Constrained and Unconstrained Optimization
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Lecture – 22
Integer Linear Programming Problem- I

In this lecture, we are going to cover the topics of integer linear programming problem. If you remember in the normal linear programming problem whatever decision variables we consider, the decision variables X capital X usually are positive. That is greater than equals 0 that is they can take the real value they can take the integer value also.

But you may have certain problems where decision variable can take integer values only. Just like suppose your decision variable is how many tables I want to purchase, how many chairs I want to purchase. Something like this way when decision variable can take only the integer values in that case the normal L.P.P will not work. Because normal L.P.P works for the data where the decision variable can take non negative real values only. So, you have to find out certain other algorithm, similar to your simplex method by which we can say that this can be giving you the integer problem only.

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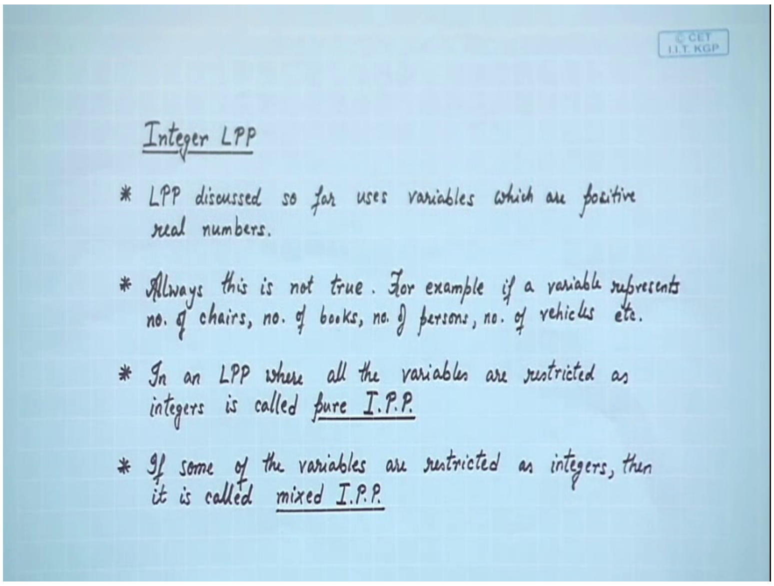

So, let us see this one in the integer LPP. So, you are going to cover this integer LPP. Linear programming problem discusses so far whatever we have discussed users variables which are positive real numbers, but this may not be always true as I was mentioning, if a variable represents number of chairs number of books number of persons number of vehicles to be find out in that case this will be integer not real number.

So, in an L.P.P where all the variables are restricted as integers is known as integer programming problem or in short we will denote it as I.P.P please note this one in an L.P.P when all the variables are restricted to only integers is known as I a p if some of the variables I.P.P. if some of the variables are restricted as integers then we call it as mixed integer programming problem. So, mixed integer programming problem means the variables may be integer or it may be real number whereas, only integer programming problem means the decision variables with which we will work they will be only integers they will take only the integer values of the variable.

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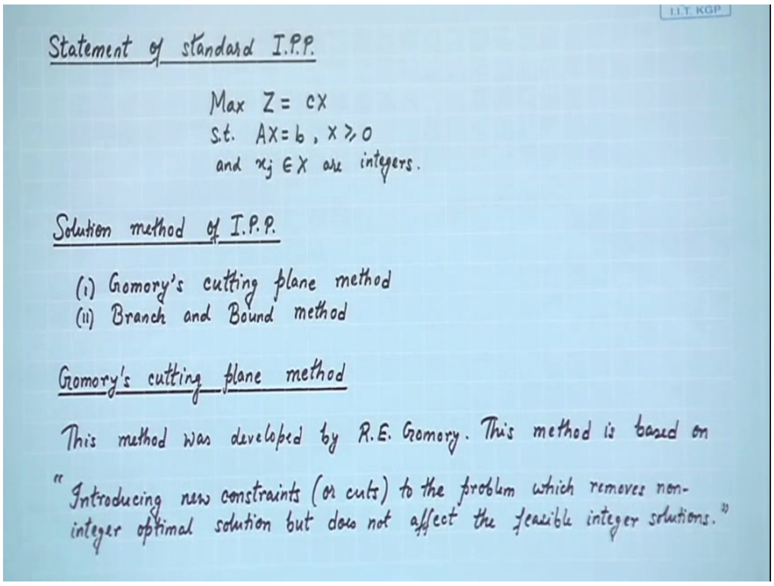

