Constrained and Unconstrained Optimization
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Lecture – 02
Assumptions & Mathematical Modeling of LPP

Now, let us start the next things. The come to the convex set. Earlier we have talked about the basis dimension and the spanning set.

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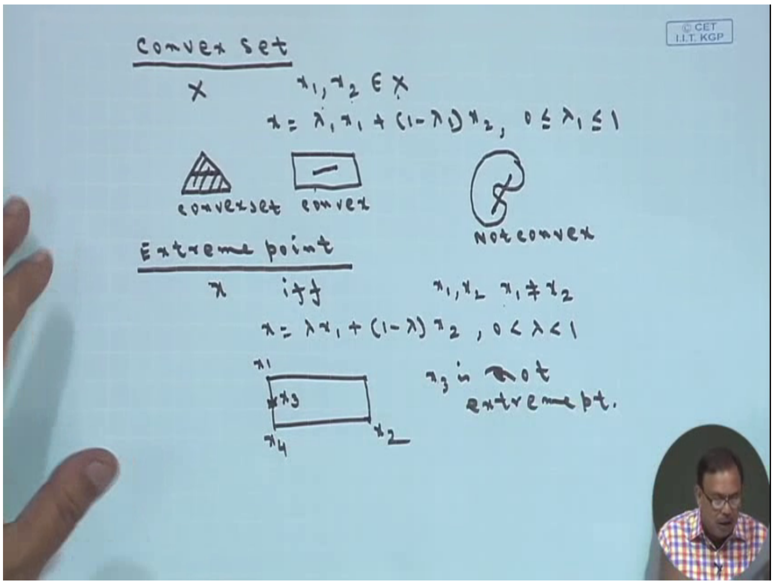

You have a set x, we say that the set x is a convex set. If we can find out the 2 points which belongs to x, and if I join these 2 points x 1 x 2, the line segment whatever you are obtaining will also lie in the set. That means, any point you take on the line segment that point also should belongs to x. Mathematically if I have to say then I will tell that if x 1 x 2 belongs to x, then if I can find out this 1 lambda 1 x 1 plus 1 minus lambda 1 into x 2, where lambda 1 lies between 0 to 1. Then x must be a member of set x. So, if this is satisfied for any 2 points x 1 x 2 belongs to x if x equals lambda 1 x 1 plus 1 minus lambda 1 into x 2, where lambda 1 lies between 0 and 1, and x must be a member of the set x.

For an example it will be easy if I take any triangle set. This is a convex set. You take any 2 points here. Suppose I am taking these and these if I join these, all the points lies inside this convex set only. Similarly, so this is a convex set. If I take a rectangle like this then any 2 points you take here you join them, they will all the intermediate points will lied inside this rectangle only. So, this will also be convex set, but if I take something like this, suppose I am taking this set. If I take a point here, if I take a point here, if I join you will see that some of the points are not in the set itself.

So, this one is not convex set. So, I hope it is clear that mathematical definition is this one and geometrically if I have to say then I will tell like this. From here one more thing comes that is extreme point. A point x is said to be a extreme point of a convex set if and only please not this one. If and only if that does not exist any 2 points x 1 and x 2 where x 1 is not equals to x 2, in the set there does not exist any 2 points x 1 comma x 2 x 1 does not belongs to x 1 not equals to x 2, such that x equals lambda x 1 plus 1 minus lambda into x 2 where lambda this holds.

So, and point one point x is and extreme point of a convex set, if and only if there does not exist any 2 points x 1 x 2 where x 1 and x 2 are not equal and they satisfy this one. That is that is not a linear combination convex combination of these 2. So, if you see this rectangle here, in this rectangle if I find let me draw it here, if I take a point x 1 here if I take a point x 2 here. These point x 1 is an extreme point of this rectangle for this point x 2 is a is any 2 points rectangle. Because I cannot find any 2 points x 1 and x 2 are some other 2 points, where if I make the join them then x 1 x 2 lies here, but you think this one x 3. If I denote this by x 4, then x 3 is not extreme point is not extreme point. Because the reason is that your x 3 lies whenever I am joining x 1 and x 4 x 3 is a combination of this x 1 and x 4. So, x 3 is a not an extreme point where as these x x 1 or x 4 or x 2 or x 3 are the extreme points.

