Constrained and Unconstrained Optimization
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Lecture - 25
Mixed Integer Programming Problem

So, in this class we will start the mixed integer programming problem. So, basically you are having the linear programming problem if you see. Your linear programming problem the decision variables can take any value any real value, which we are saying x j greater than equals 0.

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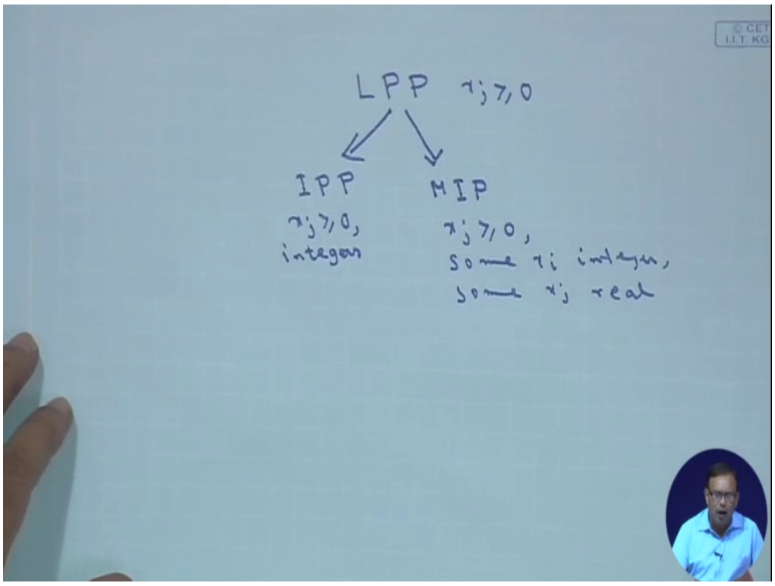

Now from this linear programming problem you are formulating integer programming problem. In integer programming problem what happens the variables x j are greater than equals 0 and they are integers. Now another type of problem which may come that is which we call as MIP mixed integer programming problem say mixed integer programming problem. Here also your x j will be greater than equals 0, but some x i be integer and some x j are real values; that means, here x j can take either integer value or real value this type of problems we call it as the mixed integer programming problem.

So, basically if you see initially we started with the linear programming problem then we have done the dual simplex methods, now we are saying that from I may have a special type of a linear programming problem which we call as integer programming problem where all the decision variables are positive that is greater than equals 0 and integers only. We have seen how to find out the solution and the mixed integer programming problem where x j decision variables will be greater than equals 0.

But some x some x j may be integer, some x j may be real this type of problem we call it as mixed integer programming problem.

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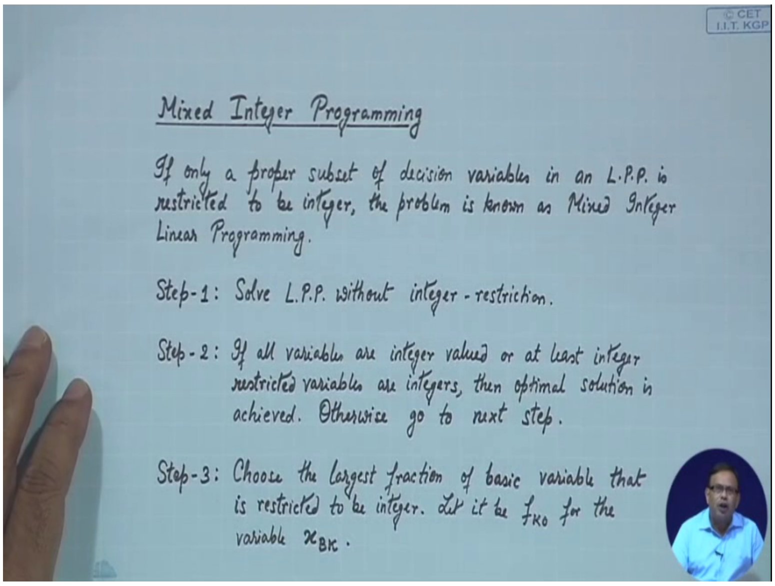

So, if only a proper subset of decision variable in an LPP is restricted to be integer the problem is known as mixed integer linear programming problem. So, if some of the subset of the decision variables restricted to integers, other variables may take real values what are the steps? Steps are solve the LPP without integer restriction first; that means, as if you do not have any restriction on it by that way you try to solve this particular problem as we have done earlier.

