**Constrained and Unconstrained Optimization**
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**Lecture – 23**
**Integer Linear Programming Problem – II**

So, let us start with the example which we did in the last class. For solving one integer programming problem linear programming problem.

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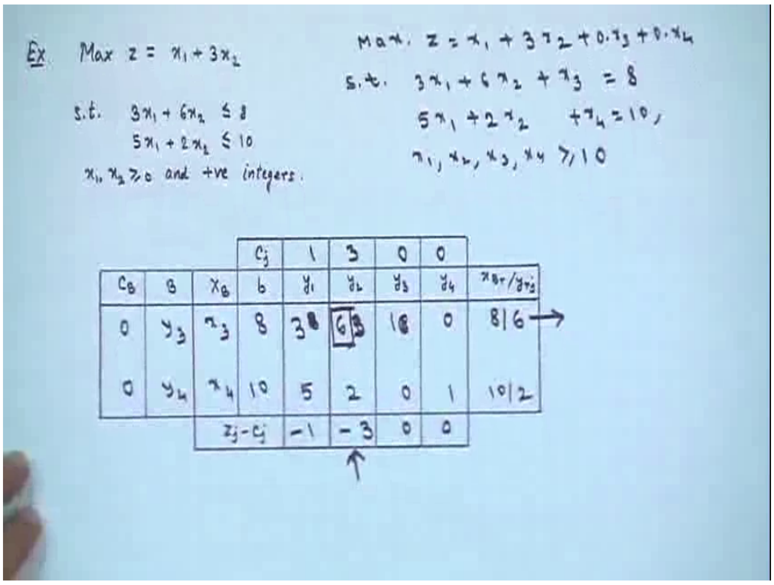


So, just to recap your problem was this using the slack variables, we made it the in the standard format. We made the simplex table there we found that z j minus c j is less than 0. So, most negative that is x 1 x 2 was the entering vector, and after calculating the ratio x 3 was the outgoing vector.

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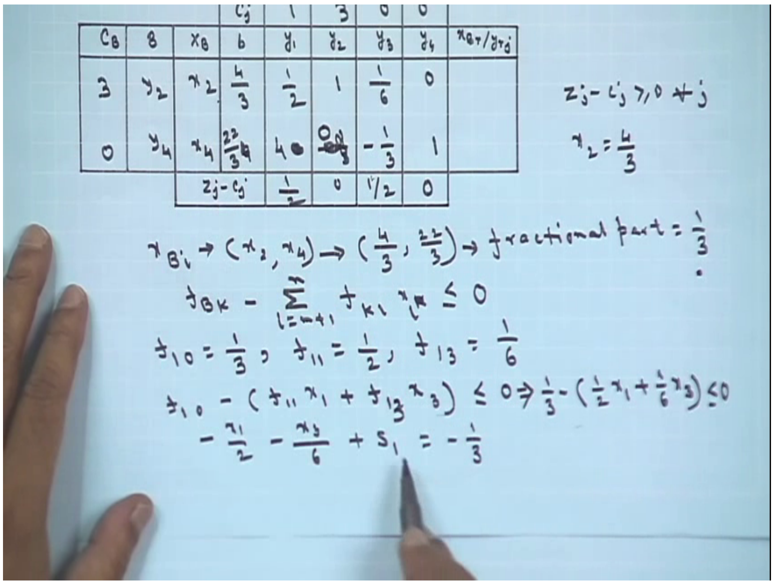


So, in the next table whatever we have formed x 2 and x 4. Then we recalculated it we calculated the, value of z j minus c j. What we are finding out is z j minus c j is greater than equals 0 for all j, but the feasibility condition is not satisfied. Since value of one of the basic variable that is x 2, this is equals to 4 by 3 that is non-integer. So, now, I have to use the gomorian method to avoid this non integer value.

So, what we are finding here is that fractional part of x bi, your fractional part of x bi, that is x bi means here it is x 2 and x 4. Fractional part of x bi is which one? 4 by 3 this value of x by 2 is 4 by 3 and it is 22 by 3. So, from here the fractional part is equals to one by 3. The fractional part is 1 by 3.

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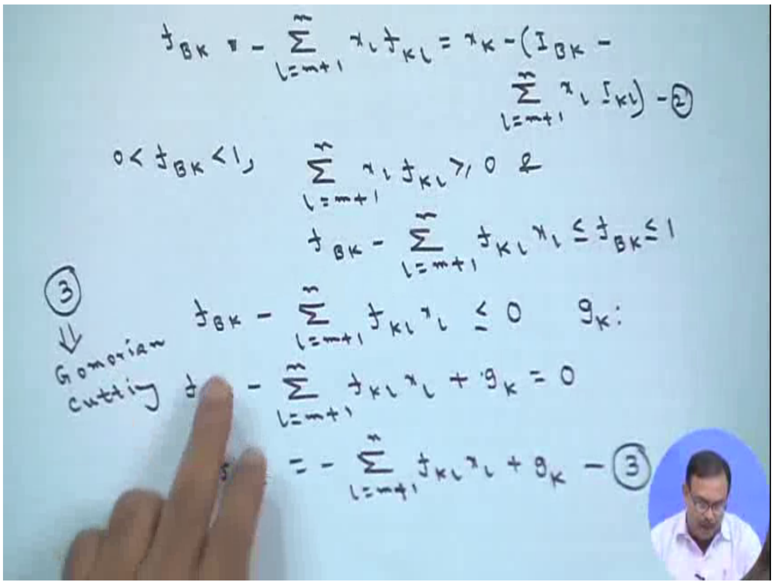