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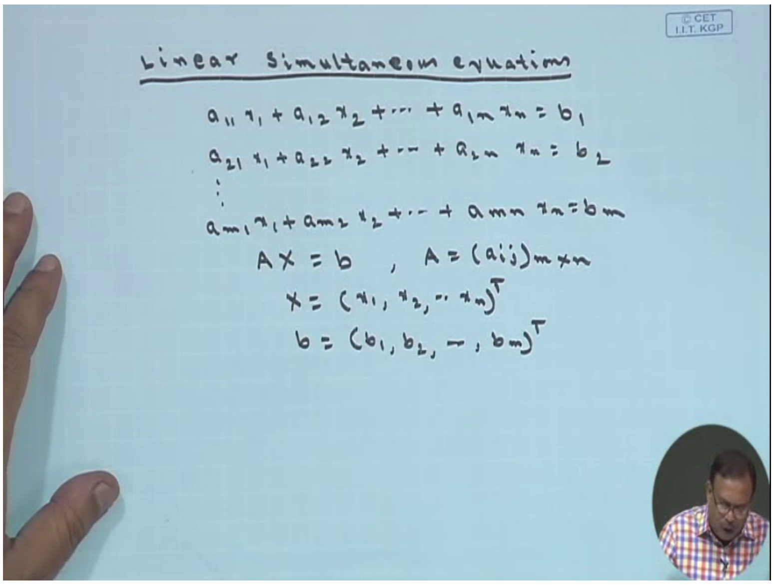

Already you have done just I am writing this one to talk about the linear simultaneous equations. Let us think about a set of equations a 1 1 x 1 plus a 1 2 x 2 plus a 1 n x n this is equals b 1 a 2 1 x 1 plus a 2 2 x 2 like this way plus a 2 n x n. This is equals b 2, and a m 1 x 1 plus a m 2 x 2 plus a m n x n equals b m; that means, you are having n variables and m equations here. M n variables x 1 x 2 x m and you have the m equations. In matrix notation I can write it in the form as A x equals set b, where a is the matrix I can write own a i j this is m cross n matrix. You are x will be nothing but the x 1 x 2 vectors and the transpose of that, and similarly your b will be b 1 b 2 and b m to the power transpose.

So, in other sense whenever you are some equations in matrix notation always we can write it in the form of A x equals b.

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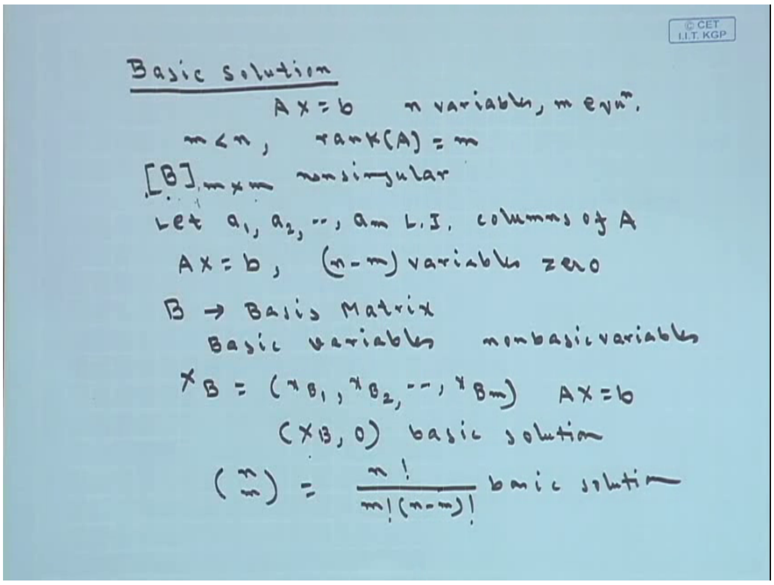

So, because now whatever we are going to talk this matrix notation will be very useful. Let us come to the basic solution of that system of equations. You have a system if equations A x equals b, with n variables I have n variables as I discussed earlier and you have m equations. So, and we are assuming that m is less than n that is number of equations a less than number of variables and rank of the matrix A is equals to m right.