Let us take one standard L.P.P, I.P.P that is integer programming problem which is of the form maximized z x subject to a x equals b x greater than equals 0. And x j belongs to capital x are integers. Please note this one, x is greater than equals0, but we are imposing one additional condition that is x j belongs to some set capital x which are integers. Now the solution methodology for integer programming problem or I.P.P. basically we use 2 particular approaches one is gomory's cutting plane method and another one is branch and bound method gomorys cutting plane method in this particular method, what happens.

It was developed by R E gomory and this method is based on this basic concept introducing new constraints or cuts to the problem which removes the non-integer optimal solution, but does not affect the feasible integer solution. So, please note this one, that whenever we have non integer optimal solution it removes or it cuts the that particular solution. And by which we are doing it you are introducing a new constraint to the problem. Please note this one we are adding a new constraint to the problem by which we are removing the non-integer optimal solution.

But the solutions which are already integer solution they will not be affected by this method.

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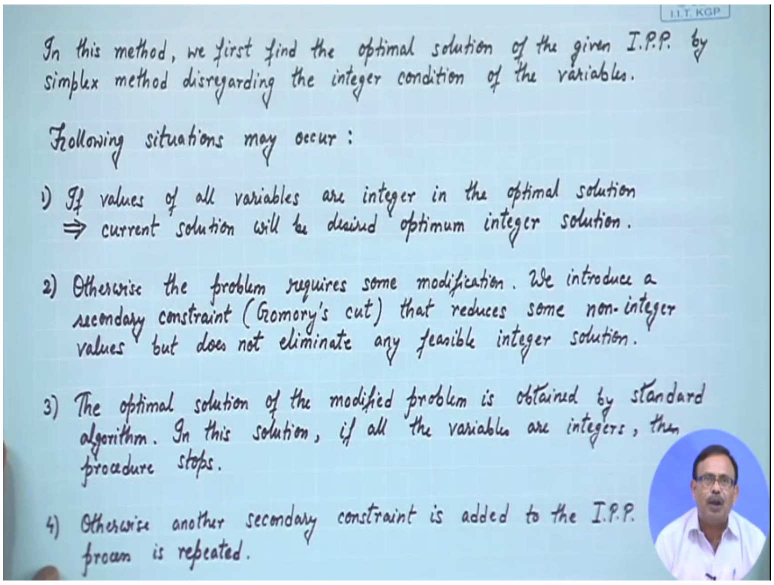

So, let us see this one, in this particular method, we first find the optimal solution. In this method we find the optimal solution of the given I.P.P by simplex method discarding the integer condition of the variables; that means, at first what we are doing we are finding the optimal solution of the integer programming problem by normal simplex method whatever we have learned earlier disregard of integer condition; that means, we do not consider now that the decision variables are integers.

So, whenever we are solving the integer programming problem by normal simplex method by removing the integer condition some situations may arise. The first situation is if the values of all variables are integer in the optimal solution. So, when if the values of all variables are integer in the optimal solution; obviously, that will imply current solution will be the derived integer solution. The current solution will be the integer solution.

Number 2 is otherwise the problem requires some modification; that means, the solution whatever you have obtained their values of some decision variables are non-integers. So, what we do we introduce a secondary constraint which we call as gomory cut that reduces some non-integer values, but does not eliminate any feasible integer solution, which I emphasized earlier. So, you are introducing one constraint which is known as gomory's constraint or gomory's cut which basically will remove some non-integer values, but it will not affect the already integer solutions which are feasible.

