Constrained and Unconstrained Optimization

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Lecture - 24
IPP Branch & Bound Method

So, in the last class we started the integer linear programming problem where the decision variables are the integers and they take the positive values. We have done the Gomory's cutting plane method. If you remember, we have told in the last class that I P P problems can be solved by two ways; one is Gomorian cutting plane method, where whenever you have one variable decision. Variable in the basis whose value is non integer. You are adding one constant and you are adding one Gomorian slack variable to remove that particular decision variable from the basis.

So, that we have seen in the last class. Today, in this class we will discuss the branch and bound method for solving the I P P. It is similar to this earlier one there is a basic difference in branch and bound method in branch and bound method. At first, we try to find the solution and we check that which variables in the basis has non integer values from there we try to find out the upper bound and lower bound of the problem and then we divide the original problem into two sub problems by adding two more constants, using the lower bound and using the upper bound of the problem. So, that we are doing it. So, let us see what the branch and bound method uses in branch and bound method.

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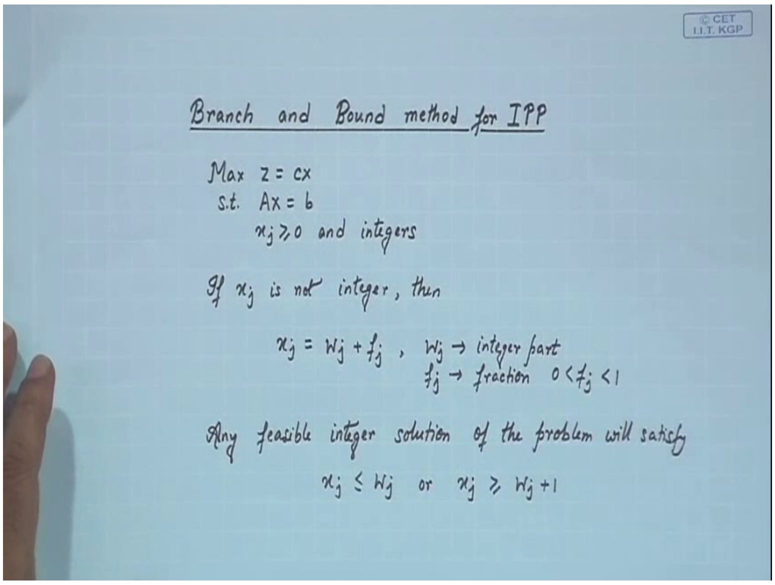

Suppose, you have a problem maximize z equal c x subject to A x equals b x j greater than equals 0 and this is integers. Now x, if x j is not an integer, in that case I can write down x j equals w j plus f j where w j is the integer part and f j is the fractional part where f j lies between 0 and 1; obviously. So, you have the problem original, problem maximized. z equals c x subject to a x equals b x j greater than equals 0. If x j is not an integer, in that case we are writing x j equals w j plus f j form, where w j is integer part of the variable and f j which lies between 0 and 1 is the fractional part of the problem.he problem. If you take any, it will must satisfy any one of these two conditions. I think it is fine, x j less than equals w j or x j greater than equals w j plus 1, because the bound it will be either exactly less than equals w j or it will be plus 1 since your f j is the fractional part whose value lies between 0 and 1. So, for any feasible integer solution of the problem, it must satisfy any one of these two constants x j less than equals w j or x j greater than equals w j.

So, basically what we are doing; since your x j is less than equals w j or x j greater than equals w j. So, we are dividing this original problem into two sub problems in one sub problem. We will include the new constant x j less than equals w j and in the another sub problem. We will include the other inequality condition x j greater than equals w j plus 1 or in other sense I can reformulate.