So, since rank of the matrix is equals to m. So, therefore, there will exist one matrix B of order there will be one matrix B of order m cross m, which will be since rank is m please note that you may be knowing the definition of the rank. So, it will be nonsingular. So, I am getting a matrix B of order m cross m since the; I am assuming the rank of the matrix is m. So, I must get one m cross m square matrix which will be nonsingular. So, let a 1 a 2 a m these are the linearly independent columns. Since the rank is m therefore, I must get m linearly independent columns of a. So, a 1 a 2 a m are linearly independent columns. So now, your A x equals b, can be transformed into a system with m equations and m unknowns which I am representing by b. So, this is equivalent to a matrix B with m equations and m unknowns and the remaining n minus m variables what you are having, this n minus m variables will be 0.

So, 1 1 linearly independent columns are there. So, I have m decision variables in the matrix, from where I can obtain the solution of those decision variables. And the remaining variables we will make n minus m matrix. So, this matrix B we call it as the basis matrix, this matrix B is known as the basis matrix. Please note this one, matrix B how you are forming since the rank is m rank of the matrix A is m I must get n linearly independent column vectors and using those I am forming one matrix B which will be automatically nonsingular and this is non as the basic matrix.

Now, the this linearly independent columns are there. And corresponding decision variables are known as basic variables. So, in one sense the variable attach to the linearly independent columns are known as basic variables. And remaining n minus m variables we call it as non basic variables. So, I have some basic variables, I have some non basic variables, basic variables are the variables which are attached to the corresponding to the linearly independent columns. And the remaining variables n minus m variables are known as non basic variables.

So, basically I can say that x b equals x b 1 x b 2 something like this way x b m. These are the basic variables of the system A x equals b. And the solution can be achieved from here A x equals b solution whatever you are obtaining that will be x b comma 0. So, x b comma 0 is known as the basic solution, x b comma 0 is known as the basic solution of the system of equations A x equals b. So, I am not saying that this is the only solution this is the solution. I am saying this is a basic solutions how you are obtaining the basic solution? The basic solution we are obtaining by taking the rank that is n linearly independent variables from there, and I am forming a matrix B. I am associating the variables which are attached to these linearly independent columns which I am saying as the basic variables. And all other n minus m variables will be the non basic variables. And this x b comma 0 is basic solution.

So, if you have n variables and m equations you will have factorial n by factorial m into factorial n minus m basic solutions. So, maximum; what you can get that is by making all the possible combinations you can obtain factorial n by factorial n into factorial n minus m basic solutions? Just let us take a example because just we are giving the solution of a system of equations.

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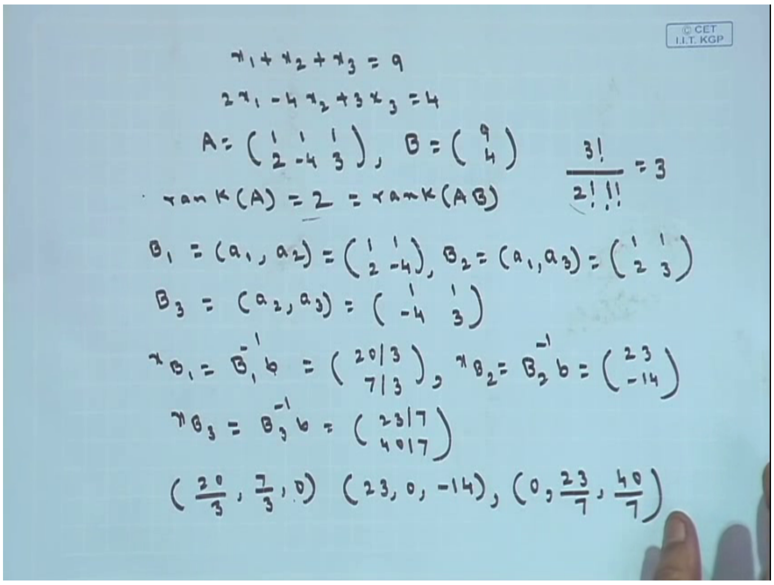

Let us take the equations like this x 1 plus x 2 plus 3 say this is equals 9. 2 x 1 minus 4 x 2 plus 3 x 3 this is equals say 4. So, in matrix notation if I have to write down your a will be 1 1 1; 2 minus 4 and 3. Your b is 9 and 4. This one b is 9 and 4. If you find the rank of this matrix A your rank of a will be 2, that I am leaving for you. you just check it very easy. So, rank of a and this is equals rank of A B. Also rank of a equals rank of A B if I take and which is equals to 2.