The third is the optimal solution of the modified problem is obtained by the standard algorithm. In this solution if all the variables are integers then you your procedure stops here; that means, after introducing the new constraint you are restructuring or recreating your table. And again you are using normal simplex algorithm to solve the problem and if the solution if you find that all the variables are integer; that means, you have obtained your optimal solution, which is feasible and whose values are integers values of the decision variables are integers.

Then you stop the process otherwise another secondary constant has to be added to the current I.P.P and the process will be repeated; that means, you will go on introducing the constants till you have the values of the decision variables of the basis is real number that is non integer and you will repeat the process.

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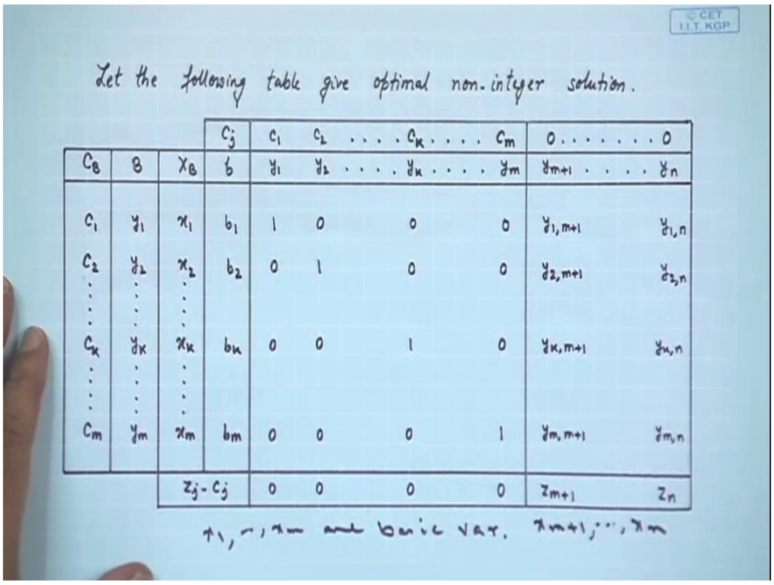

			c_j	c_1	c_2	$\dots c_k \dots$	c_m	$0 \dots$	$\dots 0$
C_B	B	X_B	b	y_1	y_2	$\dots y_k \dots$	y_m	$y_{m+1} \dots$	$\dots y_n$
c_1	y_1	x_1	b_1	1	0	0	0	$y_{1,m+1}$	$y_{1,n}$
c_2	y_2	x_2	b_2	0	1	0	0	$y_{2,m+1}$	$y_{2,n}$
$\vdots$	$\vdots$	$\vdots$							
c_k	y_k	x_k	b_k	0	0	1	0	$y_{k,m+1}$	$y_{k,n}$
$\vdots$	$\vdots$	$\vdots$							
c_m	y_m	x_m	b_m	0	0	0	1	$y_{m,m+1}$	$y_{m,n}$
		$z_j - c_j$		0	0	0	0	z_{m+1}	z_n

$x_1, \dots, x_m$ are basic var. $x_{m+1}, \dots, x_n$

So, actually what happens you consider this optimal non integer solution. Suppose you have c j c k c j s are there c 1 c 2 c m you have the values of y 1 y 2 y m here, you have x b b c b b values are there.

I have divided into 2 parts one is c m other values are there 0 to y m plus 1. So, here x 1 to x m are the basic variables. Please note this one, here x 1 to x m are basic variables in this case are basic variables. And remaining n minus m variables that is x m plus 1 to x n x m plus 1 to x n are non-basic variables. So, you are assuming this thing that on this you have the basic variables x 1 to x m, and remaining n minus m variables that is x plus x m plus 1 to x n are non-basic variables.

