Constrained and Unconstrained Optimization

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Lecture – 17
Problems on Sensitivity Analysis

So, let us continue with the earlier class, where we have done the theories of the sensitivity anilities, analysis that is for changes in different parameters. What are the ranges for which the optimum solution and the feasibility will remain? Unchanged, also if I add one variable what happens? Or if I add one constraint what happens? So, let us take one example in the quickly let me look after. Already I have discussed these problem.

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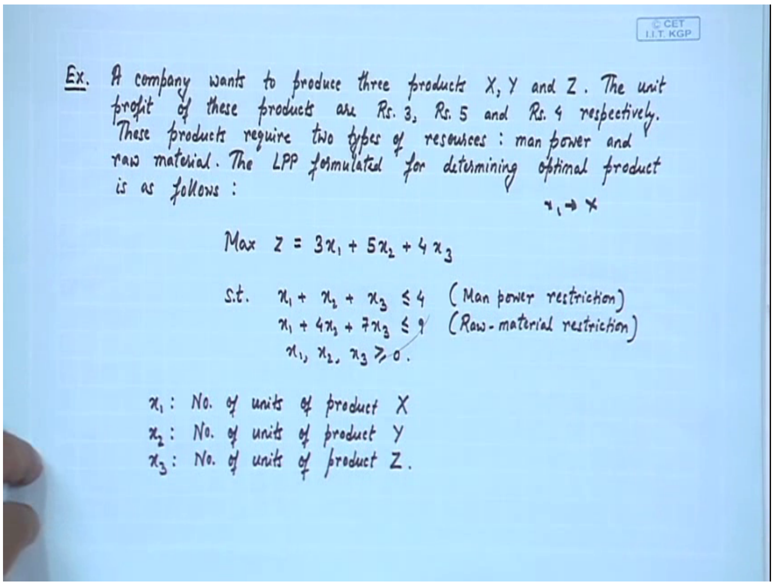

So, this is your problem maximize the subject to this map these we have told maximize z equals this one subject to this. And I have to we have given the 5 different questions over here. Let us take the questions one after another.

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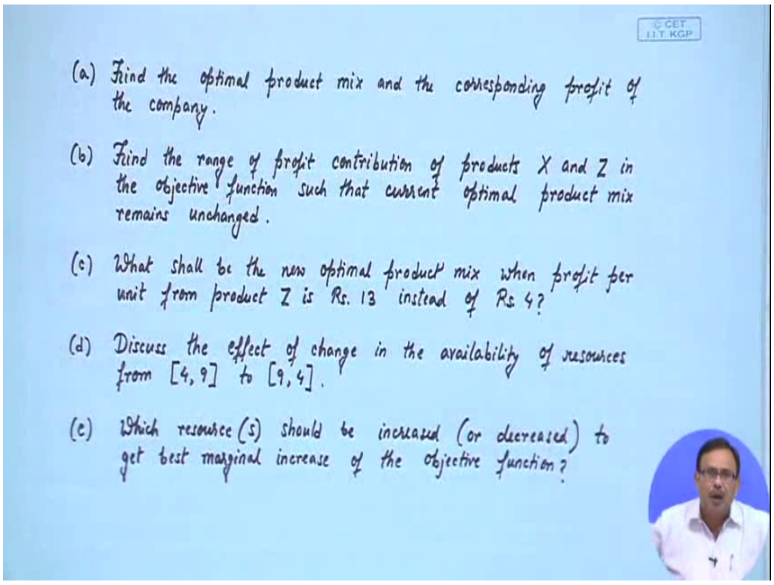

The first one if you see the first one is find the optimal product of the mix, optimal product mix And the corresponding profit of the company.