Now, you have formulated this equation, f B minus this one should be less than 0, to make it integer type. After that we introduce the gomorian slack variable. So, I am writing that equation itself f of B k minus summation over l equals m plus 1 to n f of k l x l which is this is x l less than equals 0, where f B k is the fractional part of the x B k.

So, I have to find out what are the values over here. What is f1 0? Your f1 0 is the fractional part of b. Fractional part of B means x bi fractional part of x bi already we have shown as one-third. So, f of 1 0 is one-third. Once I am doing this one then on this see in both cases it is one-third. So, anyone I can take I am considering the first row. Then how many more fractional parts are there. Here you have 2 more fractional parts corresponding to this. This we have not considered because this is negative.

So, your f1 1 this will be equals to half and f1 3 this is 1 1 1 2 1 3 f1 3. This is equals 1 by 6. So, what happens here f B k. That is f1 0 minus this is, f1 1 x 1 corresponding value will be x 1 plus f1 2 x 2 sorry this is 3. So, it will be f1 3 x 3 this is less than equals 0. So, if you substitute the value here, from here you will get one-third minus half x 1 plus 1 by 6 x 3. This is less than equals 0.

So, if you note this one, that please see this thing at first I am finding out the fractional part of these fractional part of this is one-third, of this one is one-third otherwise maximum one I will take. Now I have to use this inequality whatever we have shown earlier, on this inequality now f1 0 will be equals to B that is the fractional part. If I consider this row, because both are one-third then I have to take the fractional parts on other columns. Here it is if it is f1 0, this will be f1 1 this will be f1 3. So, f1 1 half f1 3 6.

So, this inequality can be written as this from here we are getting this less than equals 0. Now this constant, I will make it equality constant by applying the gomorian slack variable, and I will reconstruct the table from here. I will reconstruct the table from here that is, I will write down minus x 1 by 2 minus x 3 by 6 plus s 1 this is equals minus one-third, where s 1 is the gomorian slack variable. And this particular constraint now will go to the next one. This constraint will now go to the next one, next one means what we will go into this one; that means, already you are having only 2 variables in the basis x 2 and x 4.

Now, one more constraint basic variable s 1 will be added over here and this row also will be added. So, in the next iteration, next simplex table will contain 3 basic variables x 2 x 4 the earlier ones and new one is s 1.

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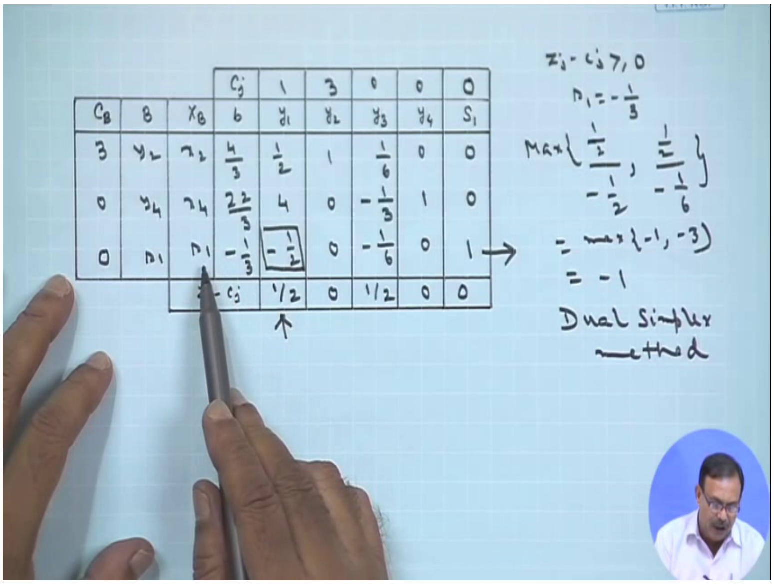


So, we are constructing now the new table, which will contain x 2 x 4 and s 1. So, this will be y 2 y 4 and s 1. Values are 1 3 0 0. And please note that for slack variable also the value will be the coefficient will be 0. So, that here you will obtain 3 0 and 0.

So, once we are doing it, we are writing this one 4 by 3 half 1 1 by 6 0 and 0. Please note that these 2 rows will remain as it is there will be no change. These 2 rows will remain as it is and one more row will be added with the coefficients of this one. So, next row will be 22 by 3, 4 0 minus one-third one and 0. And the third one is value of B here is minus one-third. So, therefore, it will be minus one-third, minus half 0 minus 1 by 6 0 and 1.