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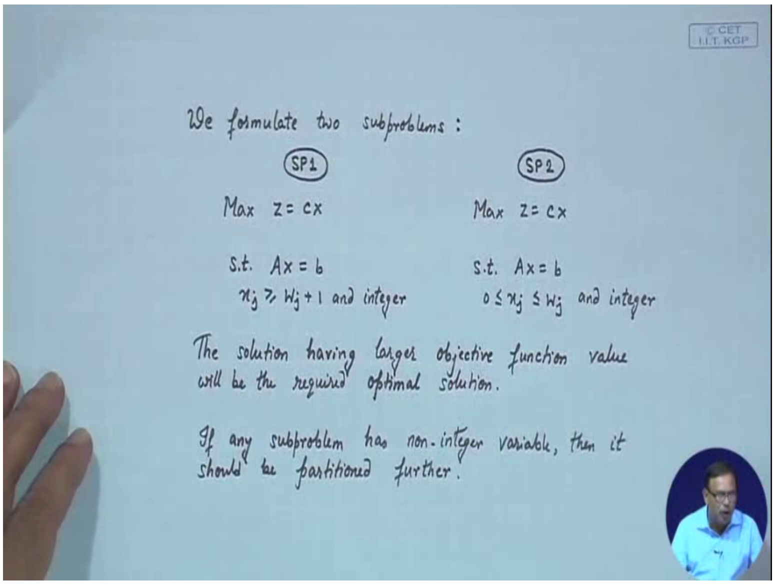

two sub problems like this one; I am telling SP 1, where we are telling maximized z equals c x subject to A x equals b x j greater than equals w j plus 1 and integer.

So, one condition we are writing here x j greater than equals w j plus 1 andobviously, all x j will be integer and the sub problem 2 SP 2, which are denoting as SP 2. This we are writing as maximized z equals c x subject to A x equals b along with 0 less than equals x j less than equals w j and x j's are integer. So, whenever I am getting the lower bound and the upper bound of an non integer variable in the basis from there I am making two conditions x j less than equals w j and x j greater than equals w j plus 1 and from there we are formulating two sub problems like this SP 1 and SP 2.

Now, the solution having larger objective function value will be the required optimal solution. So, there are some possibilities one case is both the problems SP 1 and SP 2 has optimal solution and they satisfy the feasibility condition that is you are getting the integer solution for both the problems SP 1 and SP 2. In that situation the solution having larger objective function value will be the required optimal solution. Now, if any sub problem is has non integer variable. If any of these two sub problem has non integer variable, then again it should be partitioned further.

That means you have the original problem SP. Once you have the original problem and for this particular problem, if the optimal solution contains non integer values for the basis variable divide it into two sub problems SP 1 and SP 2. One will contain the lower bound constant, the other one will contain the upper bound constant, then again if required you can divide these and like this way it will continue until you are obtaining, your required solution. So, basic idea is this thing.

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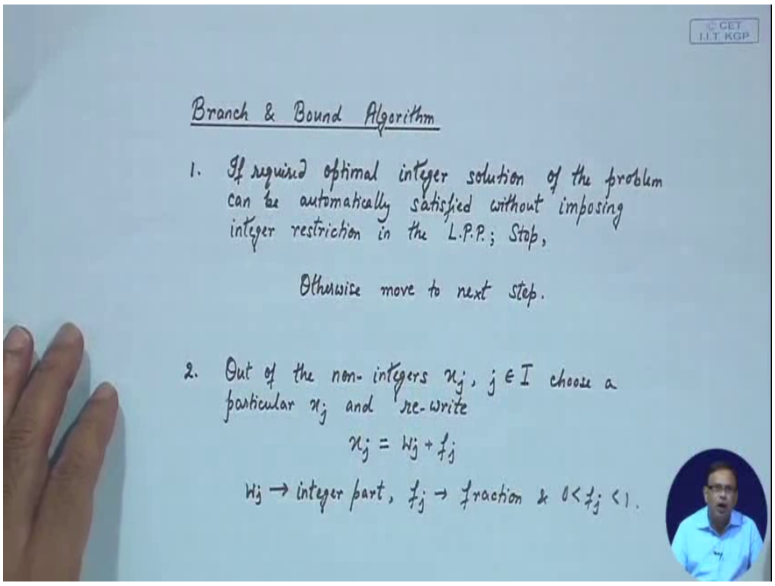

So, this in terms of the algorithm we can write it like this branch and bound algorithm; if required optimal integer solution of the problem can be automatically satisfied without imposing the integer restriction in the LPP in that case stop; that means, without imposing integer restriction. If I solve the problem and if I obtain that the solution is optimal and providing me the integer solution in that case, that solution will be the optimal 1 and I will stop otherwise I will go to the next stop, whatever we have told out of non integer x j s j belongs to I 1 to n or whatever it may be choose one particular x j and rewrite it, in the form of x j equals w j plus f j. Where w j is the integer part and f j is the fractional part of this x j and obviously, your f j lies between 0 and 1. So, in that case after that in step.

