Constrained and Unconstrained Optimization
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Lecture - 18
Introduction to Duality Theory- I

So, let us continue with the previous class; in the previous class, we were doing one example on sensitivity analysis that for changes in the parameters in the coefficients of the LPP problem; what can be the possible effect on the optimal solution of that particular problem. The last one which we have is which resources should be increased or decreased to get the marginal increase of the objective function.

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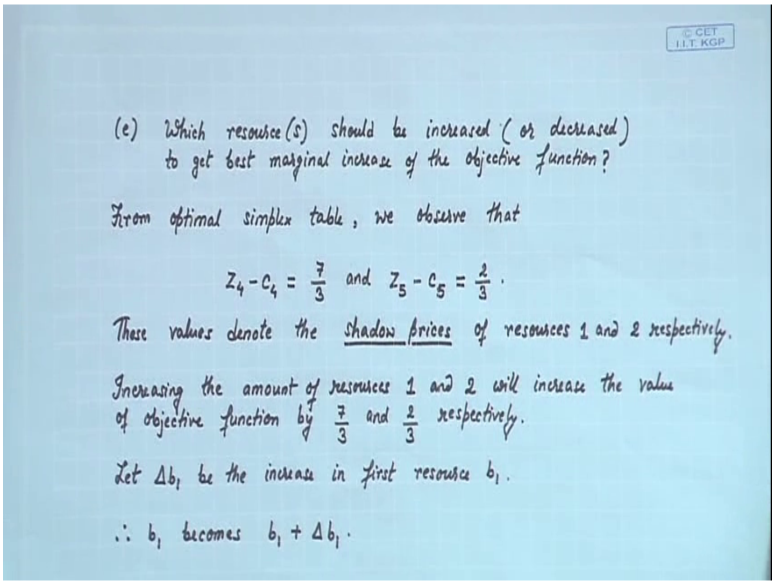

So, the which resources should be increased or decreased to guess the optimal marginal index of the increase of the objective function.

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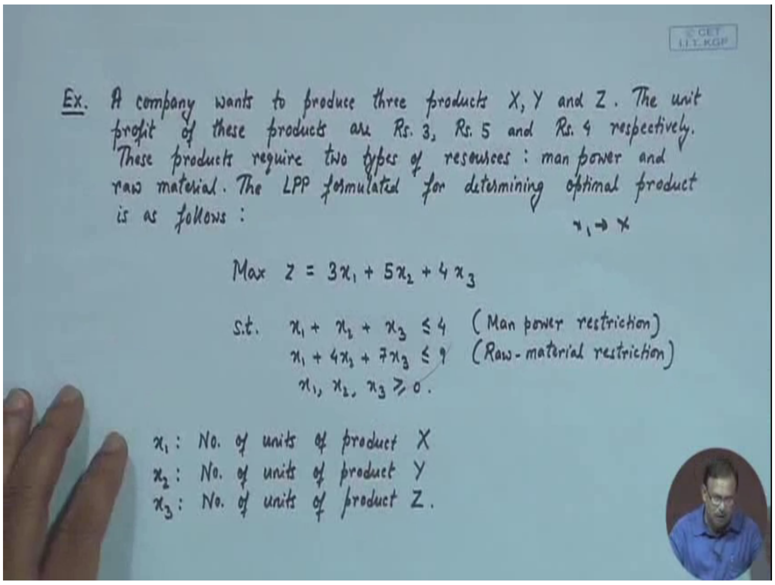

If you just quickly let us recall this was the original problem maximize z equals this 3 x 1, 5 x 2 plus 4 x 3 subject to this constraints.

Basically you are producing 3 type of products capital X, capital Y and capital Z; their unit prices are given as 3, 5 and 4 and for this particular problem if you recall your solution was this table the last table was the solution that is x 1 equals 7 by 3 x 2 equals 5 by 3 x 3 equals 0 and z star was 46 by 3.