So, now like this way, whenever you are having one table where z j minus c j is greater than equals 0, but the solution is infeasible since one of the value of the one of the decision variable or basic variable is fractional. So, by this way you are calculating what is the fractional part, maximum fractional part. And then you are this inequality you are calculating like this way. And you are formulating the table. So, now, your z j minus c j value will be equals to half, 0 half 0 and 0. Here if you find your z j minus c j this is greater than equals 0, but the solution is infeasible. Since s 1 equals minus one-third. As we mentioned earlier whenever you will introduce the new constraint and you will add the gomorian slack variable always the value of the gomorian slack variable will be the negative one.

Now, if it is negative we know how to find the how to find out or eliminate the negative value in the basis using dual simplex method. Using dual simplex method always we can do this one. So, in this case you’re what happens here maximum of here negative is coming on this thing. So, maximum of this divided by this because negative is here negative is here. So, maximum of half divided by minus half and half divided by minus 1 by 6. If you do it, you will get maximum of minus 1 minus 3 and this is equals minus 1.

So, this is equals to minus 1. So, therefore, without going in details because this thing we have discussed earlier, this will be your departing vector. And this will be your entering vector; that means, this will be the your pivot element. So, please note that on this since your basic variable value is negative. So, we are using dual simplex method. We are using dual simplex method to eliminate this one. And from there we are finding what is the outgoing or departing vector. Your departing vector will be s 1 and your entering vector will be y 1.

So, therefore, in the next one, s 1 will be replaced by x 1. So, let us formulate the next table.

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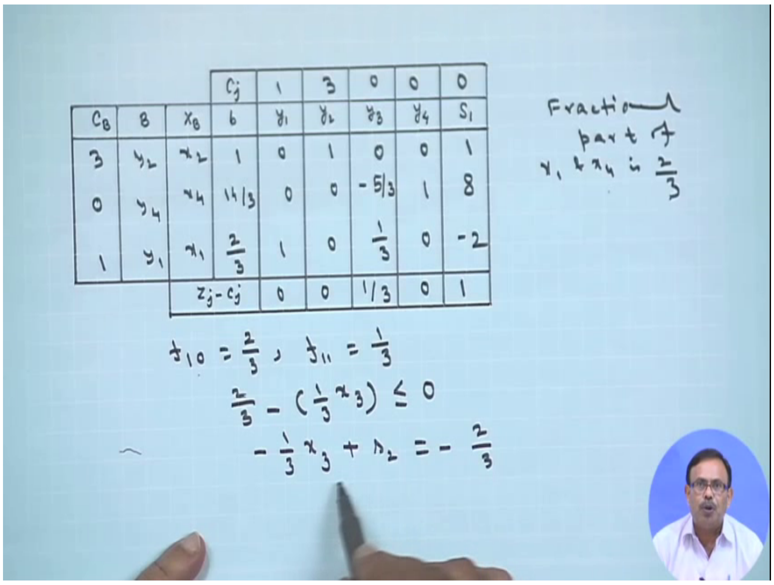


In the next table now, you are having x 2 x 4 and x 1. Because x 4 has been replaced by x 1, you are having y 2 y 4 and y 1. Here it will come 1 3 no change 0 0 0. So, c B s value will be 3 0 corresponding to y 1 it is 1. So, now, it is becoming 1 0 1 0 0 1. Then there will be 14 by 3 0 0 minus 5 by 3 1 and 8. It will come as two-third 1 0 one-third 0 and minus 2.

Calculate the z j minus c j, it will be 0 0 one-third 0 and 1. So, again if you find here, it is z j minus c j is always again greater than equals 0, but if you find here x 1 is two-third x 4 is 14 by 3. So, these are non integer values. So, therefore, I have to eliminate one from this the fractional part for both case x 1 and x 4 is two-third the fractional part of x 1 and x 4 is two-third. So, therefore, we can use anyone of these 2. I am using say x 1 there is a tie both are same. If I am using x 1 then f1 0 will be two-third, I am again repeating this f1 0 will be two-third I would have taken this one also.

So, f1 0 is two-third and on this row you have only one fractional part that is one-third. So, f1 1 will be equals to one-third therefore, your inequality will be two-third minus one-third into one-third belongs to x 3. So, it will be x 3 which is less than equals 0 or this I can write down minus one-third x 3 plus s 2 this is equals to minus 2 by 3. So, in the next one what happens, since there was a fractional part in both x 1 and x 4 therefore, what we have done the fractional part is 2 by 3.

So, I have to use a new constraint gomorian constraint to remove the fractional part since there is a tie I can use anyone I am using say x 1. Then your f1 0 will be two-third there is only one fractional part one-third. So, you are getting this inequality by adding the gomorian slack variable s 2 you are getting this constraint equality constraint. Now this will be added on the existing table. That is on this table it will be added over here. So, once this is added on the basis now you will have x 2 x 4 x 1 and s 2 this will be added.