Step 2 if all variables are integer valued or at least integer restricted variables are integers then optimal solution is achieved otherwise go to next step. What do you mean by this step 2? If all the variables are integer valued then; obviously, your optimal solution is your feasible also because all are integer valued because we have told some variable will be integer, some may take real values, some may take integer value and if at least integer restricted variables that is the variables whose values should be integer in the optimal solution their value is becoming integer only.

Then the optimal solution you obtained in step one that is the optimal solution of the original problem and then you terminate otherwise you have to go to the next step. In step 3 this is just like your Gomorian cutting plane method, what we did the first one that is IPP problem. So, here you chose the largest fraction of basic variable, choose the largest fraction of basic variable that is restricted to be integer let it be f k 0 for the variable x BK.

So, whenever you are coming in step 3 the meaning is that your sum of the variables are taking non integer values which should take the integer values. So, out of those variables which are taking the non integer values you choose that variable basic variable which has largest fractional part. So, you are assuming that if f k 0 be the largest fractional part corresponding basic variable is x BK.

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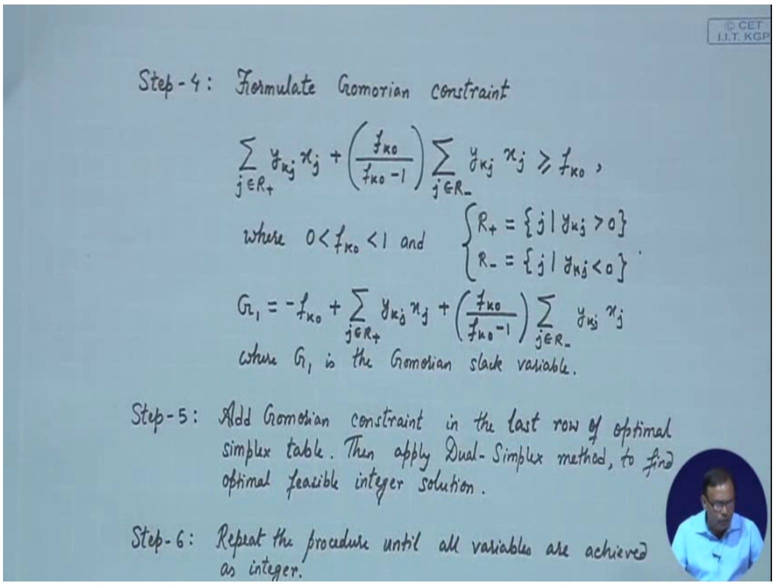

So, once I am obtaining this one then I am formulating the Gomorian constraint like this, that is summation j belongs to R plus y k j x j this is summation over j belongs to R plus I am writing your R plus equals j such that y k greater than 0. Because whenever you have y k j values that is in a table whenever we are considering in a table y values you are taking y values may be positive y values may be negative also.

So, R plus you are telling whenever you are taking R plus, in that case you will take those j values where y k j will be greater than 0 and R minus will be those values of j where y k j will be less than 0. So, I am formulating it like this summation over j belongs to R plus y k j x j, plus f k 0 again remember this f k j is nothing, but this largest fractional value corresponding to the variable x BK.

So, f k 0 by f k 0 minus 1, summation over j belongs to R minus y k j x j this must be greater than equals f k 0 where f k 0 lies between 0 and 1. So, if you remember for integer problem integer linear programming problem these Gomorian constant value was little different and for fractional part it is different. In the, for the case of integer linear programming problem I have shown how we derive this equation here since we have done it.

So, I am leaving it to you that how we can come to this particular inequality. So, once I am obtaining this inequality from this inequality, I can write down the corresponding equality by adding the Gomorian slack variable G 1 which I am specifying. So, your G 1 is this one that is minus f k 0 plus summation j belongs to R plus, y k j x j plus f k 0 by f k 0 minus 1 summation over j belongs to R minus, y k j x j where G 1 is the Gomorian slack variable.