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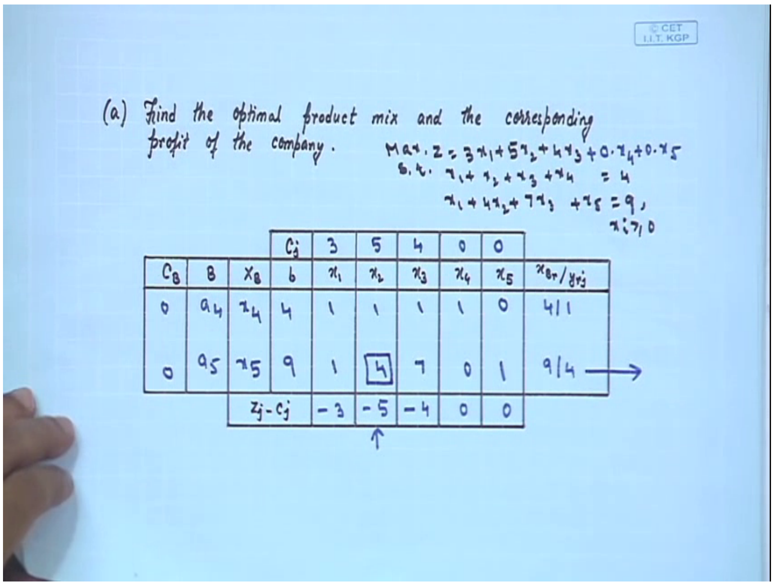

Or in other sense if I have to say I will tell that you just find out the solution of the simplex problem whatever we have given. So, your simplex problem was this. We have 2 less than equals constraints. So, whenever I try to write it in the standard form. I have to add 2 slack variables over here x 4 and x 5. So, your problem is maximize z equals 3 x 1 plus 5 x 2 plus 4 x 3, subject to x 1 plus x 2 plus x 3 plus x 4 this is equals 4. Second one is x 1 plus 4 x 2 x 1 plus 4 x 2 plus 7 x 3 plus x 5 this is equals 9. X i is are greater than equals 0, and in this case if you see here it will be 0 into x 4 plus 0 into x 5.

So, your original problem is this which you have converted into standard from maximize z equals this by introducing this one. So, as usual initial basic feasible solution will be x 1 equals 4 x 5 equals 9. So, that your basis will be the basis we will have 0 variables x 4 and x 5. So, once I am writing these 2 variables x 4 and x 5 say, this will be your a 4 and a 5. We will not spend more time on this because we have done all. These things earlier 3 5 4 objective coefficients we are writing here. So, a 4 a 5 it will be 0 0. Your b is 4 and this is 9. So, b is 4 and 9. So, it will become 1 1 1 1 and 0. Second one is 1 4 7 0 1. So, you calculate zj minus cj, zj minus cj will be minus 3 minus 5 this will be minus 4, this is 0 this is 0. So, most negative is this one therefore, your entering vector will be x 2.

So, we have to calculate the ratio this is 4 by 1, this is 9 by 4. So, your outgoing vector is x 5. So, your entering vector is x outgoing vector is x 5 and your pivot element is 4. So, I have to make this element as 1 and the other elements as 0.

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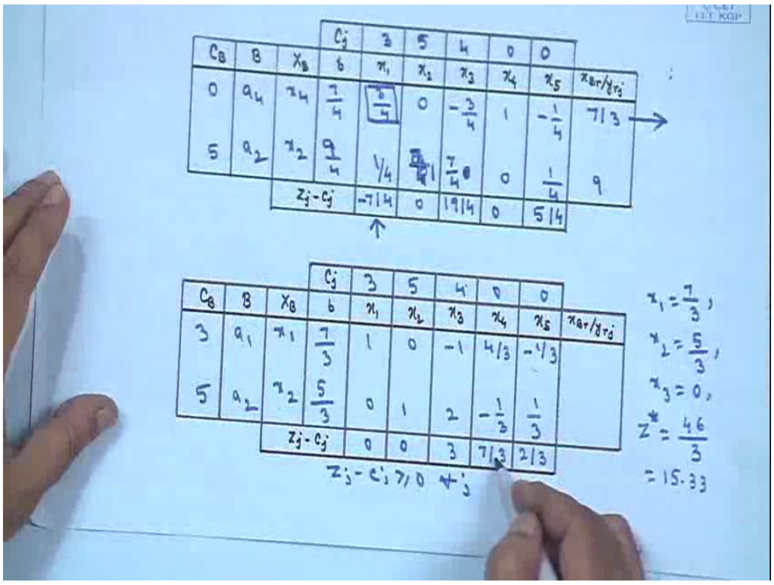

So, in the next table your x 5 will be replaced by x 2. So, let us write down the next table in the next table your x 4 will remain as it is you are having now x 2. So, it is a 4 and a 2 your C B is 3 5 4 0 0. So, x 4 is 0 x 2 is 5, your vectors will be the rows will be after manipulation 7 by 4 3 by 4 this will be 0, minus 3 by 4 1 minus 1 by 4 and this will become 1 by 4 1 7 by 4 0 1 by 4. Sorry the b is 9 by 4. So, 9 by 4 means this will be 1 by

4. This will become 1, this will be 7 by 4 because this has to be 1, x 2 was the entering vector 7 by 4 this will be 0 this is 1 by 4.