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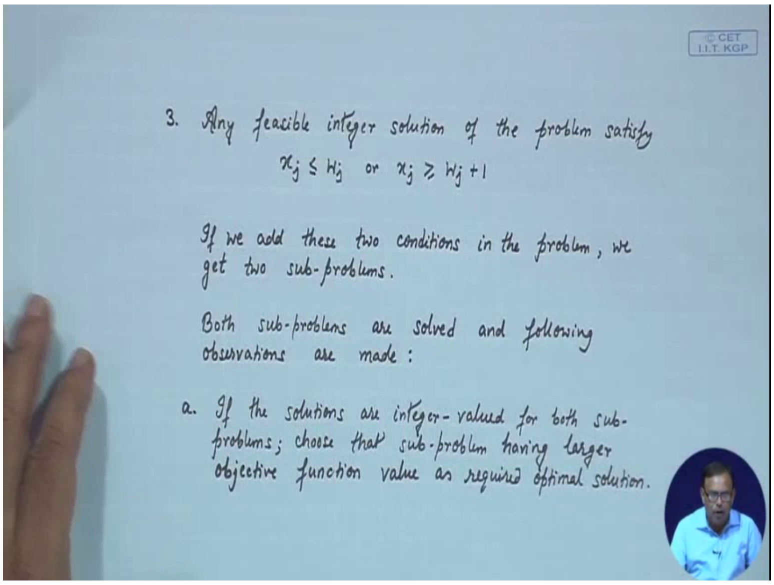

Three, any feasible integer solution of the problem which will satisfies will satisfy either x j less than equals w j or x j greater than equals w j plus 1. So, if we add these two conditions in the problem. We will obtain two sub problems, both sub problems. You have to solve and following observations can be made both sub problems whenever we are solving. The observations can be made if the solutions are integer valued for both sub problems. If the solutions are integer valued for both sub problem choose that sub problem having larger objective function value as required optimal solution.

So, please note this one, if the solutions are integer valued for both sub problems choose the sub problem having larger objective, function value as required optimal solution.

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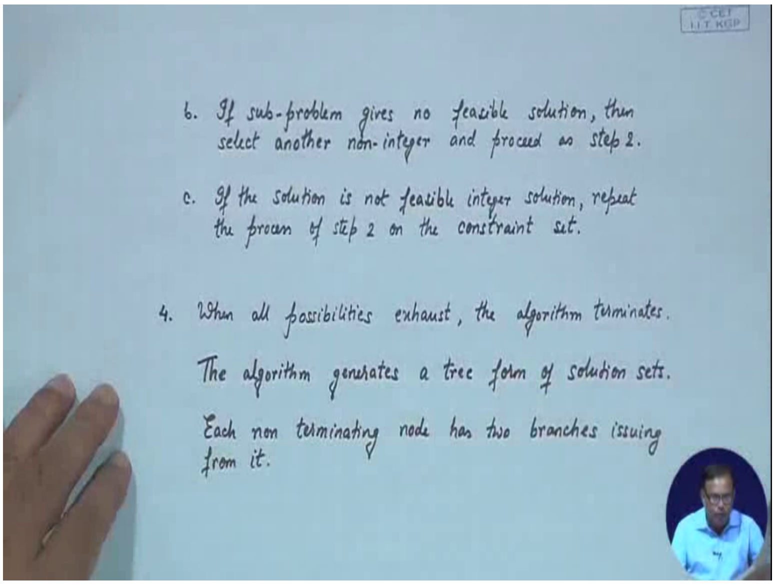

So, this is one case which may arise the next case will be if the sub problem gives no feasible solution then select another non integer decision variable in the basis another non integer variable and proceed as step 2, whatever we told means if the non integer variable which you have taken after that you are dividing it into 2 sub problems and if both the sub problems having no feasible solution then you have to choose another non integer variable and repeat the process 2 as we are saying as in steps 2.

If number c, if the solution is not feasible integer solution repeat the process of step 2 on the constants rate; that means, again follow the same process sub divide the problem when all possibilities exhausted the algorithm will terminate; that means, it has no solution. So, this algorithm generates a tree form of solution set. Please note this one as I have shown earlier, I will, it will have a root from the root, it will have some branches, each branch may have sub branches like this way it will continue.