So, by on this inequality constraint we are this at first we are making the non negative sorry; this is already greater than equals type of inequality, this first we are making the less than equals type inequality by multiplying minus sign on both side of this, then you are adding the Gomorian slack variable G 1 and you are getting this particular equation. So, now, as usual you have to add this Gomorian constraint in the last row of the optimal simplex table, that is last optimal simplex table whatever you have derived earlier for that one for that last simplex table, you add the Gomorian slack constraint and then you apply dual simplex method. Already I have explained I have to use the dual simplex method because the basic variable which will be entered the value of that basic variable will be negative.

Therefore to find the departing vector and entering vector I have to use the dual simplex method, which we have discussed earlier to find the optimal feasible integer solution and once I have obtained it, then step 6 repeat the procedure until all variables or I should say all required variables are integers. So, this I have to go on adding the constraints until I have the fractions, until I am obtaining the values of the decision variables as integers which are required.

So, now let us take one example and let us see how we solve this particular problem.

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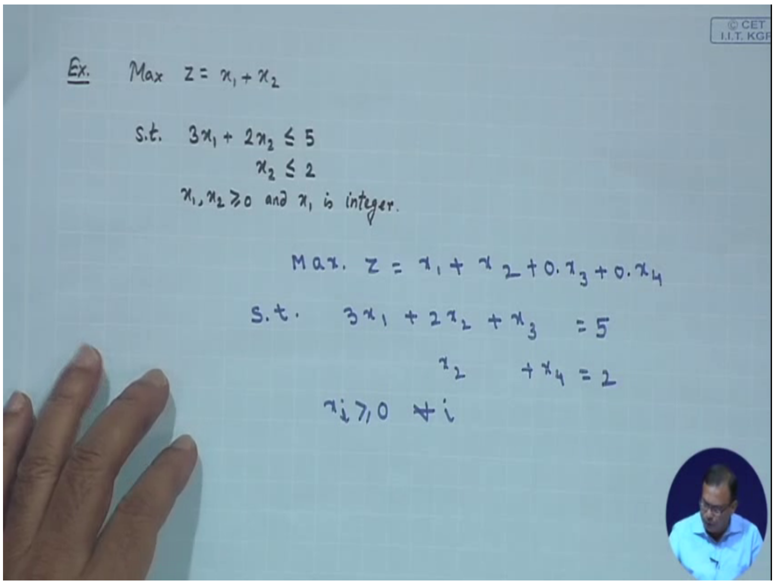

So again I am taking a simple small problem, maximized z equals x 1 plus x 2 subject to 3 x 1 plus 2 x 2 less than equals 5, x 2 less than equals 2 x 1 x 2 greater than equals 0, x 1 is an integer. So, basically here you see the difference, the difference is x 1 we are specifically saying it is an integer and x 2 we are not mentioning anything we have told x 2 will be greater than equals 0.

So, x 2 can take real value as well as integer value. So, I have one variable whose optimum value should be integer and I have another variable whose optimum value maybe integer maybe real. So, for this problem as we have told earlier in step one what I have to do at first? I have to solve the problem forgetting about the integer constraint that is first I will write this problem into the standard form. The standard form will be maximized z equals x 1 plus x 2, subject to 3 x 1 plus 2 x 2 plus x 3 this is equals 5 next one is x 2 plus x 4, this is equals 2, x i greater than equals 0 for all i, and your x 1 is integer.

So, here your x 3 and x 4 these are slack variables, which we have used to make these less than equals type problems into equality problems. So, I have used this thing. So, at first I will formulate from these the initial simplex table. So, let us formulate the initial simplex table from here; obviously, the coefficients in the objective function for the slack variables x 3 and x 4 will be 0. So, and in the basis 2 variables will enter that is x 3 and x 4. These 2 will enter because if I make x 1 x 2 as 0, your x 3 will be 5, x 4 will be 2.