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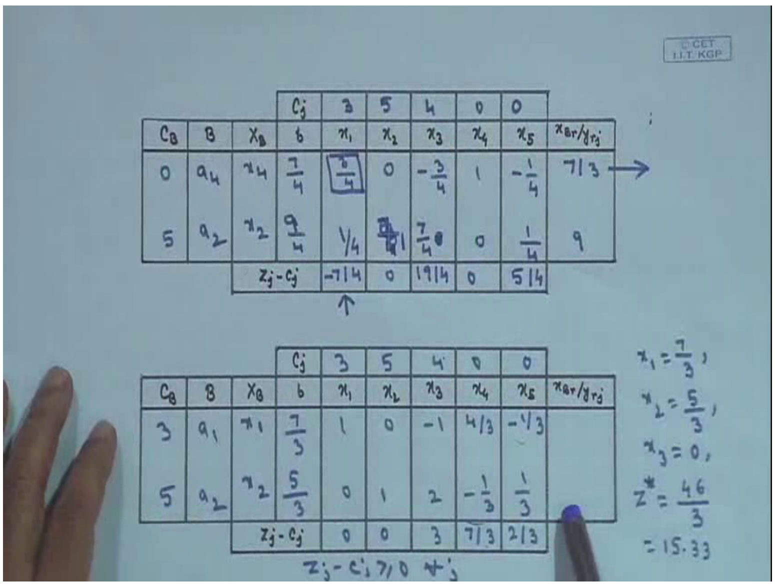

The basic variable was here x 1 and x 2 and the others were the non basic variables. So, this was the initial optimal solution. So, from here; if you see the problem now which resources should be increased or should be decreased to get the best marginal increase in objective function now from optimal simplex table if you see from optimal simplex table the value of z 4 minus c 4 is 7 by 3 and z 5 minus c 5 is 2 by 3.

From the optimal table; so, here we have written z 4 minus c 4 is 7 by 3 and z 5 minus c 5 equals 2 by 3 these values we are denoting as shadow of resource 1 and 2 respectively. So, these are the positive values. So, we have taken these and we are saying these are the shadow price of the resources 1 and 2 respectively; now increasing the amount of resource 1 and 2, we will increase the value of the objective function by 7 by 2 and 2 by 3 respectively.

So, if I increase the amount resource one then the objective function will increased by 7 by 3; whereas, for resource 2 it will be increased by 2 by 3. Now suppose delta b 1 be increase in the first resource of the b 1 that is if a b 1 is the initial resource and we are increasing it by delta b 1.

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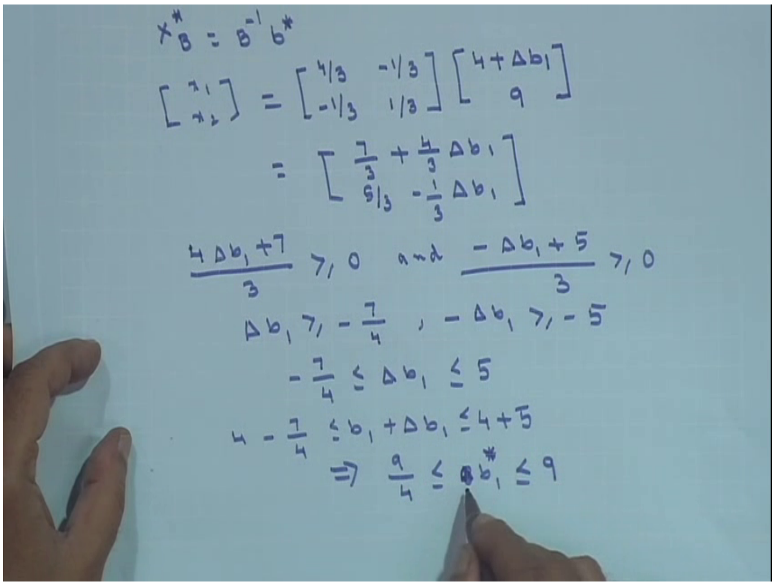

So, that b 1 becomes b 1 plus delta b 1. So, in this case, you have to in the increase in data b 1 is the increase in b. Now your x B star x B star this is equals b inverse b star this value of b star or in other sense x 1 x 2; this I can; this equals I can write down from here the corresponding value I have to take here; this I have taken as x 4 and x 5. So, b inverse matrix is nothing, but this one this is your b inverse 4 by 3 minus 1 by 3 minus 1 by 3 and 3.