So, this algorithm generates a tree form of the solution set each non terminating node will have two branches which will be issued from here; that means; from root there will be always two branches. Now, let us take one example and let us see how the branch and bound method exactly works for this particular problem let us take this problem;

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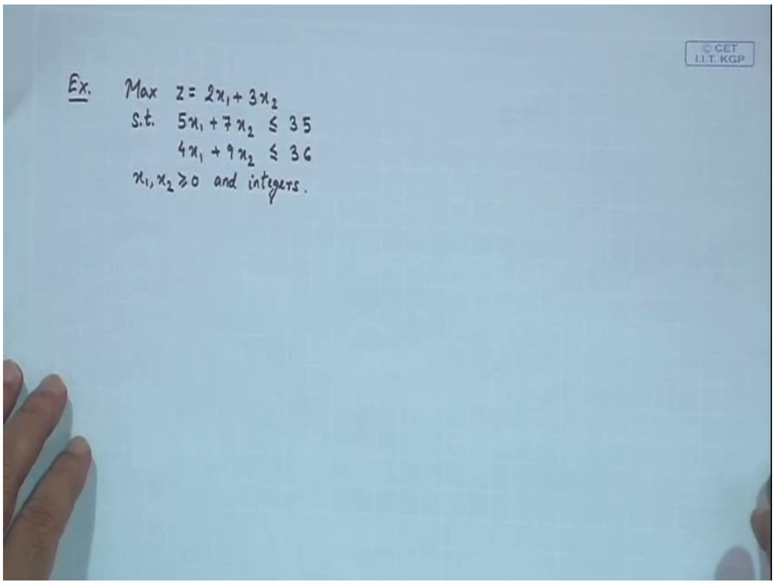

Maximize z equals 2 x 1 plus 3 x 2 subject to 5 x 1 plus 7 x 2 less than equals 30 5 4 x 1 plus 9 x 2 less than equals 36 and x 1 x 2 greater than equals 0 and both are integer.

Now, for this problem I can find the solution in two ways; one is, I can use the normal simplex method I can add the slack variables on it and if I wish then I can write down the initial simplex method, initial simplex table and from there I can go on iterating until I am obtaining the solution instead of doing the simplex algorithm. Since it is a problem of 2 variables only we will use the graphical method with respect to this one. So, you see these two problems, these two constants for the graphical method. What I have to do I have to draw the line for 5 x 1 plus 7 x 2 equals 35 and 4 x 1 plus 9 x 2 equals 36 and x 1 x 2 greater than equals 0 and they are integers.

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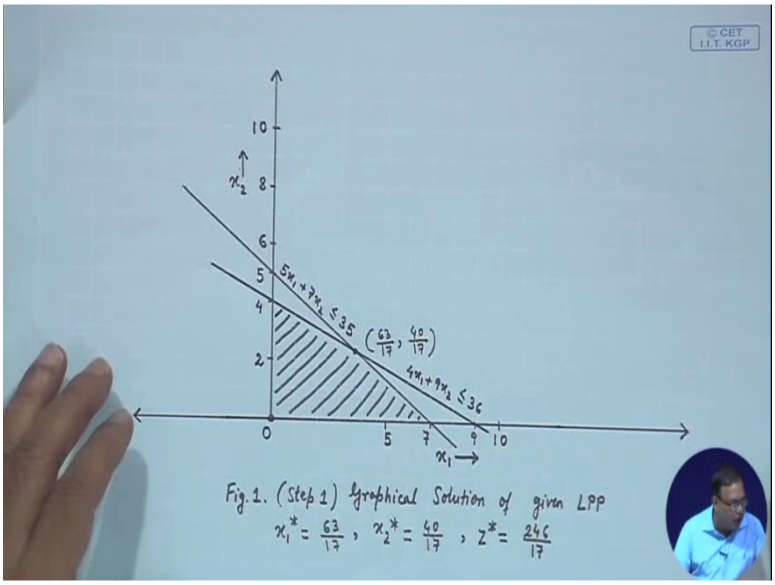

So, please see from here that, in this line represents 5 x 1 plus 7 x 2 less than equals 35 that is equals 35. It crosses the 0.7 on the x axis and 5 on the y axis whereas, 4 x 1 plus 9 x 2 equals 36. This will meet the point at 9 on the x axis and 4 on the y axis. So, you are getting the second line 4 x 1 plus 9 x 2 less than equals 36. Since they are both less than equal, direction will be for both of them will be this 1.

