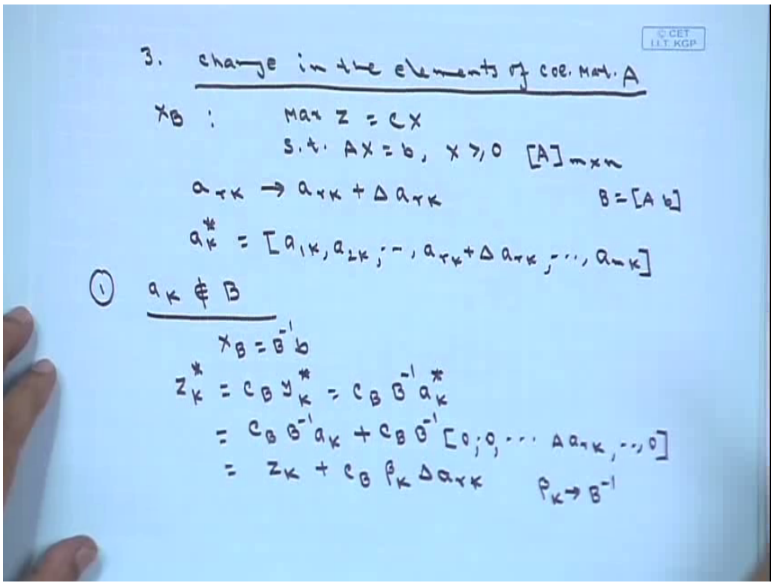
Constrained and Unconstrained Optimization

Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 16
Sensitivity Analysis – II

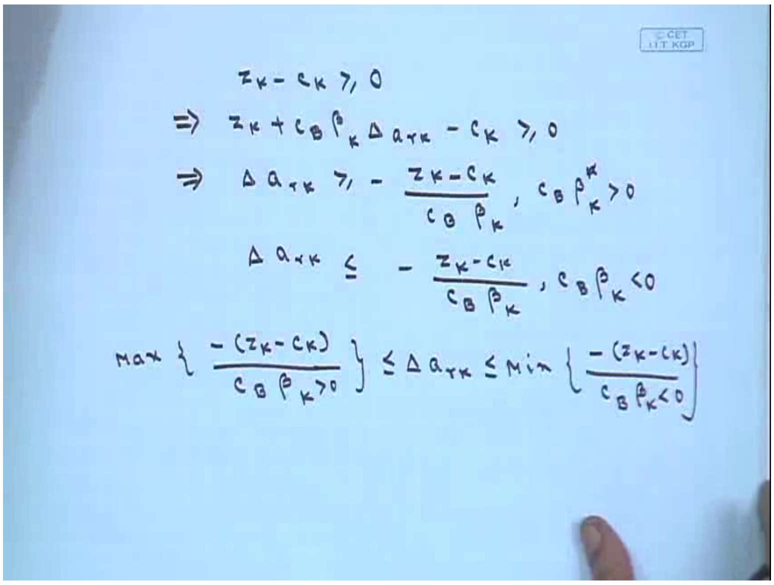
So, in this class, we are continuing from the last lecture, where we were checking the variation in the parameters the effect of the changes in the parameters on the optimal solution.

(Refer Slide Time: 00:39)