So, here we are writing this one as 4 by 3 minus 1 by 3 into minus 1 by 3 and the last one is 1 by 3 and your b value to be increased; B will be increased this is 4 and this one will be 4 plus delta sorry into 4 plus delta b 1 and this one will come as 9. So, this is your b inverse into b star b star you are writing the 4 plus delta b 1 original value was for b plus delta b 1 and for the second one the value from this table it is this is 9; here the second resource was 9 and first resource was 4. So, these 4 has been increase to 4 plus data b 1 and this has become 9. So, that is sequence you can write down 7 by 3 plus 4 by 3 into delta b 1 and 5 by 3 minus 1 by 3 into delta b 1.

Now, this x 1 x 2; this value should be greater than equals 0; that means, individually these 2 should greater than equals 0. So, that 4 delta b 1 plus 7 divided by 3 should be greater than equals 0 and minus delta b 1 plus 5 by 3 should be greater than equals 0. So, from here you are getting delta b 1 is greater than equals minus 7 by 4 and this one is minus delta b 1 is greater than equals minus 5 if I combined these 2, then minus 7 by 4 is less than equals delta b 1 less than equals 5 .

So, I can down original value was 4. So, 4 minus 7 by 4 less than equals b 1 plus delta b 1 which is less than equals 4 plus 5 or in other sense, you are this implies your 9 by 4 less than equals delta b 1 less than equals 9. So, from here it is quite clear that since the range is 9 by 4 to 9; if the value of the first resource b 1 is increased from 4 units to maximum units of 9 units, then the objective function the optimal solution will remain the same there will be no change.

So, for the first resource from the original value 4; this was the original value was 4 from the original value 4; if we increase it up to 9 there will be no change in the optimal solution.

In the same way now let us check the second one that is b 2 resource b 2 means this value if this value how much we can change.

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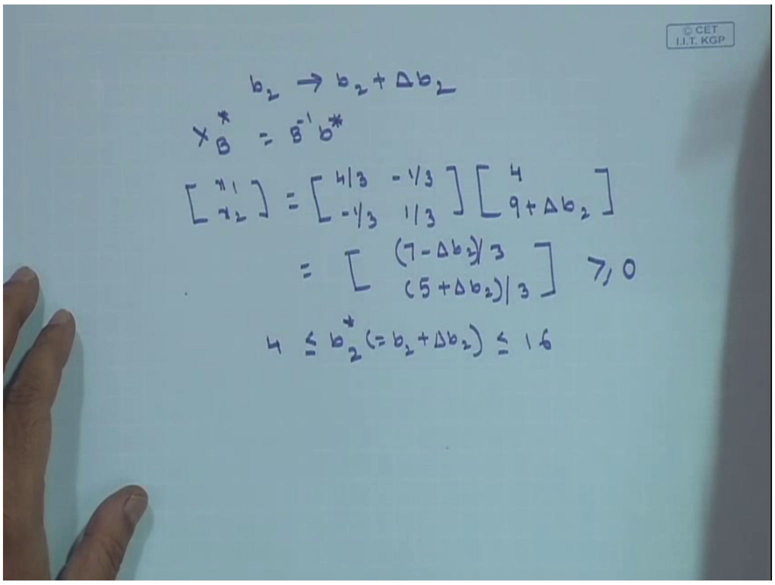

So, your b 2 set becomes b 2 is changed to b 2 plus delta b 2 suppose I am making the change. So, again following the same steps x B star equals b inverse b star from here I can write down x 1 x 2, this is equals to again this one will come 4 by 3 minus 1 by 3 minus 1 by 3 and 1 by 3. So, this will be 4 by 3 minus 1 by 3 minus 1 by 3 into 1 by 3 here for the earlier case original value of resource 1 is 4.