So, and the region will be this shaded region bounded by this two lines and the coordinate axis that is x axis and y axis. So, from the graphical solution it is quite obvious these are the extreme points and if you calculate, you will obtain the maximum value of these at this point, that is 16 by 63 by 17 and 40 by 17. Therefore, after graphical solution of the original problem, what we are obtaining x 1 equals 63 by 17 x 2 star is 40 by 17 and z star equals 246 by 17.

So, using this graphical method, what happens your solution is like this x 1 star equals 63 by 17 your x 2 start equals 40 by 17 and your z star or z max is becoming 246 by 17, although this is the optimal the solution which you obtained, but this solution is not feasible, since both of them x 1 and x 2 are non integer now. So, since both x 1 and x 2 are non integer. So, I have to choose anyone of these two.

So, we are choosing say choose x 1, we are choosing x 1, because x 1 has the largest fractional value largest fractional value. So, what is the upper bound of z initial upper bound of z is initial upper bound of z. This is equals to 246 by 17, which is approximately if I make this, will be 14.0. This now, similarly the bound of x initial or bound of x is from 3 to 4. From here your x is 3.0 something.

So, x can take x, not x, but the bound of x 1. If I have to write bound of x 1 is 3 and 4. This I got from this value 63 by 17. This is 3.0 something. So, lower bound of x 1 is 3 and upper bound of x 1 is 4. Similarly, lower bound of x 2 will be 2 and in the same case this is 34. So, lower bound of x 2 is 2 and upper bound of x 3 is 3.

So, we are finding for the integer, for the variables which contains non integer values. What is the lower bound and what is the upper bound; So, for x 1 lower bound is 3 and upper bound is 4. So, from here now, i can formulate two problems in one case your x 1 will lie in between 0 and 3 in other case your x 1 will be greater than equals 4. So, now, this original problem, say this was your S P what we denote this thing as S P.

So, your S P this problem is divided into two sub problems which we are denoting as SP 1 S P 2.

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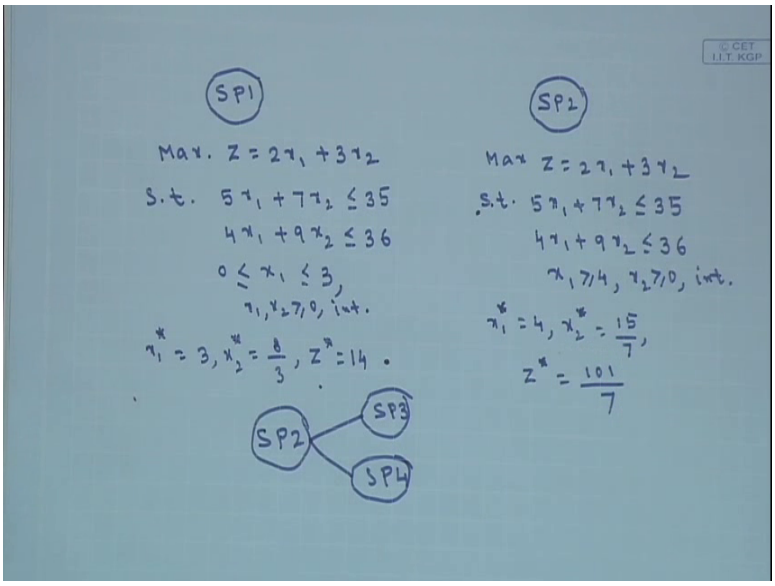

Your SP 1 will be maximized z equals 2 x 1 plus 3 x 2 subject to 5 x 1 plus 7 x 2 less than equals 35, 4 x 1 plus 9 x 2 less than equals 36 and 0 less than equals x 1 less than equals 3. This extra condition now we are adding compared to the earlier problem. If you see and this we are adding since the lower bound is 3 and upper bound is 4 for the variable x 1.

So, x 1 lies between 0 to 3 x 1 x 2 greater than equals 0 and; obviously, they are integers, and similarly, yours problem 2 will be SP 2 will be maximized z equals 2 x 1 plus 3 x 2 subject to 5 x 1 plus 7 x 2 less than equals 35. First two constants will remain same, that is second one will be 4 x 1 plus 9 x 2 less than equals 36 x 1 greater than equals 4 x 2 greater than equals 0 and both the variables should be integers.