So, if you calculate zj minus cj that is C B, B, C B into b minus cj this will be minus 7 by 4 0 19 by 4 0 and 5 by 4. So, again still your a zj minus cj is less than equals 0. So, entering vector will be x 1. Now calculate the ratio this will be 7 by 3 and this will be 9. So, outgoing vector will be x 4. So, your pivot element is 3 by 4. So, I have to make this element as one and this element as 0. So, your x 4 will be replaced by x 1 here. So, you are coming as x 1 x 2, your b values are a 1 a 2. So, your cj are 3 5 4 0 0. This is 3 and 5. So, the rows will be 7 by 3 this will be 5 by 3 1 0 minus 1 4 by 3 minus 1 by 3. And this will be 0 1 2 minus 1 by 3 1 by 3. You should check whether the values are coming correctly or not afterwards.

So now, if we calculate zj minus cj zj minus cj will be 0 0 3 7 by 3 and 2 by 3. So, zj minus cj if you see this zj minus cj is greater than equals 0. For this case zj minus cj is greater than equals 0 for this and it is 0 for basic variables greater than 0 for non basic variables. So, you are getting the unique optimal solution, x 1 x 2 is present. So, your optimal solution is x 1 equals 7 by 3. X 2 equals 5 by 3, x 3 is not present therefore, x 3 will be 0 and maximum value of z z star is equals to this, into this, that is 3 into 7 by 3 plus 5 into 5 by 3 other term will be 0. So, that value you will get as 46 by 3 and this is nothing but tentatively I can write down 15.33.

So, one can say that to obtain the maximum profit of rupees 15.33 the product is x has to be produce 7 by 3, a product why should be produce 5 by 3 equal units. And no product should be produced for the product z. So, please note one note that one that to obtain the maximum profit 15.33, one has to produce 7.3 unit of product capital it is 5.3 unit of product capital Y. And no your no unit should be produced for this one. Please keep this one in mind this particular table the last table. Because in the next 4 problems we will use this thing frequently we will use this optimal table.

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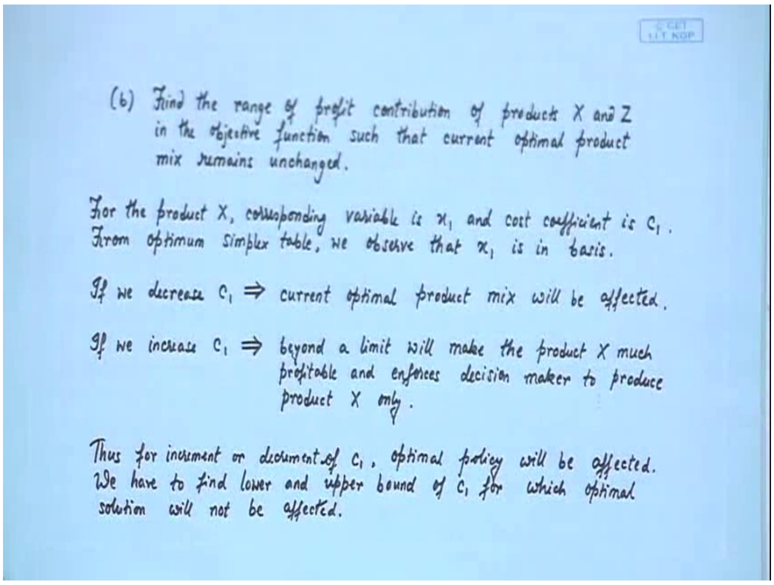

Now, come to the second problem. The second problem is, find the range of profit contribution of product x and z in the objective function, such that current optimal product mix remains unchanged; that means, what should be the variations in or what range of profit contributions of x and z. So, that it remains unchanged. For the product x what happens? Corresponding variable as you know it is x 1 and the coefficient is c 1. So, up from optimums a simplex table that is what I was telling your x 1 is present in the basis, right. So, your cost is c 1. So, if we decrease c 1 what will happen? Current optimal product mix will be affected. And if we increase c 1 in that case beyond the limit it will make the product x much profitable. And enforces decision makers to produce product x only.