Whereas original value of resource 4 is 9 and there is an increment of delta b 2. So, that you will get this thing; so, this value will be equals to 7 minus delta b 2 divided by 3 7 minus delta b 2 divided by 3 and next one will be 5 plus delta b 2 divided by 3. So, we are getting this thing. Now for feasibility this should be greater than equals 0 and if we get this one from here just following the earlier method that is flowing the same method you can get the lower bound and upper bound and you can if you combined both of them then ultimately you will find 4 less than equals b 2 star which is nothing, but n which is equals to b 2 plus delta b 2 is less than equals 16.

So, that you can say that these value always actually this should not be delta b 1 plus delta b 1. So, this I can say b 1 star for the earlier case it will be delta 1, but b 1 star and here it is b 2 star. So, b 2 star can take the value from 4 to 16 or in other sense its original value of resource 2 was 9 if you increase it up to 16 then also the problem the optimal solution of the original problem will be as it is. So, like this way whenever I have the original problem I can see how much deviations can be given. So, that the optimal solution will remain same I do not have to calculate it again and you have to find out the optimal solution now next come to the next topics that is introduction to duality theory.

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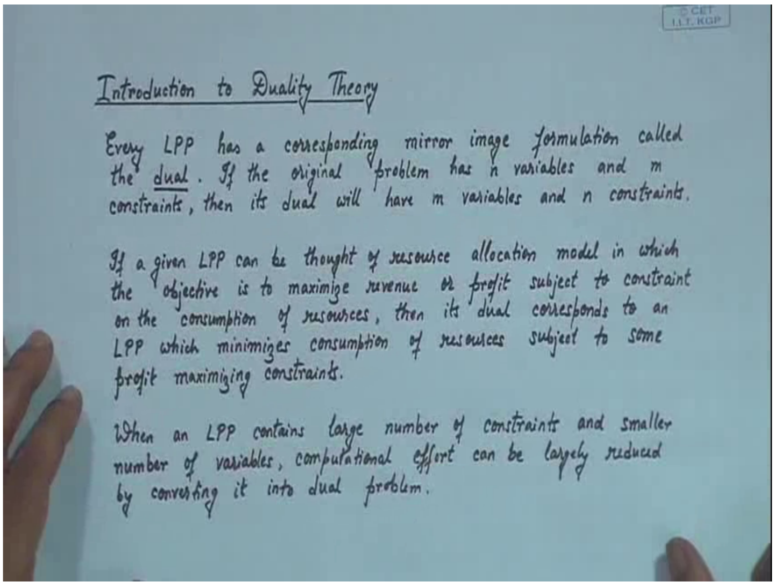

Every LPP has a corresponding mirror image formulation which is known as the dual of that LPP. So, please note this one we are saying that every LPP as a mirror image formulation that is if you keep it on the mirror then image formulation will be like that if the original problem has n variables and m constraints then its dual will have m variables and m constraints as we have written here. So, please note this one that if the original problem has n variables and m constraints then its dual will have m variables and n constraints if a given LPP can be thought of a resource allocation model that is in which the objective is to maximize the return appropriate subject to some constraints.

This is the original LPP usually we always say LPP as a resource allocation model where I have to optimize maximize the objective function which may be the return or the profit subject to certain constraints on the completion of resources subject to this one then its dual will correspond to an LPP which will minimize the consumption of resources subject to some profit maximizing constraints. So, basically if you are maximizing a return or profit subject to certain constraints of on consumption of sources then its dual will be the opposite one that is minimization of the consumption of resources subject to profit maximizing constraints. So, when an LPP contains a large number of constraints and smaller number of variables.

Then computational effort can be largely reduced by converting it into an dual problem please note this one that why should I go for the dual problem because ultimately if you see in the dual problem as well as in the original LPP in both case your optimal solution will remain same optimal solution will not change. So, one reason for going for dual LPP is that if I have a large number of constraints then to find the feasible solution of that large number of constraints is time consuming and computationally it takes more time whereas, if I have smaller number of variables; if I converted into that responding well problem then what will happen my large number of constraints will go to the minimization one and since there was smaller number of variables.