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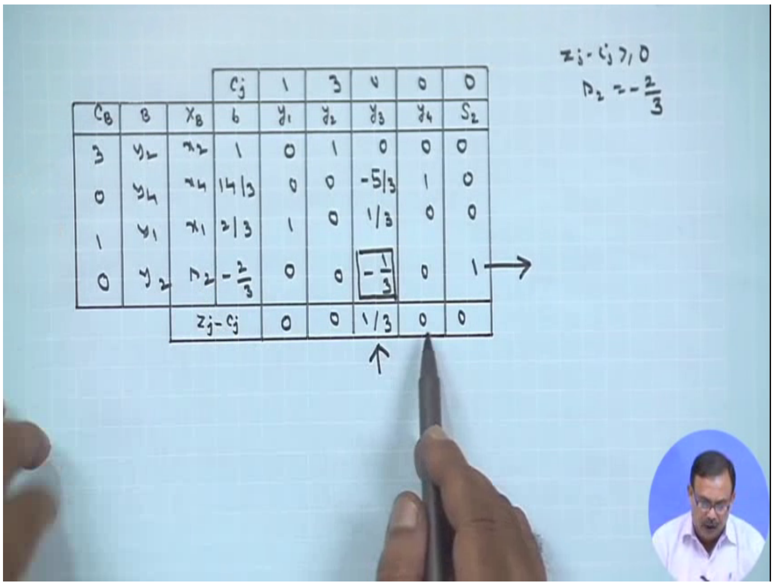


So, once we are adding this, in the next table what you will obtain x 2 x 4 x 1 and s 2. These are y 2 y 4 y 1 and y 2. Here the values are same 1 3 0 0 0. The slack variable you see we are removing from here because slack variable is not necessary. So, it is 3 0 1 0. Corresponding c j values and again what you will do you will write down this entire table along with that you will add this row together.

So, it will become 1 0 1 0 0 0 14 by 3 0 0 minus 5 by 3 1 0. For x 1 it is 2 by 3 1 0 one-third 0 0 then minus two-third, 0 0 minus two-third 0 0 minus one-third 0 and 1, calculate the z j minus c j value you will obtain that this one. So, your z j minus c j is greater than equals 0 for all j, but the solution is infeasible since s 2 equals minus two-third.

Therefore, you will use the dual simplex method, since the value of a basic variable is negative. So, if you use dual simplex method which I am not explaining. You will find that this will be the outgoing vector and; that means, s 2 will be the outgoing vector and your y 3 that is x 3 will be the entering vector. So, that your pivot element is this one. So, in the next basis s 2 will be replaced by y 3. So, once I am doing this thing, therefore, what you will obtain in the next table is you are having this one, x 2 x 4 x 1 and s 2 is replaced by s 3.

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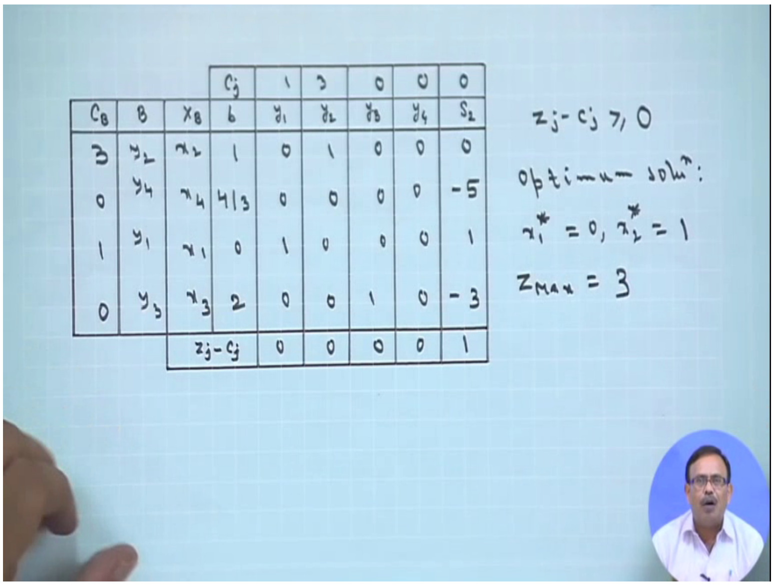


So, this is y 2, y 4, y 1, y 3, c j values are 1 3 0 0 0. And this is 3 0 1 0. The values you will get as 1 0 1 0 0 0, 4 by 3 0 0 0 0 minus 5. For x 1 it is 0 1 0 0 0 1, this is x 3 it is 2 0 0 1 0 minus 3. And if you calculate the z j minus c j value it will be 0 0 0 0. This is 0 this is 0 this is one. So, you will get 1.

So, your z j minus c j greater than equals 0, and for the decision variables your original problem was maximized z equals x 1 plus 3 x 2. So, therefore, your 2 decision variables were there. And here what you are obtaining that the value of the decision both decision variables are integer. Therefore, this solution is optimum and you can write down your optimum solution as like this. Your optimum solution is x 1 star equals 0 x 2 star equals 1. And z max maximum value of z is equals to 3 into 1 plus 0 into 1. So, this is 3.