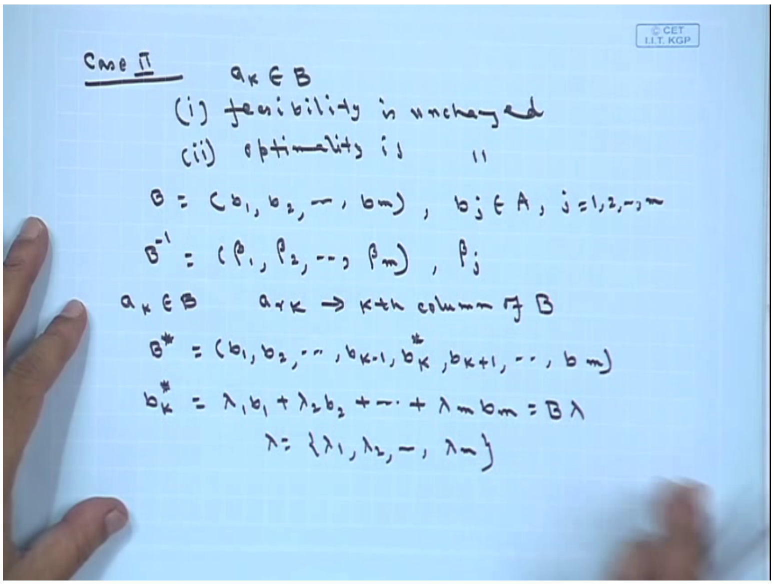
So, we were talking about the changes in the coefficient matrix that is A. The first part we have done that is if a k does not belongs to B. In this case we have observed that if it lies in this range in that case there will be no change in the optimal solution.

(Refer Slide Time: 00:56)



Now, let us see the second part that is whenever your a k lies between a k belongs to B.

(Refer Slide Time: 01:03)

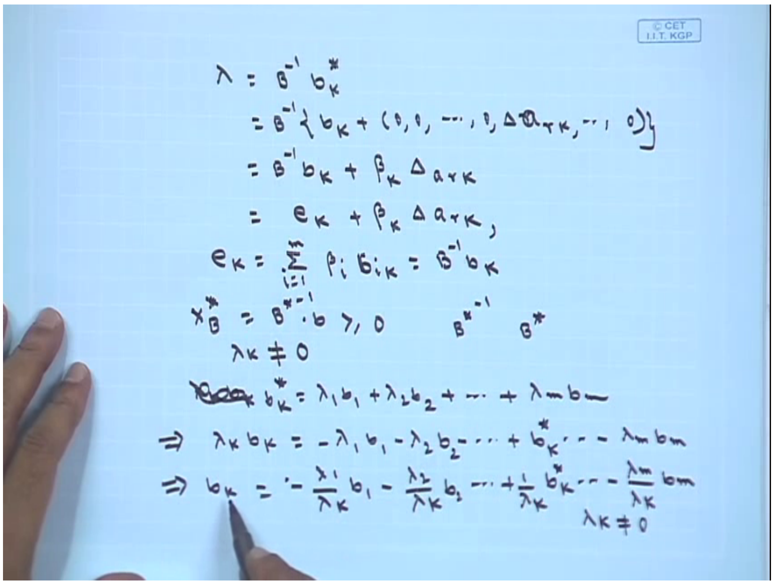


That is your case 2 case 2 is when your a k belongs to B.

So, in this case since a k belongs to B and x B equals B inverse b. So, it may happen 2 cases may arise. One case is feasibility remains unchanged, and second case is feasibility will be changing. So, optima sorry feasibility is unchanged 2 cases. I mention, that is feasibility is unchanged, and number 2 the optimality is unchanged. So, whenever a k belongs to B since x B equals B inverse b, your feasible feasibility may be changed. So, once the optimal solution or feasibility is changing. So, what we will try to check, that for what range for what variation feasibility will remain unchanged, and for what variation of a k your optimality will remain unchanged. So, that is our basic criteria because otherwise again we have to recalculate the problem. So, the first case let us come, suppose your b is b 1 b 2 bm b 1 is b 2 bm, bm where bj belongs to A. I am just writing once here j equals 1 2 m.

And your B inverse we are assuming as beta 1, beta 2 like this way say beta m, where beta j is nothing but the jth column vector of B inverse. Beta j is the jth column vector of B inverse. Now since your a k belongs to B we are doing this case. So, if I make a change in ark what it will affect? It will affect kth column of b, it will affect the kth column of b. So, there will be a change from B to B star. So, I can write down there will be a change from B to B star and B star I can write it as b 1 b 2 like this way b k minus 1 say b k star b k plus 1 unlike this way b m. So, this vector can be written as a linear combination of the basic vectors of b this entire vector can be written as this one this b k star. So, b k star I can write it as a linear combination of others that is lambda 1 b 1 plus lambda 2 b 2 plus lambda m bm. Which is equals to I can write down nothing but b into lambda where your lambda is equals to lambda 1 lambda 2 and like this way lambda m.