So, number of constraints will reduce. So, our aim is if I have less number of constraints then my solution; I can get quickly or computationally it will take much less time that is one of the basic reason for converting one original LPP into its dual form and then find out the solution of the dual problem and then again try to find out the solution of the original problem from the dual problem. So, please note this one if I have a large number of constraints whereas, number of variables are less in that case it is beneficial to convert the original problem into its corresponding dual problem because whenever I am converting it into the corresponding dual problem then number of constraints in that case will reduce and number of variables will increase and since number of variable constraints are less.

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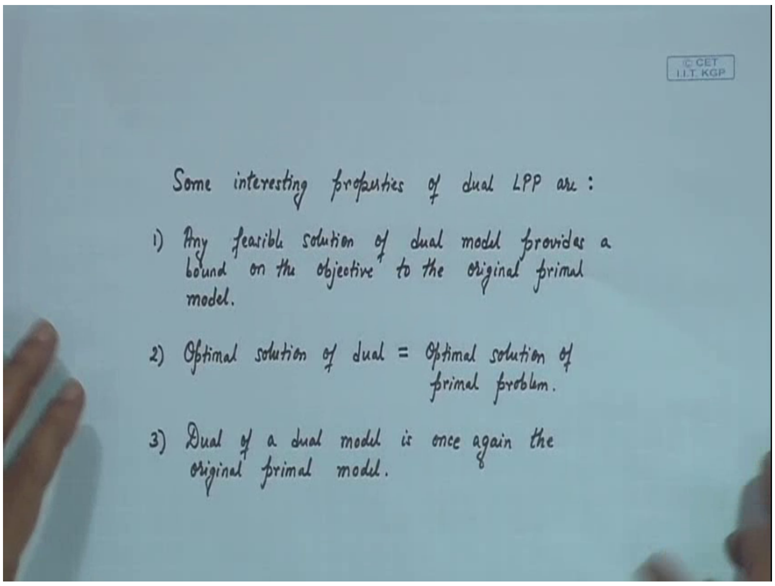

Therefore computational time will be much less. So, some interesting properties on dual LPP number one any feasible solution of a dual model provides a bound of the objective to the original primal problem. So, any feasible solution of dual model provides a bound on the objective to the original primal problem optimal solution of dual equals to optimal solution of the primal problem. Please note this one optimal solution of a dual is equals to the optimal solution of a primal problem third point is dual of a dual model is once again the original primal model that is you have the original model you are taking dual of the problem then on the dual again.

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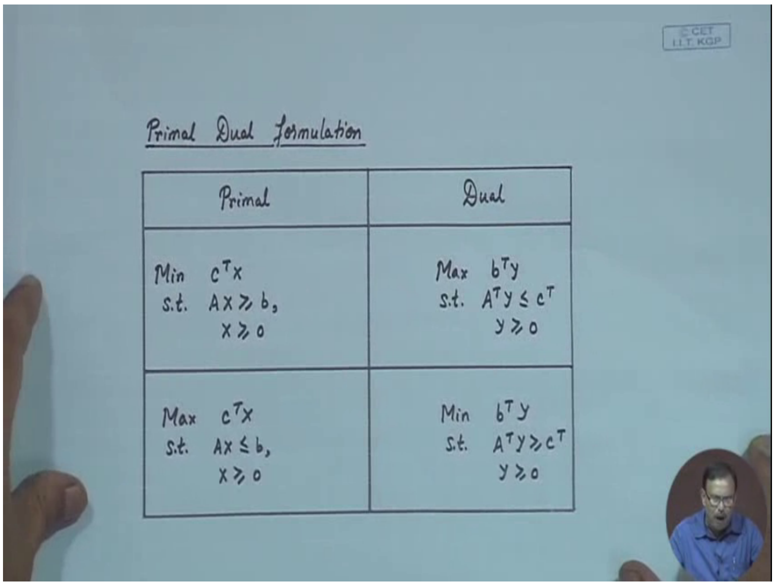

If you take dual you will get at back the original model. So, we can say that your primal dual formation.