So, number of basics solutions here will be number of basic solutions will be factorial 3 by this one factorial 2 into factorial and whose value will be 3. So, what can be the possible basic solutions for here I can write down possible basic solutions like this. One can be a 1 comma a 2 a 1 comma a 2 means, I am telling this column as a 1 a 2 and a 3. So, this will be 1 1 2 minus 4. One can be this, since the rank is 2. So, I have to take 2 linearly independent columns b 2 can be a 1 comma a 3, b 2 will be then a 1 a 2 means one 2 1 3. And the third one which you can obtain b 3 as a 2 a 3 this 3 combinations are possible only this is 1 minus 4 and 1 3.

So, if you see I am obtaining this basic solutions b 1 b 2 b 3 like this. Since the rank was 2. So, the maximum number of basic solutions I can obtain using the formula factorial n by factorial m into factorial n minus m, this is equals to 3. So, I obtain b 1 b 2 and b 3 like this. So, what can be the basic solutions now? I can write down x b 1 equals b 1 inverse b. You know what is b 1? What is b? You know it if you calculate you can obtain these value these I am leaving for you because of shortage of time. Similarly your x b 2 if you see it will be b 2 inverse into b the result will be 23 minus 14. And for the third one x b 3 this will be equals to b 3 inverse b, and this is will be 23 by 7 and 40 by 7.

Therefore we can say that the basic solution one is this is which one this corresponds to a 1 a 2. So, a 3 will be 0. So, one solution for this problem can be 20 by 3, 7 by 3 and 0 a 3 is 0. For this case it is a 1 a 3 that is a 2 0. So, it will be 23 0 and minus 14 this is another solution, and the third solution this is a a 2 a 3 that is a 1 is 0. So, this will be 0 23 by 7 and 40 by 7. So, if you see here; what is a happening, in the process of finding the solution basically we are trying to find out the basic solution at first. How to find the basic solution? For that we are using the linearly independent columns or the rank from the rank we can obtain it. Once I am getting the linearly independent columns of the matrix in that case I am forming a matrix B 1 B 2 B 3 like this. Where your b's are combination of these linearly independent columns, a 1 a 2 and a 3 and from there we are obtaining the solution.

So, this we call afterword you will see that this is an initial solution. And from this initial solution we have to improve it and we have to obtain the optimum solutions. So now, let us come to the original thing.

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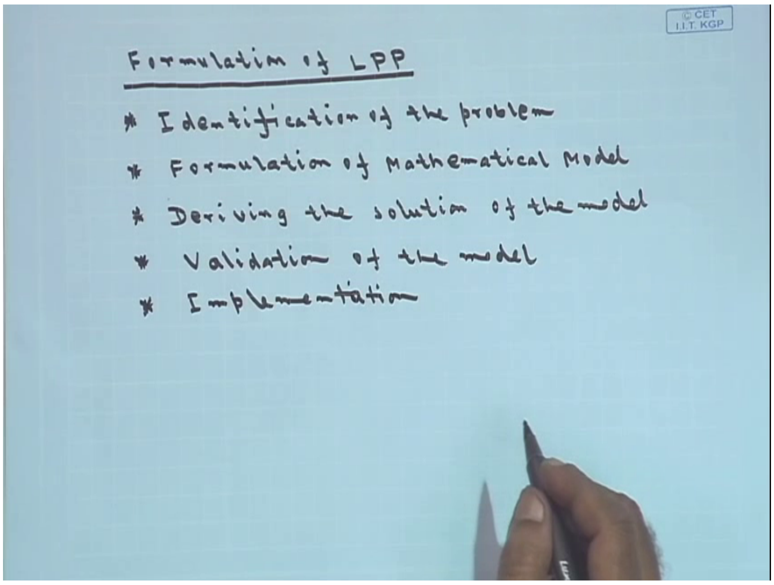

That is formation of formulation of linear programming problem I am not writing the full form. In the linear programming problem basically I have to take certain decision, subject to certain constraints whatever we are having. Suppose I want to set up one factory to set up the factory I need the man power I need the land I need that technical though, how I need there may be have certain constraint on my cash flow.