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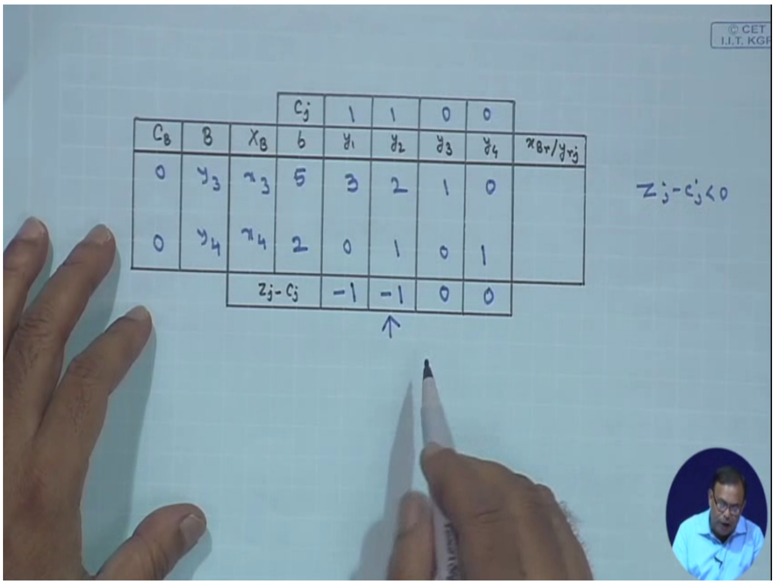

So, once I am doing it in the table first table your x 3 and x 4 is appearing, here I have y 3 and y 4 the values of the coefficients are 1 1 0 and 0, x 3 x 4 corresponding objective function coefficients are 0 your b values for this particular problem is 5 and 2. So, once I am writing b values b value is 5 and 2, and then you are having 3 2 1 and 0, 0 1 0 and x 4 is coefficient is 1. So, you are forming the initial simplex table from the initial simplex table now calculate the z j minus c j value, this will be 0 into 0 minus 1, this will be 0 into 0, 0 into 2 plus 0 into 1 minus 1.

So, it will be minus 1 this will be 0 since 0 is there this will be 0 now z j minus c j is greater than equals 0 sorry z j minus c j for this case is less than 0. So, you can choose both are having same value, anyone you can choose I am choosing say this one. So, x 2 will be the entering vector over here. So, since x 2 will be the entering vector therefore, you have to calculate the ratio that is 5 by 2 and this is 2 by 1 for this case, b value by y value. So, this gives the minimum value therefore, your departing variable will be in this case is x 4.

So, as usual whatever you have done earlier your x 2 will be the entering variable in the basis and your x 1 will be the sorry x 4 will be the departing vector, and the pivot element is one; that means, this is already one I have to make this component as 0 in the next table. So, in the next table x 4 will be replaced by x 2. So, once I am doing it your x 3 will be replaced by x 2.

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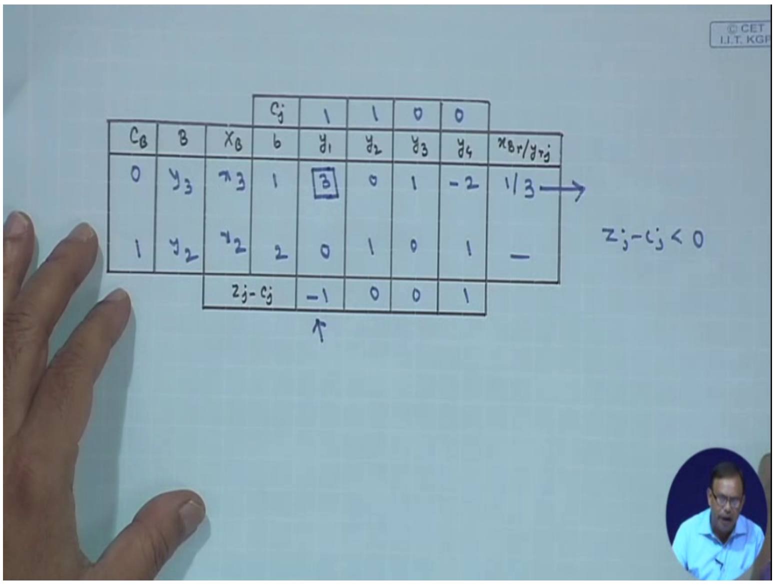

So, here now x 3 x 2 will come. So, you have y 3 and y 4 this is 1 1 0 and 0.