(Refer Slide Time: 04:55)

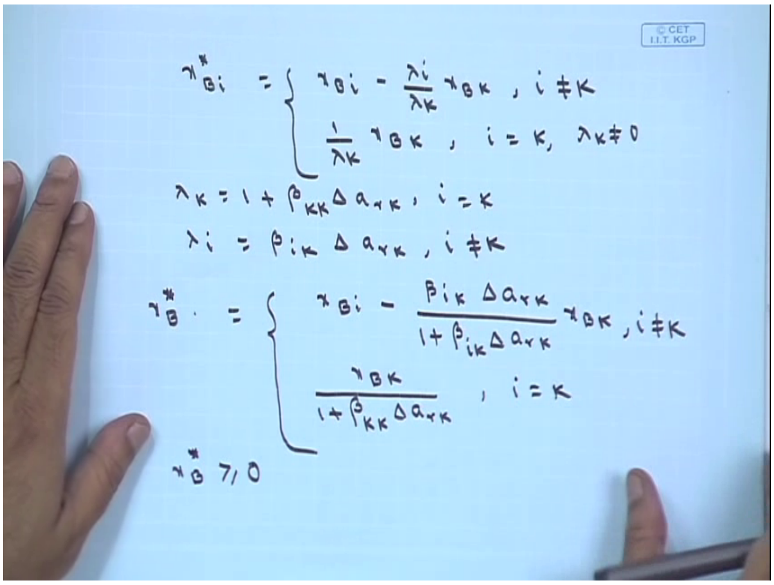


So, since b k star equals b into lambda. So, I can find out the value or I can write down the value of lambda as from the earlier equation B inverse into b k star B inverse into b k star and what is b k star. B k star I can write down b k plus this vector in the kth place it will be delta a rk like that way it is going to 0. So, it is nothing but B inverse b k plus beta k into delta ark. Where again beta k is the kth column vector of B inverse. So, B inverse b k is nothing but the ek, plus beta k into delta ark where your ek. You can write down ek is nothing but summation over i equals 1 to m beta I into bi k.

So, this is equals B inverse into bk. So, basically ek is nothing but the unit vector whose kth element is 0 one and others are 0 it is the unit vector, whose kth element will be one and others are 0. Now for feasibility condition your value of x B star should be greater than equals 0, that is for feasibility condition your x B star which is nothing but B star inverse into b, this should be greater than equals 0. Now when B star will be B star inverse this one B star inverse this will exist only when B star is non singular assume now if B star is non singular then only B star inverse will be existing.

So, we must have this delta k should be not equals to 0. Now from the earlier one that is from b k star equals this thing from b k star equals b lambda from here you can write down lambda k b k, lambda k into b k sorry, before that b k star this is equals lambda 1 b 1 lambda 2 b 2 like this way lambda m into b m. So, once I am writing this, this implies lambda k b k this is nothing but minus lambda 1 b 1 minus lambda 2 b 2 like this way it is going plus b k star and the last one will be minus lambda m into bm. Which implies your b k will be minus lambda 1 by lambda k into b 1 minus lambda 2 into lambda k into b 2, like this way plus 1 by lambda k into b k star and like this way minus lambda m by lambda k into b m and Please note that here your lambda k star is not equals to 0. Your lambda k star lambda k is not equals 0, from this we have told since B star is non singular. So, we can write down this therefore, once I obtain this bk.