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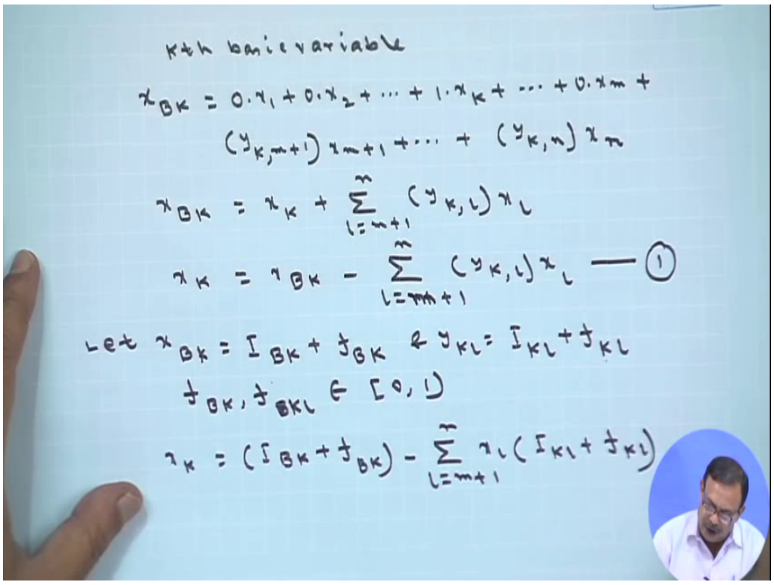

We assume that kth basic variable, kth basic variable is corresponding to the non-integer value, kth basic variable corresponds to the non-integer value in the solution; that means, you have the sum value x b k which takes some value like this 0 into x 1 plus 0 into x 2 plus like that way one into x k you are going on plus 0 into x m plus, we are writing y k m plus 1 into x m plus 1. Like this way the last one will be y k comma n into x n.

So, you are assuming that kth basic variable in the basis corresponds to the non-integer value and that we are writing as x b k equals this thing. So, x b k you can write down x b k equals x k plus all other will be 0. So, summation l equals m plus 1 to n y k l x l. And from here you can write down x k equals x b k minus this quantity summation l equals 1 to m plus 1 to n y k comma l into x l.

So, if the variable x k is basically non integer. And which we have written as x b k minus thing. Now let x b k this is equals to i b k plus f b k, it has 2 components basically x b k and this y k l. So, I am writing and y k l equals i k l plus f of k l. Well i b k and i k l are integer part of x b k and y k l respectively. So, basically each of x b k and y k l you are writing as in terms of integer part plus the fractional part. So, f b k and f k l are the fractional part of x b k and y k l.

Since we are assuming that x k is non integer. So, the coefficients here x b k and y k l we are writing in terms of the integer part plus the fractional part like this way, i b k plus f b k where of course, this f b k and f k l f b k and f of k l f of k l both belongs to 0 and 1.

Both belongs to 0 and 1. Both belongs to 0 and 1 note this one is the open bracket. So, therefore, from one I can write down x k equals just substituting the values of x b k and y k l equals i b k plus f of b k minus summation l equals m plus 1 to n x l into i k l plus f of k l. This we can write down and from here.

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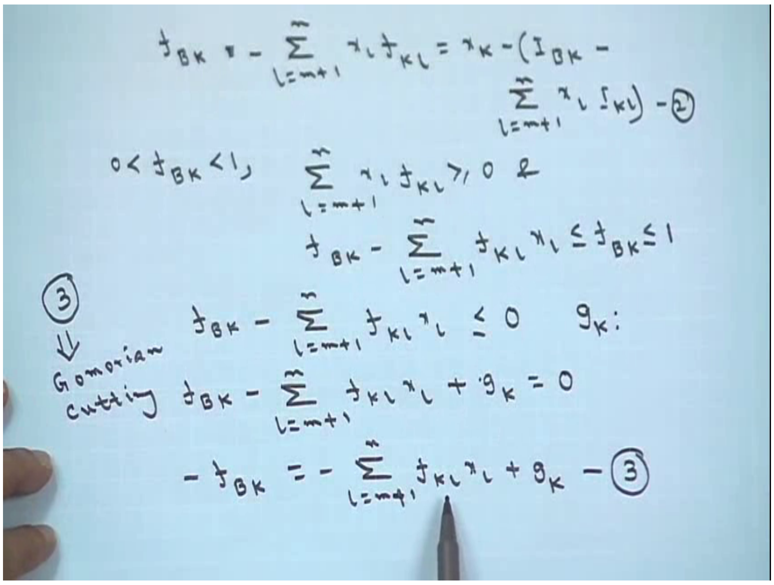