So, y 3 is 0, but y 2 coefficient is this. So, if I calculate it and write down after row manipulation I will obtain 1 3 0 1 minus 2 and for this case x 2 this will be 2 0 1 0 and 1. So, basically this is 1 and this is 0, y 2 was the entering vector. So, z j minus c j if you calculate minus 1 0 0 and 1 again for this particular case your z j minus c j is less than 0, I have only one negative value.

So, therefore, this one is entering is this one, calculate the ratio b by y value here it is one third this will be infinity. So, we cannot consider therefore, your departing variable in this case will be x 3. So, your x 3 will go out and x 1 will enter into the basis and your pivot element is this pivot element is 3 this one. So, I hope it is clear, this is the pivot element and x 3 will go out and x 1 will enter in the next iterative table.

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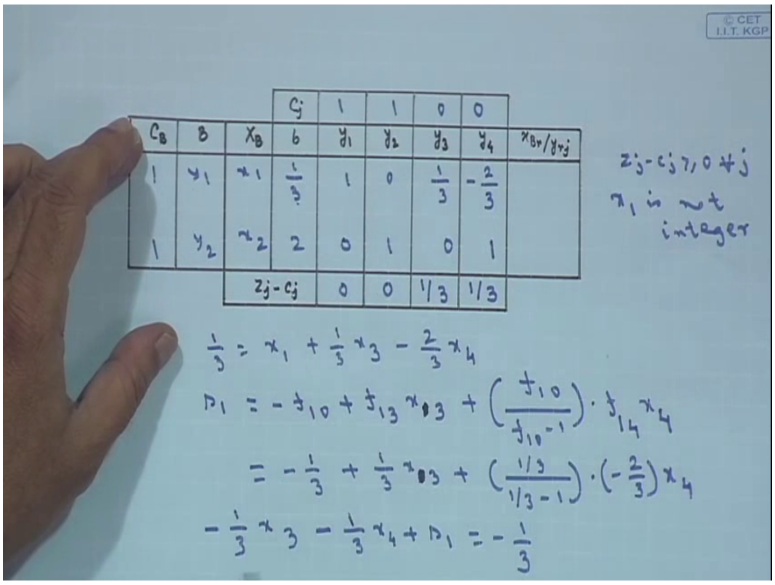

So, let us formulate the next iterative table. In the next iterative table x 3 has been replaced by x 1 and this is x 2 is already there. So, this will be y 1 y 2 the coefficients of the objective functions c j s are 1 1 0 and 0, x 1 x 2 both are one. So, if I am writing it is one third 1 0, 1 third minus 2 by 3, and for this case it is 2 0 1 0 and 1. If you calculate the z j minus c j values you will obtain 1 minus 1 0, 1 minus 1 0, in this case one third minus 0, 1 third this is equals minus 2 third plus 1. So, it is also one third.

So, for this particular case if you find your z j minus c j is greater than equals 0 for this problem, your z j minus c j greater than equals 0 for all j, but what happens x 1 if you see value of x 1 is one third, which is not integer. Since x 1 is not integer therefore, I have to formulate now the Gomorian constraint from this and this one I have to formulate the Gomorian constraint if you write down. So, now, basically I have to formulate this G 1 equals this thing, your corresponding to this, this equation I can write down as one third equals x 1 plus 0 into x 2, plus one third into x 3, minus 2 third into x 4 the corresponding to the row where x 1 is having fractional value.

So, x 4 has the most negative coefficient in the mixed fractional part. So, using this Gomorian constraint now G 1, this now I have to calculate. So, your s 1 will be equals to in place of G 1 I am writing s 1, minus f 1 0 plus f 1 3 will come only f 1 3 x 3 because this is the positive one if you remember I have told it has 2 parts, y k j x j where j belongs to R plus, R plus is y k j greater than 0. So, y k j is greater than 0 on this case here only.