So, please note this one, since x is line in for the product x corresponding variable is x 1 and it is lying in the optimal solution in the basis it is present; therefore, if which decrease the value of c 1 then the current product mix or the basis will be affected. Whereas, if we increase c 1 in that case it will be much profitable and it will enforce to decision maker to produce maybe x only, thus what I can say for increment or decrement of c 1 optimal policy will be effected optimal policy will be affected we have to find the lower and upper bound of cone for which optimal solution will not be affected. So, by these 0 what we are saying that whether I increase or decrease the value of c 1 optimal policy will be affected. So, for what range of c 1 optimal policy will not be affected that we will calculate now.

Range of delta c 1 if you remember This is from the earlier lecture already we have told, maximum of minus delta j minus zj minus cj by y 1 j here it will be which is greater than 0 less than equals delta c 1 less than equals minimum of minus zj minus cj divided by y 1 j, where y 1 j will be less than 0.

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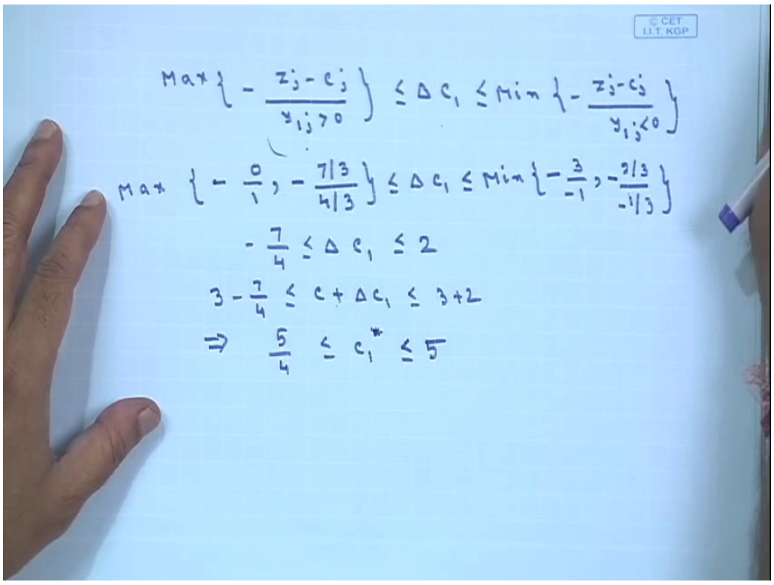

So, what are these values maximum of you are doing it for which one for product 1. See this table in this particular table you try to find out. Maximum of what you will obtain, this will be 0 by what where it is most positive most pati is positive is coming here your j minus cj for the variable it is 0, and here it is 1 the positive value here it is coming as 1. So, therefore, zj minus cj and for this case y 1 j to be I will take only that I for which y 1 j is greater than 0.

So, therefore, maximum is 0 divided by one. So, the value will be maximum of minus there is no need 0 by 1 from where you are getting 0 by 1 from here only you are getting this is 0 value divided by one and there will be another one where the value of y j is positive for this positive value zj minus cj is 7 by 3 and this yj value is positive greater than 0. So, therefore, the other one will be minus 7 by 3 divided by 4 by 3. This should be less than equals delta c 1 less than equals minimum of here, we I will take only those y ija ratios where y ij is less than 0 in the optimal solution. Where yij is less than 0 1 is this one that is zj minus cj by yij. So, this will be minus of 3 divided by minus 1 this is your 3 divided by minus 1, other one it is 0 on this. So, it will be minus or 0 to 2 third by minus 1 third. So, this will be minus 2 third by minus 1 by 3.