So, I hope it is clear to you now, that how we can solve one integer programming problem. Let us take quickly one more example.

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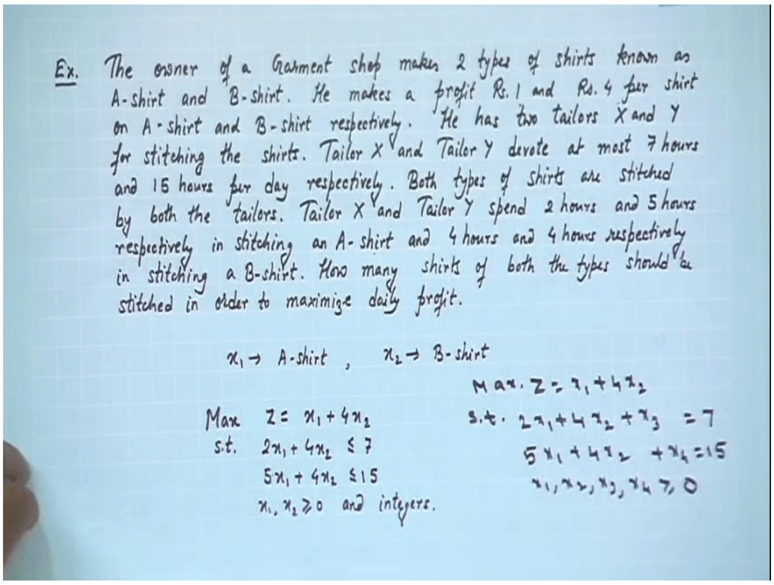


So, that it becomes clear to us. I am not reading the entire thing. The owner of a garment shop makes 2 type of shirts, known as A shirt and B shirt. He makes a profit of rupees one and rupees 4 for shirts of A shirt B shirt respectively he has 2 tailors x and y tailor x and y devote at most 7 hours and 15 hours per day, which you are coming here. Both type of shirts are stitched by both tailors tailor X and Y spend 2 hours and 5 hours respectively in stitching. The A shirt and 4 hours and 4 hours for a B shirt.

So, how many shirt of both types should be stitched in order to maximize the daily profit. So, here x 1 and x 2, I am assuming number of a shirts will be produced is x 1 number of B shirts will be produced is x 2. Since we are producing both the shirts therefore, I have to maximize x 1 profit for a type shirt is rupees one and profit for B type shirt is rupees 4. So, objective function will be z equals x 1 plus 4 x 2 and they can first one they can stitch 2 and 4. So, this will be 2 x 1 plus 4, x 2 less than equals 7 and 5 x 1 plus 4 x 2 is less than equals 15. This 2 are the constraints x 1 x 2 are greater than equals 0. So, once I am doing this one, here since I have to prepare the shirts both of x 1 and x 2 are shirt type therefore, they must be integers.

So, once you are writing it in the standard form, you can write down in the maximum of z equals x 1 plus 4 x 2 subject to again use the slack variables that is 2 x 1 plus 4 x 2 plus x 3. This is equals 7 5 x 1 plus 4 x 2 plus x 4, this is equals to 15. And x 1 x 2 x 3 x 4 is greater than equals 0. Where x and x 2 has to be integers. So, your here your x 3 and x 4 x 3 and x 4 are the slack variables.

So, after writing the it in the standard form, now we will write the corresponding tables as usual from here. Again in the basis only 2 variables will go that is x 3 and x 4 these are the variables which will go in the basis.

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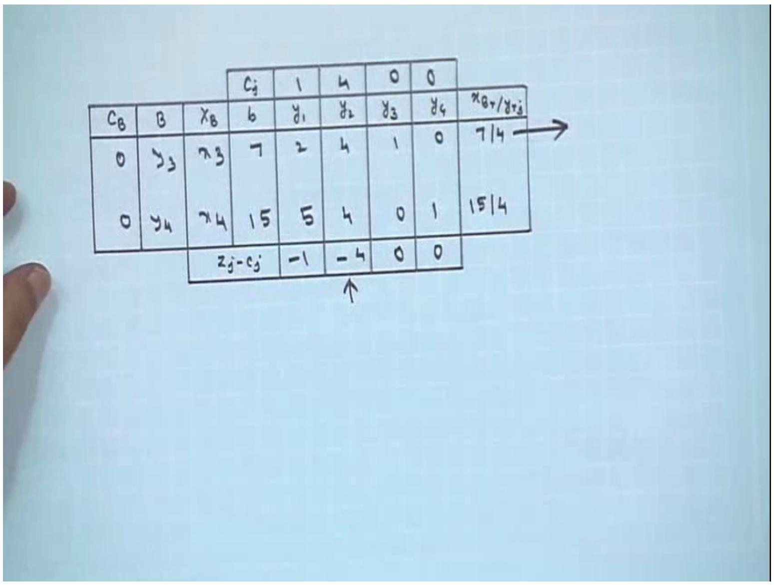

So, in the basis you have x 3 and x 4. So, this is y 3 and y 4. And here values of c j are 1 4 0 0; obviously, here we are writing 0 into x 3 plus 0 into x 4. Because in this case the corresponding coefficients or the slack variables are 0.