You have the primal problem either you can write it in this form minimize C T X that is minimize C T X subject to x greater than equals 0 where its dual will be maximize b transpose y; some new variable we are writing. So, the here the resource will be this one b transpose y subject to a transpose less than equals this c transpose. Similarly if we have the maximization problem whatever we are doing in general maximize c transpose x subject to x less than equals b x greater than equals 0; this will be converted into a minimization problem minimize b transpose y subject to a transpose y is greater than equals c transpose.

So, please note that if the original primal problem is maximization its dual will be minimization whereas, in the subjective condition if x less than equals b in that case it will be greater than equals constraints will come and this coefficient will be from the objective function that is a transpose y is greater than equals c transpose y; so, for the canonical form of the primal and dual problem can be written like this.

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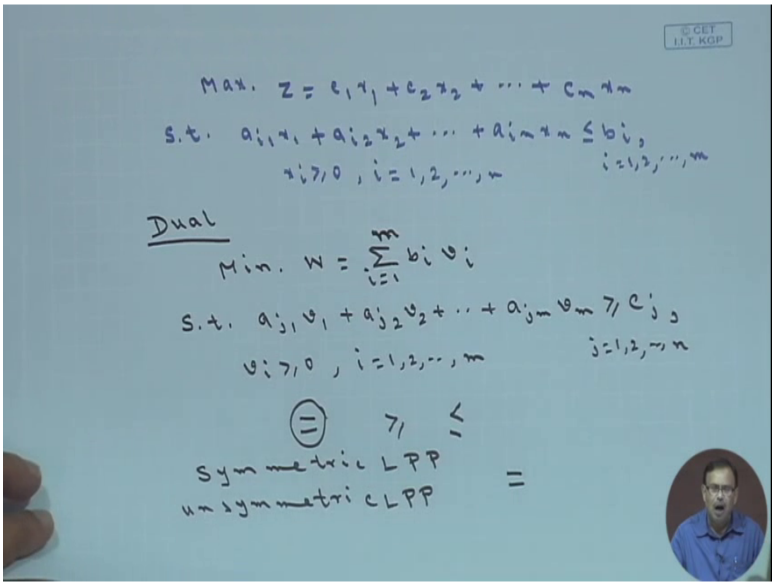

You have the problem maximize z equals c 1 x 1 plus c 2 x 2 like this way plus c n x n subject to a i 1 x one plus a i 2 x 2 plus a i n x n less than equals b i where your i lies between 1 2 n i lies from one to m and x y greater than equals 0 where your i is equals one to like this way it is n.

So, x 1 x 2 are called are unknown as the primal variables x 1 x 2 x n; we call it as primal variables and this z we call it as the primal objective function. So, this variables x one x 2 x n are the primal variables where as z is the original objective primal objective function. So, the corresponding dual of this problem corresponding dual of the problem will be since it is maximization problem. So, it will be now minimization. So, the function will be changing w equals i can write its summation i equals 1 to n b i into v i sum new variable summation i equals 1 to n b i into v i and this one subject to this will be sub 1 2 m. So, subject to a j 1 v 1 plus a j 2 v 2 like this way a j m v m.

Since this was less than equal. So, it will be greater than equals c j where j will be from one to n. So, basically if you see and of course, v i greater than equals 0 where I is equal to 1-2 like this way to m. So, in the original primal problem you had n variables and you had m constraints like this then variables was x on1e x 2 x n since you have m constraints in the original problem therefore, in the dual problem you will have m variables which we are denoting as v one v 2 v m and the number of constraints will be n your j is varying from 1 to n. So, subject to this one; so, if the constraints are in equality type that is greater than equals or less than equals type you can always convert it as usual by changing greater than equals in to less than equals or if you have equality type then this equality type.