I can write down again from this one, I can write down f b k equals f b k minus summation l equals m, plus 1 to l x l f of k l x l f of k l this will be equals to x k minus i of b k, i of b k minus summation l equals m plus 1 to n x l i k l. Suppose this is equation 2.

So, basically just by from the last equation just checking the elements on the left side or right side, we are writing this one. Now on the right side of this equation if you see your x l your i b k your i k l your sorry this will be x k i b k. And so here, x k x l i b k and i a l are integer parts. So, I just up to this one contains the integer part. Therefore, what happens, since your f b k that is the fractional part lies between 0 and 1.

So, your you can write down summation l, equals m plus 1 to n x l f of k l this will be greater than equals 0. And your f b k minus summation l equals m plus 1 to n f f of k l x l this is less than equals f of b k less then equals 1. So, from to using this 2 conditions, we can write down f of b k minus summation l equals m plus 1 to n f of k l into x l this should be less than equals 0.

Now, this part f of b k summation l equals 1 to m plus 1 to l f of k l, this part is an integer then it should be either 0. If this part has to be an integer this should be either 0 or it must be negative. So, then there must have some inequality by which this we can introduce some non-negative gomorian slack variable to make it equality that is I can write down f b k minus summation l equals m plus 1 to n f of k l x l.

Since this is negative so, I am writing g k this is equals 0. Where we call it this one this g k as gomorian slack variable this g k we call it as gomorian slack variable.

So, that we can write down from here minus of f b k this is equals minus summation l equals m plus 1 to n, f of k l x l plus g k. This constant this equation 3 is known as this 3 we call it as gomorian cutting plane, or gomorian cutting this is known as gomorian cutting.

So, by this way as we told earlier that we will introduce one new variable over here. Sorry one new constant whenever we have one value of the decision variable is non integer. So, what would be the equation constant the constant will be basically this thing this is less then equals 0. And using gomorian on slack variable we are getting this equation. So, this we will use in this case. Now your f b quite naturally from this equation if you try to see your value of x l will be negative, from here it should be negative.

So, once the value of the decision variable is negative, if required we can use dual simplex method to solve the problem. Because if you remember already we have done in the dual simplex method if you have the infeasible solution that is the value of the basic variable is negative how to convert it into positive by using the dual simplex method. So, that we have done over here. Now let us see the flow chart for gomoris pure gomoris pure IPP.

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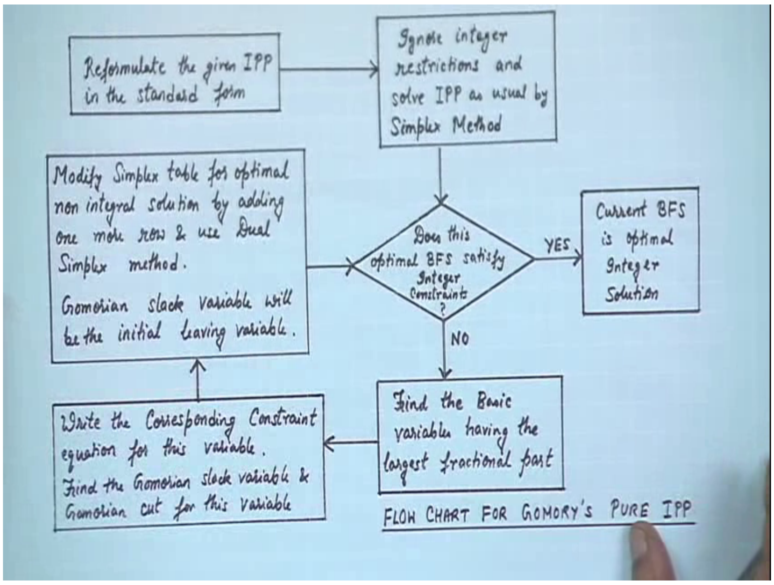