So, I have to find out the solution that what should I do or how much quantity should I produce. Which will satisfy my constraints. So, as we have seen in LPP we can formulate that thing the decision making problem basically consist of certain steps. First is just I am writing the steps. Identification of the problem. Basically you have to understand what the problem is because if you are understanding is wrong then the formulation of the model always will be long.

So, first I have to identify and I have to understand what the problem is. So, once I have identify then next step will be formulation of mathematical model. Here if you see whatever problem is been given I have to analyze it, if I want to analyze it a from the problem that is from the hardcore problem I have to convert into some model. Usually we convert it into some mathematical model, because it has been observe that mathematical models are very easy to analyze.

Therefore once we are trying getting the problem, we will formulate the corresponding mathematical model corresponding to that. This will be your second step. The third step is deriving the solution of the model. Now what technique or what methods you will find to solve the mathematical model for that I have to find out this one deriving the solution of the model. And optimization comes into picture basically at this stage. Optimization is the technique by which you can find out the solution of some mathematical model. And the last step is not last step, the validation of the model. Means you have to test the model whether it is giving proper result or not. If required you have to improve it and improve your technique. So, that you can obtain the better result. So, this is the next part is validation, and the last part will be implementation.

Implementation comes into picture, because whenever you will take about the various models in that case number of variables number of constraints can be many. Many means I may have to find out and problem where I may have 100 variables. I may have 200 equations, and which is not possible to make hand calculation. So, I have to use certain techniques and I have to computerize it, I have to write down certain algorithms. So, for that part our implementation comes into picture.

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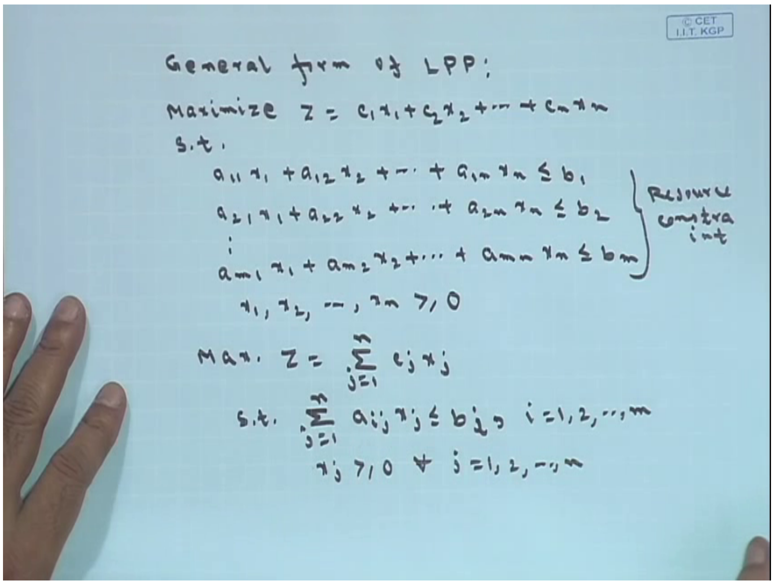

So, the general form of LPP, if I have to tell general form of linear programming problem. It is I am writing maximize it can be minimize also I will come to that maximize or minimize z equals c 1 x 1 plus c 2 x 2 plus c n x n. Subject to a 1 1 x 1 plus a 1 2 x 2 plus a 1 n x n which is less than equals b 1. A 2 1 x 1 plus a 2 2 x 2 like this way plus a 2 n x n less than equals b 2. And a m 1 x 1 plus a m 2 x 2 like this way plus a m n x n less than equals b m. And this we call as the resource constraint, resource constraint and please note this one, the decision variable should be x 1 x 2 x n greater than equals 0 which we call as non negativity constraint.