So, one 4 0 0 are the coefficients of x 1 x 2 x 3 x 4. So, the c B values are zeros. So, you will get 7 2 4 1 0. I am not discussing because this already we have solved so many problems till now, 15 5 4 0 1. Your z j minus c j is minus 1 minus 4 0 0. So, most negative is minus 4. So, this will be the entering vector, calculate the ratio it will be 7 by 4 15 by 4. So, the from here it is minimum is this.

So, therefore, since minimum is this your x 3 will go out. So, that in the second iteration your x 3 will go out and x 2 will enter into the basis.

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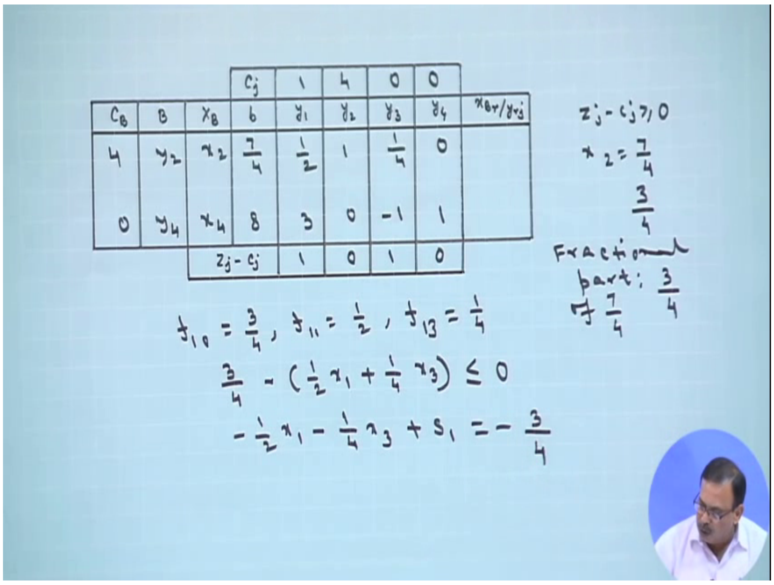

So, from here we are writing x 2 which is replaced by x 3. So, here it is y 2 and y 4. Your c j values are one 4 0 and 0. So, c B values will be x 2 is 4, this is 4 zero; obviously, here your pivot element was this; that means, I will make this element as one and this element as 0. For that I will perform the corresponding row operations.

So, it will become 7 by 4 half 1 1 by 4 and 0. For this it will be 8 3 0 minus 1 and 1. If you calculate z j minus c j you will find that z j minus c j is 1 0 1 0. So, therefore, here if you find your z j minus c j is greater than equals 0, but the solution is infeasible, since x 2 is equals to 7 by 4. Since x 2 is the fractional part, what is the fractional part the fractional part here is 3 by 4. And which is occurring here since x 2 is 7 by 4 therefore, the solution is infeasible, because you have the non integer solution.

So, the fractional part I am calculating here for only it has occurred for one case. So, fractional part of 7 by 4 is 3 by 4. Therefore, this will be your f1 0, as usual in the last time we have told. So, f1 0 is equals to 3 by 4. It has on this row you have 2 fractional parts. So, your f1 1 will be equals to half and f1 3 will be equals to 1 by 4. So, your gomorian constraint will be 3 by 4 minus half x 1 plus 1 by 4 x 3 which is less than equals 0. Now add the slack variable and we can write down half x 1 minus 1 4 x 3 plus s 1 this is equals to minus 3 by 4.

So, now add this constraint over here, that is on the next table you will have the 3 variables x 2 x 4 and x 1, and add this thing.

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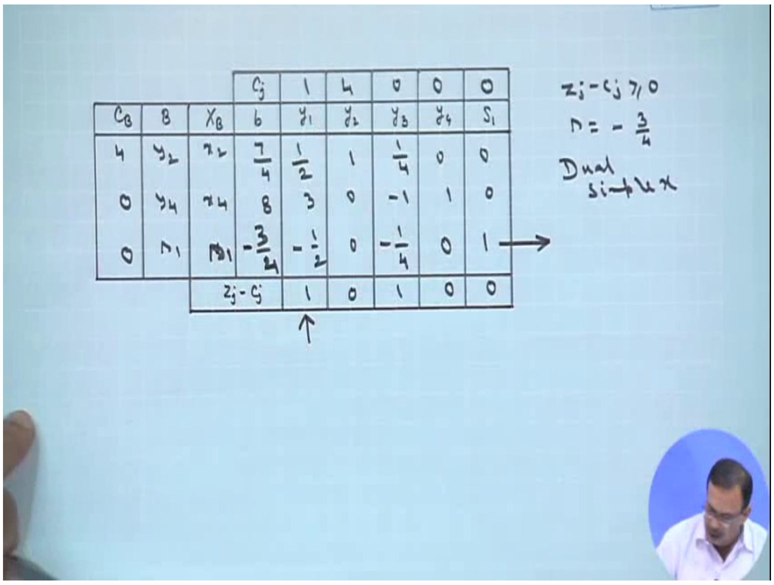