So, see come to the first one at first, because this is coming through the entire thing you are starting over here you are starting here reformulate the integer programming problem in the standard format; that means, make it into less than equals type constant and the problem should be maximization problem. Number 2 is ignore integer restrictions and solve the I.P.P as usual by simplex method. As I told you earlier ignore the integer restriction as if assume that your x j is only greater than equals 0. It may take integer value or real value and solve it by normal simplex method whether whatever we have done earlier.

So, next step is does this basic feasible solution does this optimal basic feasible solution satisfy integer condition. So, we are checking after getting the optimal solution that whether this b f s is integer satisfying integer constant or not. If yes, then for current basic feasible solution is optimal integer solution and you stop the process now if the current optimal b f s does not satisfy the integer constant. Then find the basic variable having largest fractional part please note this one.

Then you find out the basic variable having largest fractional part and once you have obtained the largest fractional part then use the corresponding gomorian constant equation for this variable, which I have written write the corresponding constant equation for this variable. Find the gomorian slack variable and gomorian cut for this variable this we will. So, wherever we are going through the example for this.

So, we are finding the gomorian slack variable and gomorian cut for this variable then modify the simplex table for optimal non integer solution. Modify the simplest table for optimal non integer solution by adding one more row and you use the dual simplex method. Because in the last one if we have shown here the value of the integer the value of the basic variable will be negative.

So, I cannot solve it by the normal simplex method. That is the reason we are writing here modify simplex table for optimal non integer solution by adding this one more row and use dual simplex method. Gomorian slack variable will be the initial leaving variable in this case. And then you repeat the process as we have done earlier. So, these are the basic steps which we will use.

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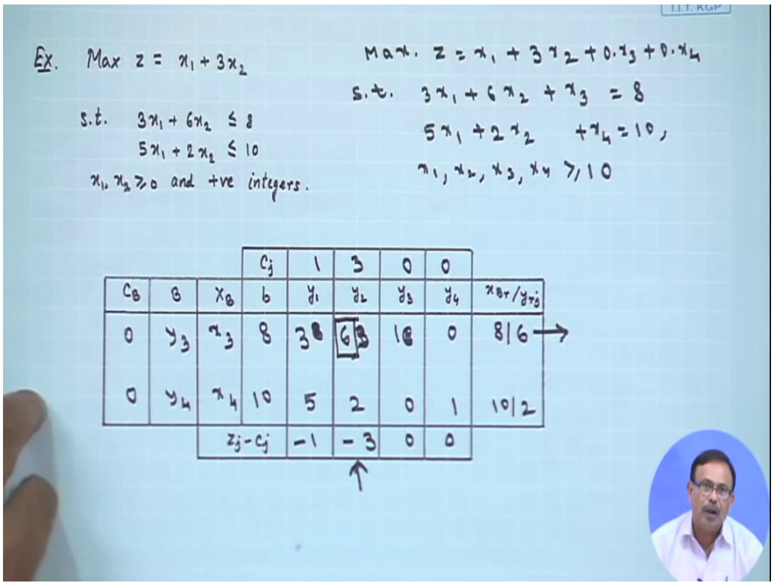

Now, let us take this example, we may not be able to finish this example, but let us start this one we have a problem maximized z equals 3 x x 1 plus 3 x 2 subject to 2 constants for simplicity we have taken one simple example and by which we will show you how to the algorithm for I.P.P works.

So, at first you write it in the standard format that is already it is maximization problem maximized z equals x 1 plus 3 x 2 subject to since both are less than equals type equation. So, introduce slack variables to make them equality. So, 3 x 1 plus 6 x 2 plus x 3 this is equals 8. And next equation is 5 b x 1 plus 2 x 2 plus x 4 this is equals to 10. And your x 1 x 2 x 3 and x 4 are greater than equals 0. So, here your slack variables are x

3 and x 4, therefore, in the basis x 3 and x 4 will go I will make x 1 x 2 as 0. Then x 3 equals 8 and x 4 equals 10.