So, like this way these values y ij zj minus cj values I have to prepare I hope it is clear now. That y 1 j means for the first one you are taking it wherever you are getting the positive values. And for the minimum value y 1 j value should be less than 0. So, if I calculate this you will get minus 7 by 4 less than equals delta c 1 less than equals 2. So, value of c is this one the as value of c for this problem is 3. Therefore, 3 minus 7 by 4 less than equals c plus delta c 1 less than equals 3 plus 2, which implies c 1 start or new value of c if it lies in this range in that case there will be no change in the optimal solution.

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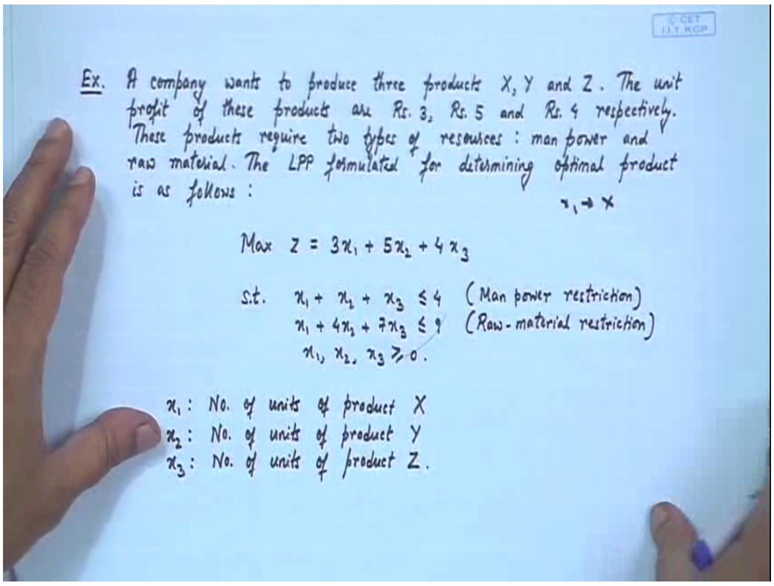

So, you see the original problem in the original problem coefficient of the product x was 3 c 1 right. So, 3 was there now if I change this 3 value coefficient from in between 5 by 4 2 5 in that case your optimal solution will remain unchanged.

So, here your advantages if I have change c in this range I do not have to recalculate the problem. Similarly for the product z what happens for the product z if you see your product z means which corresponds to the variable x 3, and x 3 is not present in the basis your x 3 is not present in this particular basis. So, therefore, the calculation which we did earlier that would be not that will not be applicable for product 3, because here this is not in the basis only thing I have to check zj value zj minus cj that is z 3 minus c 3 value should be changing. So, what I have to calculate for the product z or product z the variable is x 3.

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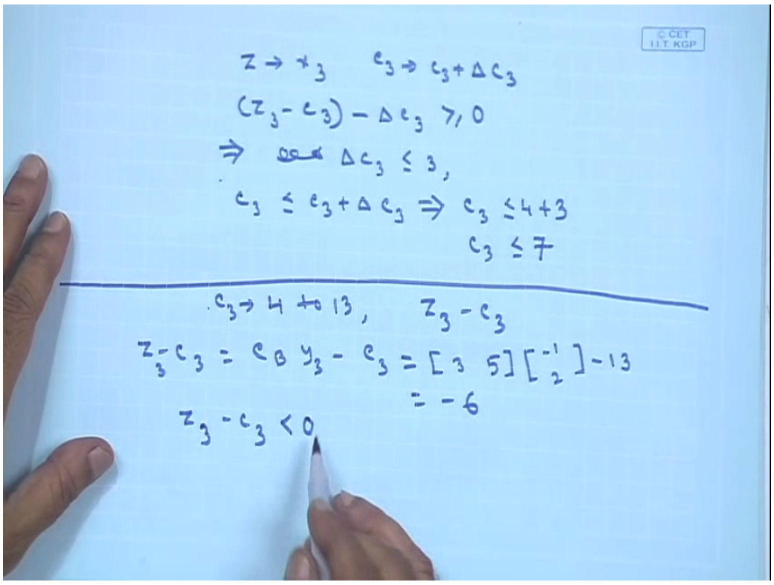

So, I have to only check the optimality, z 3 minus c 3 minus del c 3 this should be greater than equals 0. Because c 3 now has been changed to this one see 3 has been changed to see 3 plus del c 3. So, this should be greater than equals 0 which implies immediately you can tell that 0 del sorry del c 3 should be less than equals 3. Since z 3 minus c 3 is nothing but c 3. So, therefore, your c 3 less than equals c 3 plus del c 3 from here I can write down c 3 less than equals 4 plus 3 or c 3 less than equals 7. So, your original value of the coefficient for x 3 was 4. What is our conclusion now? Till the value unit price for the product z which is c 3 if it up to 7, if we change it to up to 7 there will be no change in the optimal solution. So, please note this one that there will be no change in the optimal solution.