So, you will get x 2 x 4 and x 1. Sorry s 1 you have added the slack variable as s 1 over here. So, once you are doing s 1 over here. So, once I am doing this one, it will be y 2 y 4 and s 1, c j values are 1 4 0 0 corresponding to slack also it will be 0. So, y 2 y 2 is 4 and x 4 is 0 s 1 is 0 0 0.

So, now you will obtain you will add one more constraint corresponding to s 1. These 2 rows will remain same like the other one. So, 7 by 4 half 1 1 by 4 0 and 0. Next row will be 8 3 0 minus 1 1 and 0. Next one will be half z j minus c j is sorry this will be minus 3 by 4 minus 3 by 4, minus half 0 minus 1 by 4 0 and 1, z j minus c j if you calculate 1 0 1 0 0.

So, z j minus c j is greater than equals 0, but the solution is not feasible, since s equals minus one-third minus 3 by 4. So, since this is negative. So, therefore, we will use the dual simplex method, to obtain the solution of this you know. You will find out the minimum and you will obtain what is the departing vector what is the entering vector. So, using dual simplex method your departing vector will be this one and your entering vector will be this.

So, in the next table your s 1 will be replaced by x 1. So, in the next table what you are obtaining is x 2 x 4 and x 1.

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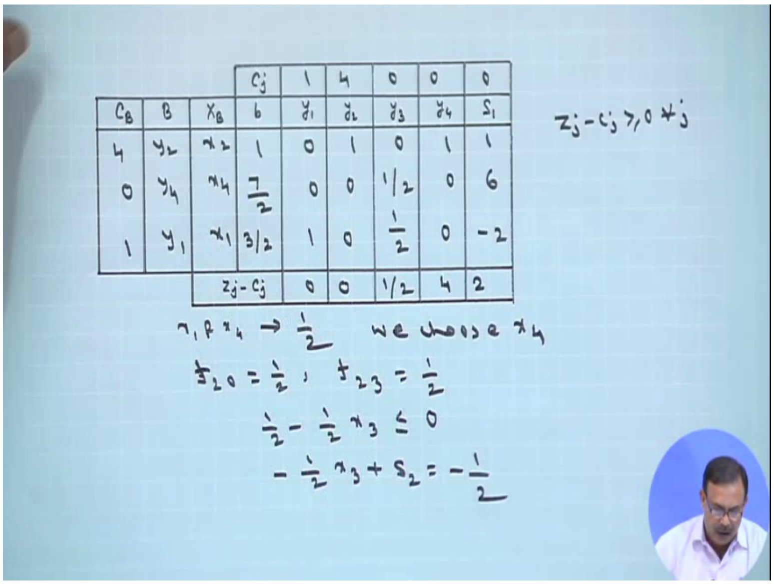

So, this is y 2 y 4 and y 1, c B values are not changed. So, to be 4 0 corresponding to y 1 it is 1. So, since it is the entering variable and departing variable your pivot element is this. I will make this an one this as 0. So, I will obtain 1 0 1 0 1 1. Then I will have 7 by 2 0 0 half 0 and 6. I will have 3 by 2 1 0 half 0 and minus 2, z j minus c j if you calculate 0 0 half 4 2.

But z j minus c j is greater than equals 0 for all j, but the solution whatever you are obtaining here that is non integer solution. If you find here corresponding to x 1 and x 4 the fractional part is half for both cases. So, I can choose anyone, if you remember in the earlier case we choosed x 1 now say I am choosing this x 4. So, we choose x 4 to eliminate. So, once I am choosing x 4. So, your f1 0 or f 2 0 if you consider this is equals to half. Corresponding to this you have only one fractional part. So, f 2 3 equals half here.

So, your inequality becomes half minus half x 3 less than equals 0. Or introducing the slack variable minus half x 3 plus s 2 this is equals minus half. And this will go into this table. So, now, in the next table what will happen, you will have you are having 3 variables. One more slack variable s 2 also will be added in the basis and I have to add these constraints and these rows will remain as it is as we have seen earlier.