So, now what you have to do. So, you see the original problem I am dividing into 2 sub problems and how we are dividing it. We are dividing it by the basis of the value of the lower bound and upper bound of the non integer variable. So, in the basis or in the solution if I have the variables, whose values are non integer from there, I will choose the variable which has more or larger fractional value as in this particular example we have chosen x 1 since x 1 has the larger fractional value, then we are finding out the lower bound and upper bound of x 1, which is 3 and 4 respectively for this particular problem.

Then we are creating two sub problems. Now, these two sub problems again can be solved by graphical method purposefully, we have chosen it.

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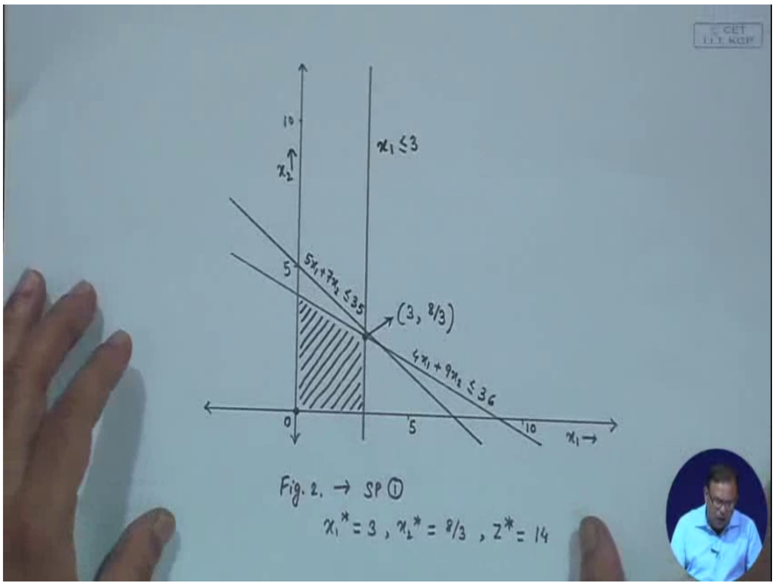

So, for the sub first sub problem if you choose then your problem will be like this your 1 is 4 x 1 plus 9 x 2 equals 36 another line 5 x 1 plus 7 x 2 equals 35 and x 1 less than equals 3, this is the bound. So, this is on this direction.

So, your feasible region is the shaded 1, and if you calculate you will find at the 0.38 by 3, you are obtaining the solution. So, therefore, your solution is for the first problem x 1 equals 3 x 2 is 8 by 3 and z star equals 14. So, please note this one that for this problem your x 1 star, this is equals to 3 your x 2 star is 8 by 3 for the first problem SP 1 and your z star is 14 8 by 3 and your z star is equals to 14, whatever you are getting now, similarly for the second problem also graphically I can find out the solution my solution set will be like this.

This is the line again 5 x 1 plus 7 x 2 and this is the line for 4 x 1 plus 9 x 2 less than equals 36 and for this case your x 1 is greater than equals 4. If you try to find out the solution you will get it at this point the maximum value. So, for this case your solution is x 1 is 4 x 2 is 15 by 7 z star. You will obtain 1 0 1 by 7. So, you are getting this two. So, for this case your solution is x 1 star equals 4 x 2 star. This is equals 15 by 7 and z star equals 1 0 1 by 7

So, if you see in this case for this problem I obtain two solutions,, but in both cases your x 2 is having non integer value and your, for the problem S P, two basically. You have more optimum value of the objective function. So, as we have told since in both the cases. I am getting non integer solution again, I have to sub divide the problem, Now, since I have to sub divide the problem; that means, now your SP 2 has to be sub divided into S P 3 and S P 4, this 2, here, if you see your x 2 is the non integer value the lower bound and upper bound from here. It is very clear lower bound of x 2 will be 2 and upper bound of x 2 will be 3.

So, in the original problem, from the original problem means I want to say on this particular problem. Now, I will impose one more condition. Now, what will be the condition? So, SP 2 I am dividing. So, these constants will remain as it is along with this. I will add one more constant for x 2 for the lower bound and for the upper bound lower bound is 2 and upper bound is 3 for this case. So, your I can write down your S P 3 as.