So, in an LPP please note the decision variables with which we want to work all must be nonnegative. And we have to optimize the function it can be maximize or minimize the function, z equals c 1 x 1 plus c 2 x 2 plus c n x n. Subject to satisfying certain constants. And please note that linear programming problem means from the name itself it is clear the objective function as well as the constraints will be linear in nature. The variables will be only (Refer Time: 25:51) non negative it may be discrete it may be continuous.

So, if I have to write down in the compact form I will say maximize z equals summation j equals 1 to n c j x j, subject to summation j equals 1 to n a i j x j less than equals b i where i equals 1 to m. And x j greater than equals 0, for all j equals 1 to n. So, this is in compact form maximize z equals summation j equals 1 to n c j x j subject to summation j equals 1 to n a i j x j less than equals b i and x j greater than equals 0. This again we can write it in the matrix notation also, which becomes much easier to represent maximize z equals.

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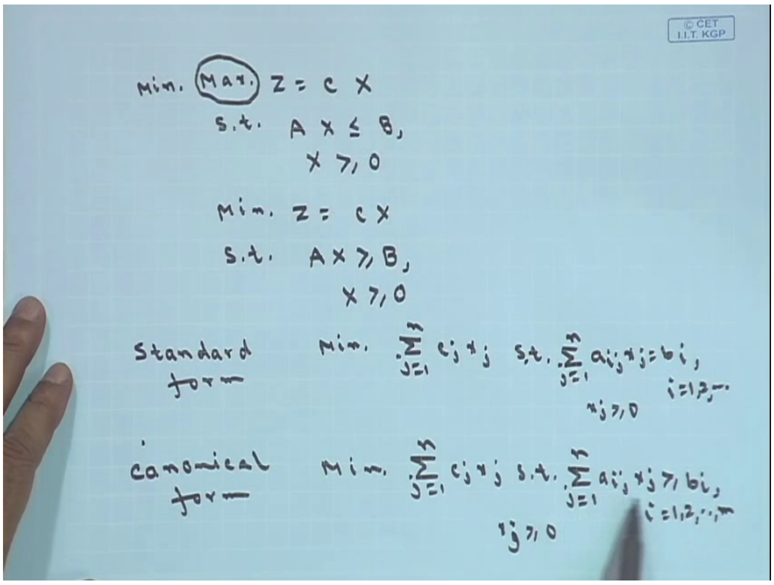

We are writing it as c into A x subject to A x less than equals b/. And x greater than equals 0, where c x A B all are the column vectors I can represent it from there. And if I have to minimize the problem then in place of maximize here it will come minimize the function, but sometimes what we do in general cases we write down minimize z equals c x subject to A x greater equals b and x greater than equals 0.

So, basically if you see for minimization the constraint we are taking greater than equals and for this one we are taking the constraint as less than equals, and if you see in LPP then basically you have m variables because here if you see we are having m equations and n variables. I have to find out the solution of this n variables which will be written which will satisfy the constraints.

So, in stand up form we can write down 2 things, I am just writing standard form and also in standard form if I have to write down I am writing minimize j equals 1 to n c j x j subject to j equals to 1 to n A x j equals b i, I will take values like this and x j greater than equals 0. So, minimize j equals 1 to n c j x j subject to a i j x j equals b i, in standard form we are means whenever I will convert it I may have inequality, but before solution I have to transform it in to the equality. And here I am writing the canonical form in canonical form it is minimization I am a just writing not the maximization c j x j subject to j equals to 1 to n a i j x j which will be greater than equals b i again I will take the values one to b will take one to m and x j greater than equals 0.

So, here in general canonical form means what whenever I will formulate the problem I the constraint will be maybe equality may be inequality. It will be greater than equals or less than equals depending upon the I am minimizing or maximizing the problem. But whenever I am trying to find out the solution then the inequality I have to convert it into equality and then we will find out the solutions. So, next we will see in the next class what kind of assumptions we have used to find the solution of LPP and then we will go for the graphical solution and other things.