothing but the dual variables corresponding to this two constraints these are the dual variables. So, this is important for us now.

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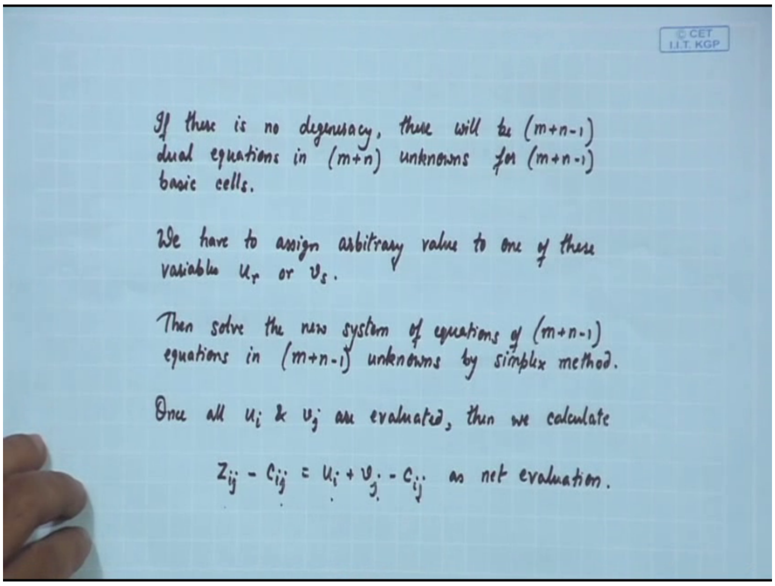

If there is no degeneracy, there will be m plus n minus 1 dual expression dual equations in m plus n unknowns for m plus n basic cells. We have to assign arbitrary values to these values u r and v s. So, basically what happens whenever I am writing this equation then I have to find out the values of u r and v s; and from there I have to proceed further. So, whenever we are writing this, then we have to solve the new system of m plus n minus 1 equations in m plus n minus 1 unknown in simplex method. And once all u i and v j are evaluated then we can calculate z ij minus c ij equals u i plus v j minus c ij as the net evaluation and which is an important equation for us for finding the optimality we will use basically this particular constraint which we will discuss in the next class.