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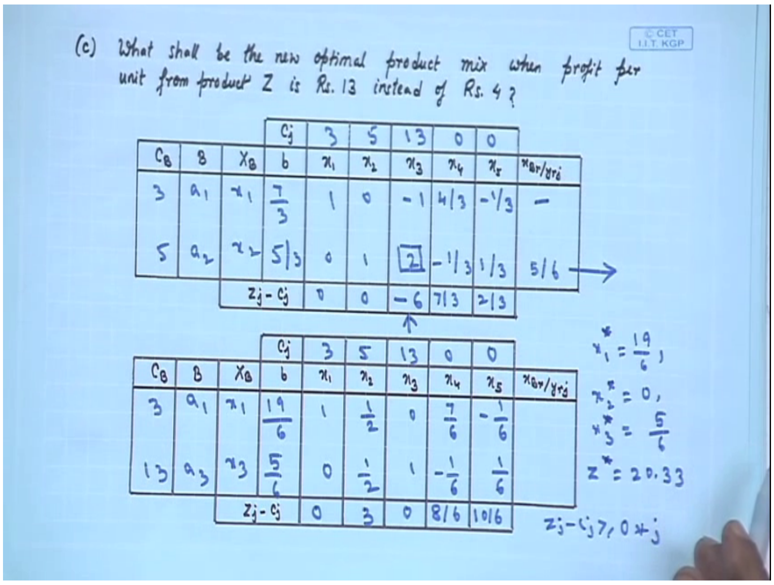

Now, come to the third one. The third one is what shall be the new optimal product mix when profit unit per product z is rupees 13 instead of 4. That is earlier for per unit cost per product z was 4, now if it increases to 13 what happens? Now if you see I am coming to this if you see whenever your c 3 is changing from 4 to 13. So obviously, your z 3 minus c 3 will also change. What is your z 3 minus c 3? Z 3 minus c 3 is nothing but C B y 3 minus c 3, and c b y 3 is 3 5 this I can obtain from this one, C B and y 3 is minus 1 2 minus 13. If I calculate this value is this one effectively I am doing from this zj minus cj is nothing but c bb minus cj. So, zj z 3 minus c 3 is less than 0.

So, correct optimal solution will not remain optimum from here, it is clear that the current optimum solution with this value we have calculated these 3 5 and minus 1 2. So, with this value minus 1 2 and after that minus new value of cj that is 13. So, it is becoming negative. And since it is becoming negative. So, current optimum value will not remain optimum in that case. So, the non basic variable x 3 will enter into basis. That is here you are having this one. So, this will be changed to 13. So, from here from this one itself I will change it to 13 and then we will calculate recalculate the value.

So, basically on this we have shown the value is becoming minus 6. So, if I write it quickly the last solution your b is x 1 x 2 a 1 a 2 3 5 this value has been changed to 13 from 4 please note this one. So, this is 3 5 this is 7 by 3 5 by 3, this one is 1 0 minus 1 4 by 3 minus 1 third, this will become 0 1 2 minus 1 third one third. So, zj minus cj values

0 0 as I have shown it will be minus 6, 7 by 3 and 2 by 3. So, basically in the last optimal solution you are only changing the value of cj from 4 to 13 and you are finding that zj minus cj is this. So, your entering vector will be this one.

So now calculate the ratio this is negative we will not calculate, this will be 5 by 6. So, outgoing vector is a 2. So, therefore, this is the pivot element. So now, you will have x 2 will be replaced by x 3. So, x 1 x 3 a 1 a 3 3 5 13 0 0. So, here it will come 3 and 13.