Similarly, your y k j is less than 0 for this and for that one it will be j belongs to R minus y k j less than 0 where the coefficient will be f k 0 by f k 0 minus 1. So, this is f one 3 into x 1 3 plus f 1 0 by f 1 0 minus 1 into f 1 4 x 4. So, once I am getting these values now I can calculate the values f 1 0 is one third this one. So, f 1 0 is minus 1 third, then f one 3 is one third, x 3 plus one third by one third minus 1 into f 1 4 f 1 4 is minus 2 third. So, you are writing minus 2 third into x 4.

So, I hope now it is clear how we are making using this Gomorian constraint over this problem. For this case the equation at first I wrote it like this and it was minus f k 0 plus this things f1 3, x 1 3, f 1 0 by f 1 0 minus 1 into f 1 4 into x 4. So, now, I am putting the values of f 1 0 which is one third f 1 3 is one third and f 1 4 is minus 1 third. So, that you will obtain s 1 equals this by simplification I can write it as minus 1 third x 3 sorry this will be x 3 this is not x 1 3 this will be x 3 and this is x 4.

So, minus 1 third x 3, minus 1 third x 4, plus s 1 equals minus 1 third. So, your Gomorian using the Gomorian slack variable s 1 you are adding this constraint. So, now, on these particular matrix, on this particular table you have to add one more row here where s 1 will be entering into the basis. So, once s 1 is entering into the basis from here; that means, these rows will remain as it is only these row will enter and one more variable s 1 will come.

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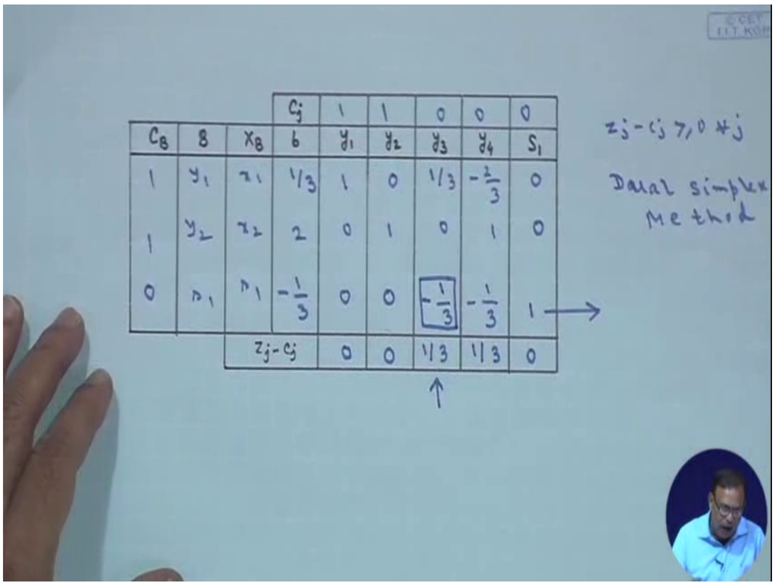

So, in the basis you will have x 1 x 2 and s 1.

So, s 1 means corresponding to this row which we are adding as a new Gomorian constraint which we have done earlier. So, this will be y 1 y 2 and s 1, c j values are 1 1 0 0 0, s 1 corresponding to slack variable s 1 coefficient in the objective function is 0. So, x 1 x 2 is 1 1 this is 0. So, I am writing there will be no change on the previous table for first 2 rows, that is one third 1 0, 1 third minus 2 third minus 2 third s 1 is 0, second one is 2 0 1, 0 1 0 and third one will be b value is minus 1 third.

Then it is x 1 x 2 is not present coefficients will be zeros for x 3 it is minus 1 third, for x 4 it is minus 1 third and corresponding to s 1 coefficient is one. So, you are writing the coefficient as one over here. Now once you are calculating this calculate the z j minus c j value first one will be 0 1 into 1 is 0, second one will also be 0 third one is one, third fourth one will be fourth one is will also become the this fourth one will be minus 1 third into 1 1 minus this. So, it will remain one third 1 minus this and then this value will be 0. If you see here your z j minus c j this is greater than equals 0 for all j, but value of the one of the basic variable is negative here.

So, already we have done these type of problems whenever I had the negative value in the basis therefore, I will use the dual simplex method which we have discussed earlier I hope you remember that thing, that in the dual simplex method first I have to find out what is the departing vector the variable which is going out that will be the departing vector and from there I have to calculate what is the entering vector. So, in this case you first now use your dual simplex method. So, once you are taking the dual simplex method this will be the outgoing vector and from here you will find that this is the entering vector.