So, in the basis your x 3 and your x 4 will come. So, you are writing 3 and x 4 your b values will be y 3 and y 4 c j s are 1 3 and corresponding to x 3 and x 4. Here it will be 0 into x 3 plus 0 into x 4. So, you are getting this one. So, this values are 0 0 your b values are 8 and 10. Now write down 2 rows as usual that is 8 3 6, sorry 8 already we have written this is 3 this is 6. This is one this is 0 and next one is 5 2 0 and 1.

So, form this original problem you are converting into the standard form. And after converting this one into the standard form you are writing the corresponding equation corresponding table you are formatting. So, z j minus c j, if you consider then minus 1 minus 3 0 and 0. Here you see z j minus c j less than equals 0, it is not satisfying the optimality condition, but x 3 and x 4 values are integer now.

So, we will use the normal simplex algorithm. So, therefore, this is the this will be the entering vector because maximum negative value occurs over here. So, from here it is 8 by 6 is the ratio this is 10 by 2 so; obviously, the minimum of these 2 is this one. So, once the minimum of these 2 is this one means your x 2 will enter into basis and x 3 will go out. So, once x 3 is going out. So, in that case your pivot element is this one.

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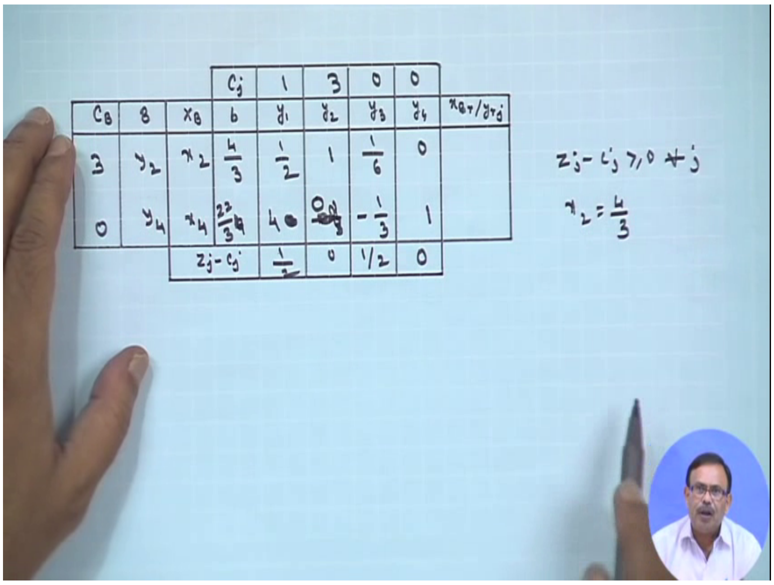

So, your pivot element is this. So, let us construct the next table where x 3 will go out and x 1 x 2 will enter. So, from here in the table here now, it will be x 2 and x 4 because x 2 is replaced by x 1 here it is coming as y 2 and y 4 c b values are like earlier that is one 3 0 0. And here it will be y 2 is 3. So, it is 3 and 0. So, like normal methods I have to make this one as one and this element as 0.

So, by doing this one what we are getting is 4 by 3 half 1 1 by 6 and 0. Whereas, this will be 4 0 minus 1 third and sorry this will be 22 by 3 this will be 4 this value you will get as 0 this will be minus 1, third this is one. So, z j minus c j is half 0 half and 0. So, here, if you find your z j minus c j z j minus c j is greater than equals 0 for all j.

So, optimality condition is satisfied, but if you see here your x 2 value is 4 by 3 which is non integer. So, since this is non integer therefore, your feasibility condition is not satisfied. So, you have to proceed or we have to check what to do next. So, what we will do in the next class, we will continue with this example.