So, again I am writing this I have to make one this has 0. So, this will be 19 by 6 1 half 0 7 by 6 and minus 1 by 6. This will be 5 by 6 0 half 1 minus 1 by 6 and 1 by 6. If you calculate zj minus cj you will find that 0 3 0 8 by 6 and 10 by 6. So, therefore, your zj minus cj is greater than equals 0 for all j. Therefore, new optimal solution is x 1 star now x 3 is how x 1 star is 19 by 6, x 2 star is not present in the basis. So, it is 0 and x 3 star is 5 by 6. And z star if you calculate 20.33. So obviously, whenever you have changed the unit price for product z that is corresponding to the variable x 3 to 13 from 4 we are finding optimal solution changes totally, and the profit also has been increased in this case. So, this is the third one.

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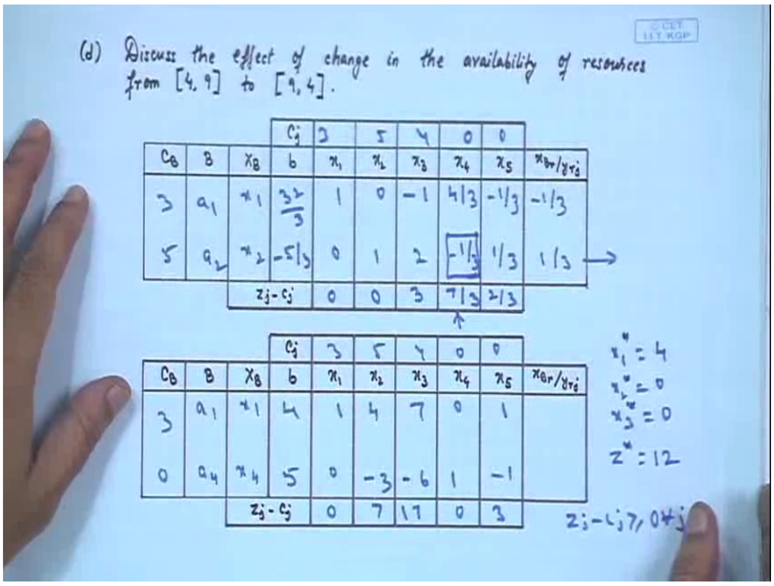

Now, let us see the 4th one, 4th one discuss the effect of changes in the availability of resources from 4 9 to 9 4 4, that is your original problem if you see it was 4 9 this I depress by 9 4 then what will be the change in the this thing.

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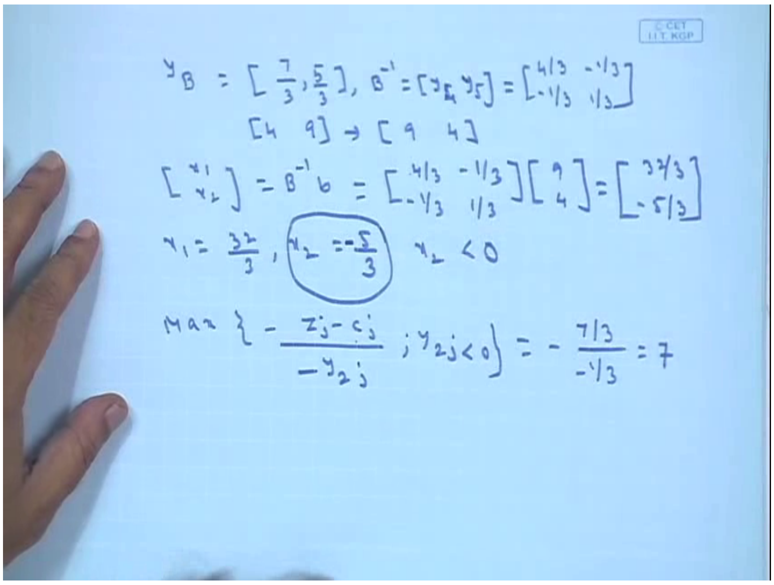

So, for optimality what happens your y B is 7 by 3 5 by 3 this is again from the last original optimal solution, this is 7 by 3 and 5 by 3. What is your B inverse? B inverse means whatever is not there in the basis that is x 4 and x 5. So, B inverse is nothing but we are writing y 4 and y 5, and y 4 is 4 by 3 minus 1 third and minus 1 third one third. So, you are getting from here.