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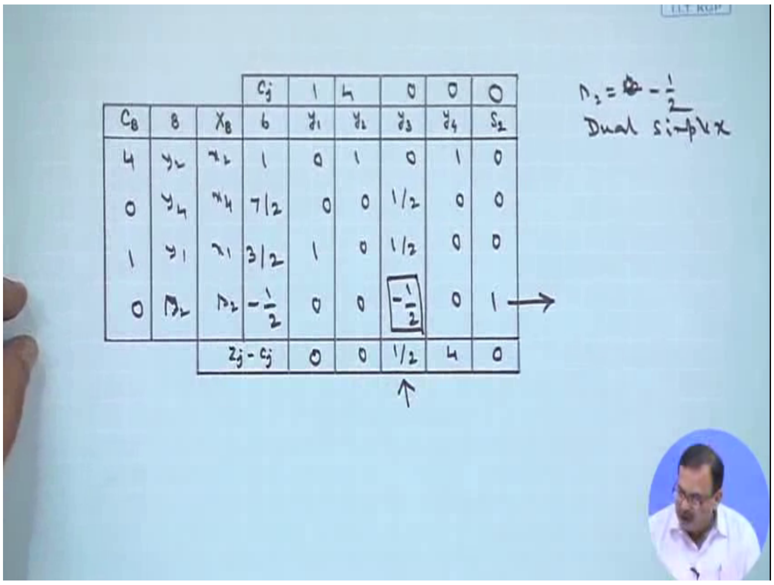

So, now in the next one, you are having x 2 x 4 x 1 and s 2. So, this is y 2 y 4 y 1 and y 2. C B values are 1 4 0 0 0, s 1 we have removed already. So, it will be 4 x 4 is 0 x 1 is 1 and s 2 it is 0. So, I am writing it 1 0 1 0 1 0. Next one is 7 by 2 0 0 half 0 0. Next one is 3 by 2 1 0 half, 0 and 0. Then s 2 is minus half 0 0 minus half 0 and 1. If you calculate z j minus c j your z j minus c j will be 0 0 half 4 and 0.

So, from here like earlier case your z j minus c j is greater than equals 0, but one of the basic variable s 2. This is equals to minus half that is negative. So, it is not satisfying the condition. Therefore, you can use the dual simplex method again on these to deciding what would be the departing vector and what will be the entering vector. If we use dual simplex method. You will find this is the departing vector and this y 3 will be the entering vector. So, that this will be your pivot element.

So, from here you make this one as one and other 2 as 0. And you will obtain this thing. So, your s 2 sorry this will be your s 2 not y 2 your s 2 will be going out and s 2 will be replaced by y 3 in the next table.

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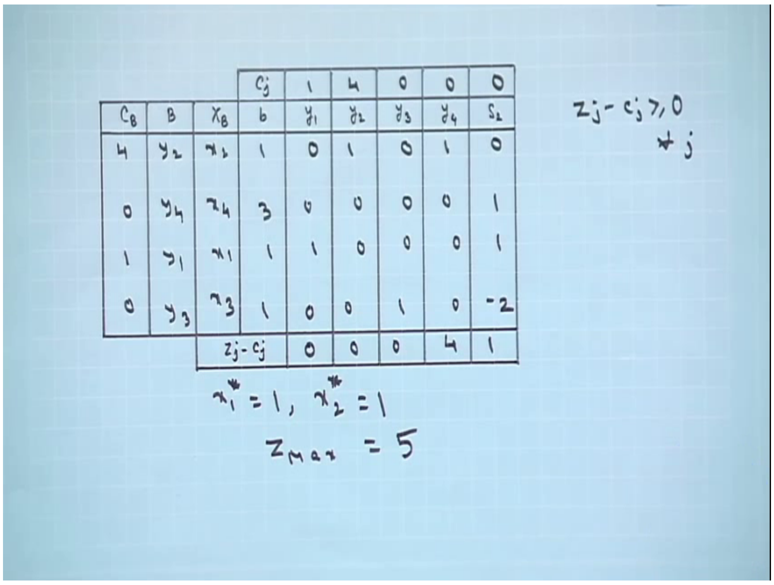


So, in the next table what happens. You are having x 2 x 4 x 1 and x 3 now. So, y 2 y 4 y 1 and y 3. The values are one 4 0 0 and 0. C B values are 4 0 1 0. So, if you use the row operations to make the element pivot element and so on. And other elements on the corresponding column as 0 you will obtain this thing 3 0 0 0 0 1 1 1 0 0 0 1 and 1 0 0 1 0 minus 2.

So, once you are getting z if you calculate z j minus c j you will get this is one. So, your z j minus c j is greater than equals 0 for all j. And if you see the values of the decision variables in the basis is integer. So, therefore, you obtained optimal integer solution. Therefore, your solution is x 1 star equals 1, x 2 star this is equals 1, x 2 is also 1. So, correspondingly z max will be equals to 4 plus 1 that is 5.

So, now I hope that it is clear to you that whenever I have to solve one integer linear programming problem and if I find that whenever I am trying to find the solution in that case in the basis some non integer solution is coming. Then using gomorian constraint and gomorian slack variable, I can remove that particular variable from basis. And then since whenever I will use gomorian slack variable, we have observed that initial value of the gomorian slack variable will be negative which is again not satisfying the feasibility condition. So, use dual simplex method to obtain the solution of that one. And by this process you can find out the solution of this.

So, this is one of the approach and there is another approach which is branch and bound method to solve one integer programming problem, which we will discuss in the next class.