You can convert into 2 types 2 inequalities one by making greater than equals another by making the less than equals type which we have discussed this one. So, this we will assume as the standard canonical form of the original problem and if I it is not in the canonical form first I will convert my LPP into canonical form and then I will write down its dual in this particular format. So, please note that this primal and dual problem they are consisting of constraints with inequalities they are always call symmetric LPP they are known as symmetric LPP. So, please note that primal and dual problems consisting of inequalities are known as the inequality constraints are known symmetrical LPP.

Whereas if the primal or du[al]- problem primal LPP has only equal type of constraint only equal type constraint that we call it as un-symmetric we call this one as un-symmetric LPP. So, please note this one if the in inequality if in the constraint you have greater than equals inequality or less than equals inequality then the primal and dual problem is known as symmetric LPP whereas, if the constraint or a contents only equality type then we call it as un-symmetric LPP now see 2 theorems over here.

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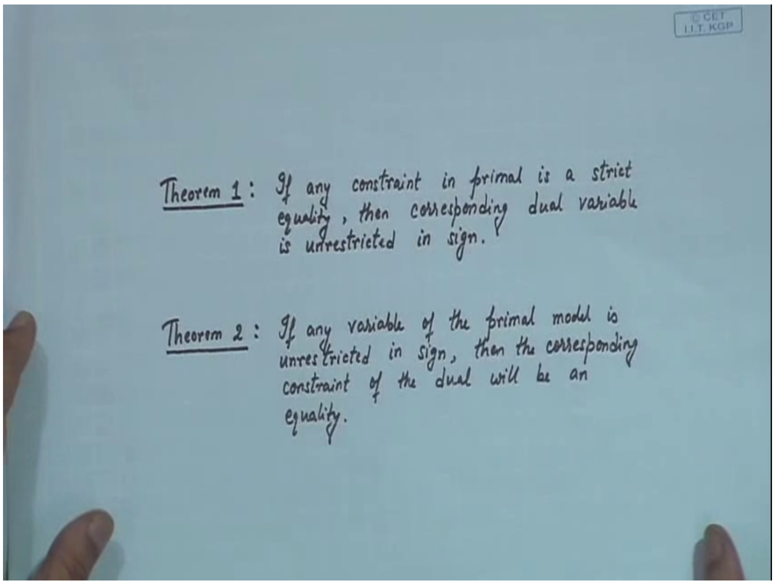

If any constraint is primal if any constraint in primal is strict equality please note this one is strict equality then the corresponding dual variable is unrestricted in sign.

So, please note this one in the primal problem if any constraint is there which of the equality type, then the corresponding dual variable will be the unrestricted in sign similarly if any variable of the primal model is unrestricted in sign, then the corresponding constraint of the dual will be of equality type; that means, these 2 are opposite once in first theorem says that if any constraint in the primal problem is of equality type, then corresponding dual variable corresponding to that constraint will be unrestricted in sign. Similarly if any variable in primal model is unrestricted in sign then the corresponding constraint of the dual problem will be of equality type.

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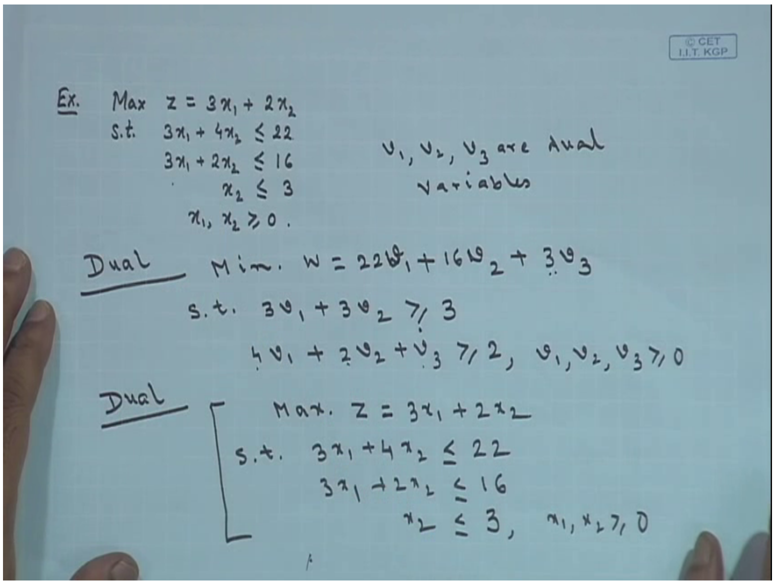