So, if I replace 9 by 9 4 I am replacing 4 9 by 9 4. So, in that case your x 1 x 2 is equals to B inverse into b. So, your B inverse is this 1 4 by 3 minus 1 third minus third one third and your 9 4. So, it will become 33 by 3 and minus 5 by 3; that means, your solution is x 1 equals 32 by 3 and x 2 equals minus 5 by 3. So, once sorry 2 is minus 5 by 3. Since your x 0 is less than 0 therefore, the original solution whatever you have obtained, that it will be remain unchanged sorry, that will change your optimal solution feasible solution will become infeasible now.

So, to remove in visi feasibility what I have to do I have to find out the maximum of minus of zj minus cj by y 2 j minus of y 2 j where y 2 j is less than 0. This will be equals to from the original table this 1 minus 7 by 3 and 1 by 3 from here you can obtain it. So, from the this table you can obtain the value of zj minus cj is this and y is minus 1 by 3. So, these 2 values if you take in that case this you take 7 by 3 which is this one and y 2 j is less than 0. So, this will be minus 7 by 3 by minus 1 by 3. So, that you are obtaining one. So, x 2 becomes non basic and x 4 becomes basically basic variable. So, from the original one whatever was there using the other way export becomes basic and non basic variable.

So, here I can calculate this thing you have x 1 x 2 you have a 1 a 2 3 5 4 0 0 3 and 5 will come over here. These values you will obtained minus 5 32 by 3 minus 5 by 3 1 0 minus 1 4 by 3 minus 1 third. I have changed this value effectively what I have done I have changed this b value now 32 2 by 3 into minus 5 by 3 in the original problem. Here it was 7 by 3 5 by 3 this I am changing it to 32 by 3 minus 5 by 3. All other things will remain same 0 1 2 minus 1 third one third zj minus cj value 0 0 0 sorry, 0 0 3 7 by 3 and 2 by 3.

Here I have to take the this 1 minus 1 third and this will be one third. So, this is the outgoing vector therefore, this is your pivot element. So, in the next one your x 1 will come x 4 will come a 1 a 4 will come. So, it is 3 5 4 0 0 this is 3 and 0. So, this becomes 4 1 4 7 0 1 and 5 0 minus 3 minus 6 1 minus 1 zj minus cj you will get 0 7 17 0 3. So, all zj minus cj is greater than equals 0 for all j. So, you obtain the optimal solution as x 1 star equals 4, and x 2 and x 3 are not present here. So, x 2 and x 3 star equals this and z t star if you calculate it is 12. So, again for this particular case also what we are finding is that the optimal solution has been changed.

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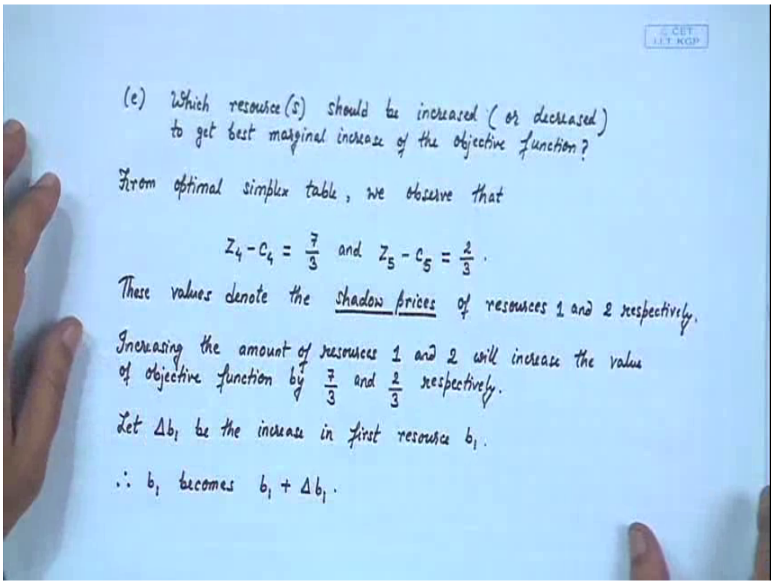

So, in the next class we will see the last problem that is this one we will discuss, which resource we should use. Which resources should be increased or decreased to get best marginal increase of the objective function. In the next lecture we will just do this and after that we will go to the dual simplex procedure.