(Refer Slide Time: 08:49)

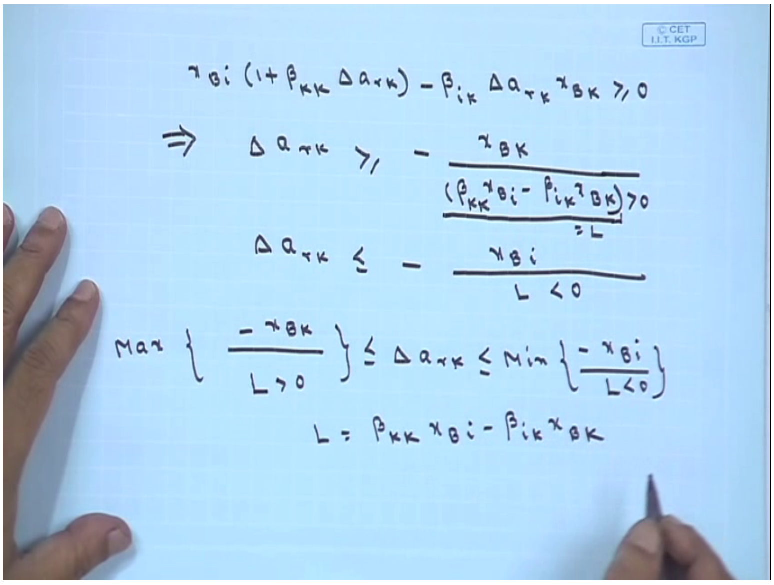


So, x B star I can rewrite like this way your x B i star this equals I can write down xbi minus lambda I by lambda k into x b k 2 things will come this will come whenever I is not equals to k. And the second one will be 1 by lambda k into x into b k where when i equals k and of course, lambda k is not equals to 0. So, x B star we are writing like this, but what is your lambda k your lambda k equals, I can write down 1 plus beta k into delta a rk whenever i equals k. And your lambda i equals beta i k from here I can write down beta i k into delta a rk.

Whenever I is not equals to k. So, that you can write down your x B i star this is equals to x bi minus beta i k delta ark from here from this calculation, divided by 1 plus beta i k delta ark into xbk, i is not equals to k. And it will be xbk divided by this 1 plus beta kk this will be sorry this will be beta kk beta kk into delta ark when i equals to k. So, I obtained this thing. Now for feasibility what happens we know that for feasible solution your x B star always should be greater than equals 0.

So, therefore, this quantity should be greater than equals 0.

(Refer Slide Time: 11:00)

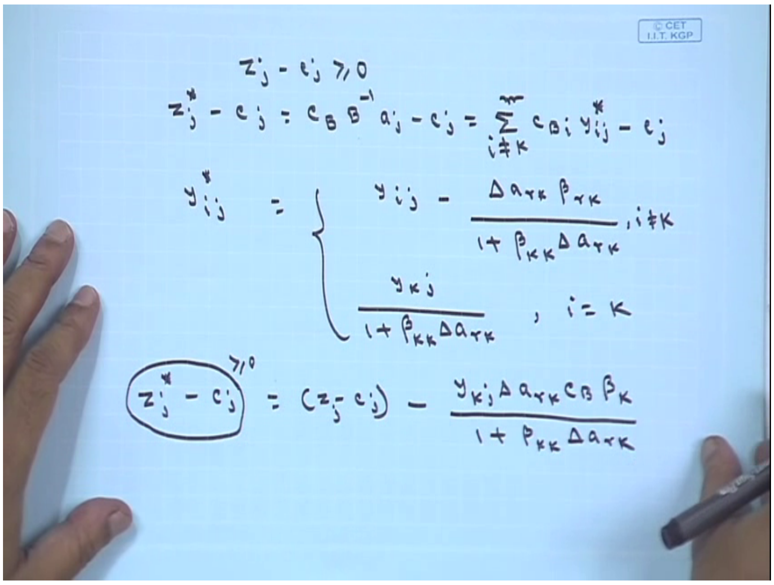


Or in other sense you can write down x B i into 1 plus beta kk, delta ark minus beta i k into delta ark xbk this is greater than equals 0. Which implies that this quantity is greater than equals x B i can go on that side. So, from here simple manipulation I can write down delta ark should be greater than equals minus xbk, divided by beta kk xbi minus beta i k into xbk. And whenever this quantity is greater than 0. And similarly delta ark will be less than equals minus xbi divided by beta kk xbi minus beta ik into xbk.