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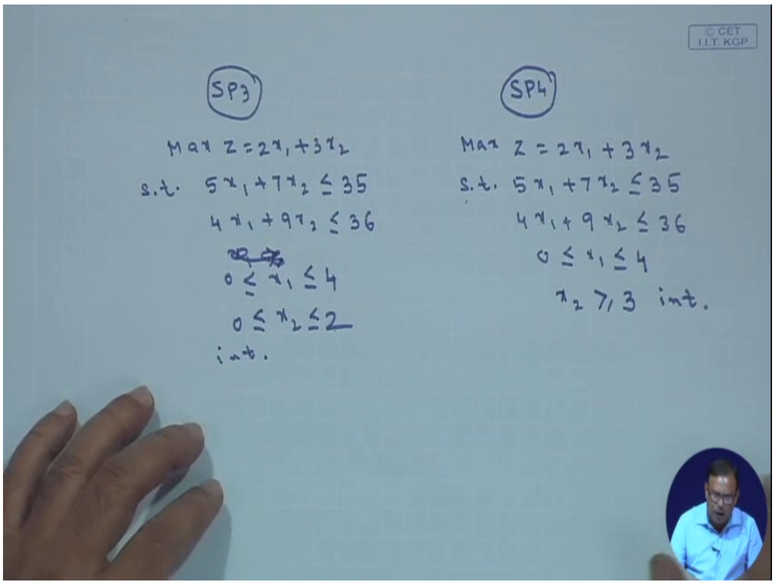

Maximized z equals 2 x 1 plus 3 x 2 subject to 5 x 1 plus 7 x 2 less than equals 35 4 x 1 plus 9 x 2 less than equals 36, your condition was x 1 already greater than equals 4; that means, one condition will be x 1 lies between 0 to 0000 less than x 1 less than equals 4. It cannot be this one and the one more condition.

Now, I will add that is 0 less than equals x 2 less than equals 2. So, whatever was there along with this your x 1 must lie in between. Now, 0 to 4, at 4 we obtained the solution. So, x 2 lower bound is 2, upper bound is 3, for that reason here I have added x 1 x 2 lies between 0 to 2 and x 1 x 2 should be integer. Similarly, your S P 4, this will become. Now, this is S P 4. So, S P 4 will be maximized, z equals 2 x 1 plus 3 x 2 subject to 5 x 1 plus 7 x 2 less than equals 35 4 x 1 plus 9 x 2 less than equals 36, your this condition is there, that is x 1 lies between 0 to 4 and your upper bound, if you see here upper bound is 3. Since it is fractional part and upper bound is 3.

So, therefore, I have to add one more thing x 2 greater than equals 3 and these are the all are integer values. So, now, as we have done earlier graphically I will solved this problem. So, I am just seeing here that this is the line 5 x 1 plus 7 x 2 equals 35. This is the line 4 x 1 plus 9 x 1 equals 36 and this is the line x 1 less than equals 4, that is this thing and what was the other 1 x 3 less than equals 2 x 2 less than equals 2 was the second one. If you would see x 2 lies between 0 to 2. So, that your solution space will be this one, this shaded region is the feasible region and if you calculate you will find at the point 4 2 we are getting the maximum value.

So, your solution in this case will be x 1 equals 4 x 2 equals 2 and z star equals 4. So, for this problem your solution then becomes x 1 star equals 4 x 2 star equals 2 and your z star equals 14 from the problem. Similarly, now I can find the solution of this 1 for this case if you find your 5 x 1 plus 7 x 2 equals 35 your 4 x 1 plus 9 x 2 equals 36 x 1 less than equals 4, you see the directions for this also direction is this for this; that means, this area, but x 2 is greater than equals 3; that means, this portion upper portion.

So, therefore, they are not intersecting anywhere or in other sense we can tell that this problem has no feasible solution and z star does not exist over here for this problem graphically it is quite clear, just see there will be no intersecting area among these 5 coordinate axis and this 4 lines. So, for this case we can write down no feasible solution for this problem no feasible solution, but for this case you see we obtained the optimal solution and they are satisfying the integer restriction also.

Therefore the solution of the original problem will be this 1 x 1 equals 4 x 2 equals 3 2 and z star equals 14. So, the original solution for this problem will be the solution of s p 3, which is x 1 star equals 4 x 2 star equals 2 and z star equals 3. So, I hope it is clear that how using branch and bound method. We can solve one integer programming problem where basically, whenever in a solution you are obtaining non integer value for a decision variable you are breaking the problem into two sub problems in one sub problem. You are using adding one more constant for the lower bound of the non integer variable in another sub problem. You are breaking it into, you are adding the constant of upper bound, and by this way again you are solving the problem and you are repeating the process until you are obtaining the desired solution.