I am not discussing again the dual simplex method here since already we have done this type of problems earlier. So, this is your outgoing vector departing vector is s 1 entering vector is your y 3 or x 3. So, your pivot element is this since your pivot element is minus 1 third; that means, now I have to make this element as one and all other elements on this particular column as 0.

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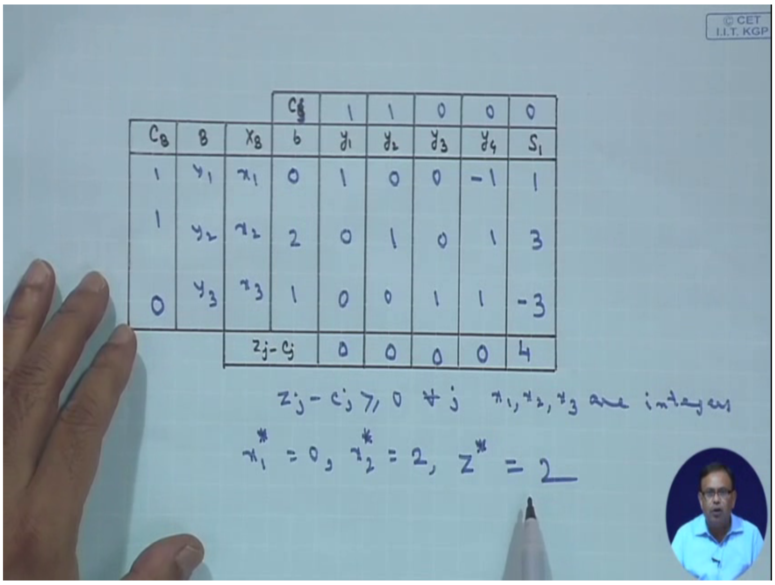

So, if I take the next one in that case your s 1 if you see this table on this table your s 1 will be replaced by x three

So, in the basis for the next table you will have x 1 x 2 and x 3. So, let us write down now this 3; x 1 x 2 and x 3 this will be y 1 y 2 and y 3, your c B values will remain as it is 1 1 0 0 0. So, this is c j not this is c j and c B values in this case will be 1 1 corresponding to x 3 it is 0. So, basically I have to make these as 1 and these elements as 0. So, this will become now after performing the row operations this will become 1 0 0 1 1 minus 3, I have to make the other elements such that on y 3 the components are 0.

So, you will obtain in the first row as 0 1 0 0 minus 1 1, and for this case you will obtain 2 0 1 0 1 3. Your z j minus c j value will be 1 minus 1 0 in that case for the second 1 1 minus 1 0 for the third 1 0 0 0 minus 0 0, for the fourth case minus 1 plus 1. So, 0 plus 0 minus 0 it is 0 for this case it is 1 plus 3 4 this is 0. So, 4 minus 0 this is 4. So, if you see here your z j minus c j is greater than equals 0 for all j, and the basic variables which are x 1 x 2 and x 3 value of these basic variables are integers.

Therefore you obtain the you have obtained the optimal solution which is also satisfying your feasibility condition. So, I can now write down from the basis my optimal solution of the original problem is x 1 star equals 0, x 2 star this is equals to 2, and my z star this will be equals to 0 into 1 plus 0 into 2 this is equals 2. So, this is the solution of the original problem. So, I hope it is clear to you now whenever I have mixed integer programming problem where some of the variables, can take only integer values some can take either real value or integer value.

So, in that case my procedure should be first solve the problem without ignoring the integer constraint, after that check the solution and find out whether there is any non integer solution is there in the basis or not. If there is any non integer solution corresponding to that non integer variable, you formulate the Gomorian constraint the you add one more constraint which we are saying as Gomorian constraint and in that Gomorian constraint and corresponding Gomorian slack variable will be added in the basis.

So, one more row will be added in the next iterative table, and then find the entering vector and departing vector using the dual simplex method and repeat the process until you are obtaining the optimal solution. So, hope that it is clear how if I have a mixed integer programming problem how to obtain the solution.