So, now let us see how to convert a particular LPP into its corresponding dual problem; let us take this problem maximize z equals 3 x 1 plus 2 x 2 subject to 3 x 1 plus 4 x 2 than equals 22 3 x 1 plus 2 x 2 less than equals 16 and x 2 less than equals 3; if you note the problem is already maximization type and all the inequalities are less than equal type; that means, this particular problem is in canonical form since I have 3 constraints in this. So, in the dual problem in the dual problem in the objective function there will be 3 variables since I have 3 constraints here. So, let v 1, v 2 and v 3 are dual variables . So, I am assuming v 1, v 2, v 3 are the dual variables. So, what is the problem now maximization will be converted into the this one minimize w equals; what will be the objective function the objective function is 22 v 1 22 v 1

Plus these v value 16; sorry, 22 v 1 plus 16 v 2 plus 3 v 3 subject 2; I have to take these rows first rows here that is first constraint is corresponding to the variable v one second con constraint corresponds to v 2 and third constraint corresponds to v 3. So, I have to write down 3 v 1 plus 3 b 2, but for x one nothing is there in the third constraint. So, so basically what you are writing the coefficients of the variable x one in each constraints and you are associating with the variables v 1 v 2 like this. So, for the first one it will be subject to 3 b 1 plus 3 v 2 greater than equals value of c j.

Here the corresponding to x one variable is 3. So, here it will be 3 similarly for the next one it will be 4 v 1 plus 2 v 2 plus sorry, 4 v 1 plus v 2 plus third constraint corresponds to the variable b 3. So, plus b 3 greater than equals the coefficient associated with the variable x 2 that is equals 2 and; obviously, v 1, v 2, v 3 greater than equals 0. So, the dual of this one original problem is this one. So, if I take this thing. So, I hope it is clear that whenever you are taking the dual problem then the minimize maximization problem will be converted to minimization and minimize w equals.

How many variables will be there it depends how many constraints you have since you have 3 constraints. So, I am writing twenty 2 b 1 plus 16 b 2 plus 3 b 3 subject 2 for the first column for the first column of the constraints you are taking you will take the coefficients here it is 3 x w 3 v 1 plus 3 v 2 for the second for the third constraint there is no coefficient for x 1; x 1 is 0. So, 3 v 1 plus 3 v 2 greater than equals 3; this 3 is coming the as the x 1 is the coefficient of x 1 in the objective function is 3; similarly, we will take the second column that is 4 v 1 plus 2 v 2 plus v 3 which is greater than equals the coefficient of x 2 that is 2 in objective function.

Again if you take the dual of this problem now; now this problem has how many constraints you will see this problem has 2 constraints. So, it is this dual of this problem will have 2 variables. So, I can write down since it was maximize minimization it will be maximization maximize z equals 3 x 1 from here, it is coming plus 2 x 2 subject 2 what happens subject 2 3 x 1 plus 4 x 2 3 x 1 plus this 4 corresponding to v 1 coefficients you are taking 3 x 1 4 2; this will be less than equals because it was greater than equals less than equals coefficient of v 1 that is 22 in the objective function. Now take the second column that is column corresponding to v 2 in the coefficient that is 3 x 1 plus this is 2 x 2.

Which is then equals the coefficient of v 2 that is 16 in the objective function. Now you take the third column corresponding to v 3 here it is only x 3 will come sorry; x 2 will come x 1 and x 2. So, in the first one v 3 is not there. So, x 2 less than equals this is coefficient of v 3 corresponding to in the objective function that is 3 and your x 1 x 2 greater than equals 0 and if you see this problem this problem is nothing, but the original problem therefore, dual of the dual gives you back the original problem. So, I hope it is little clear. In the next lecture, we will solve one to more problems on this.