If I assume that this value as equals to capital l. So, I can tell it as capital l which should be less than 0. So, I am assuming the denominator as same the denominator will be same. So, delta rk greater than equals 0 delta rk less than equals 0; so, from here for feasibility I may say that I can write down maximum of minus of xbk, by this l where l is greater than 0 less than equals delta ark, less than equals minimum of minus xbi by l which is less than 0 where your l is nothing but beta kk xbi minus beta i k into xbk.

So, we may say that if the old solution has to be remain unchanged then the variation in the; for feasibility the old a solution will remain feasible if the variation lies in this range. So, I can simply check without the variation on the coefficient matrix whatever I have made if the variation lies in this particular range, in that case I do not have to re-compute the problem I can tell that old solution will remain feasible. Now I have to check the optimality for obtaining optimality.

(Refer Slide Time: 14:09)

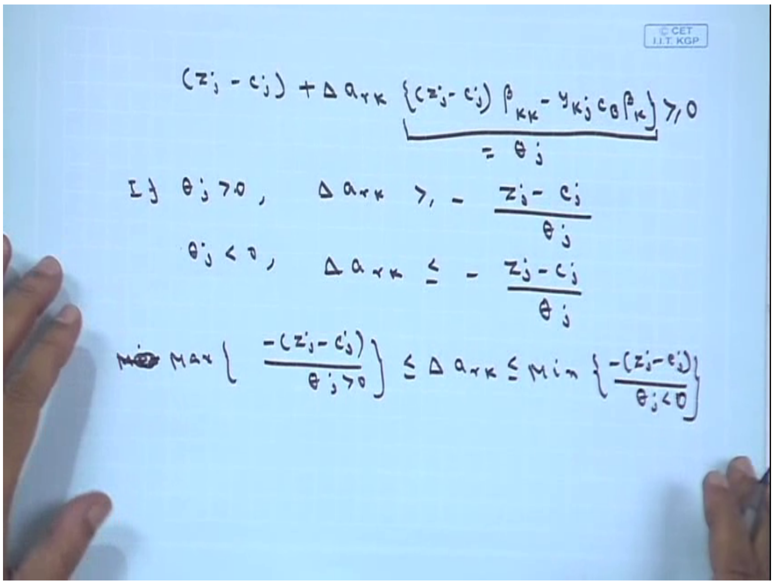


What I have to find out? We note it that for optimality this condition must be satisfied zj minus cj must be greater than equals 0. Now for changes z (Refer Time: 14:18) change to j j star say. So, it will be equals to C B B inverse aj minus cj. And which is equals you may write down summation I not equals to k 2 m c bi y ij star it will change minus cj.

Where your yij star I can write down yii yij minus delta ark into beta rk, divided by 1 plus beta kk this again from this calculation just like we did the earlier calculation; on the similar way if you do the calculation since it is same similar type; so, I am not showing the entire calculation I not equal k, and this will be ykj divided by 1 plus beta kk into delta ark whenever i equals k your y j star will remain will become this one. Now zj minus cj or zj star minus cj then will become zj star minus cj will then become zj minus cj minus ykj delta ark C B into beta k divided by 1 plus.

From here I am getting it 1 plus beta kk into delta ark. So, once I am getting zj star minus this and for optimality your zj minus cj star this quantity is greater than equals 0. Or in other sense this one zj minus cj into delta rk this should be greater than equals 0 implies your zj minus cj plus delta ark into zj minus cj into beta kk from that itself, minus ykj C B into beta k.

(Refer Slide Time: 16:36)

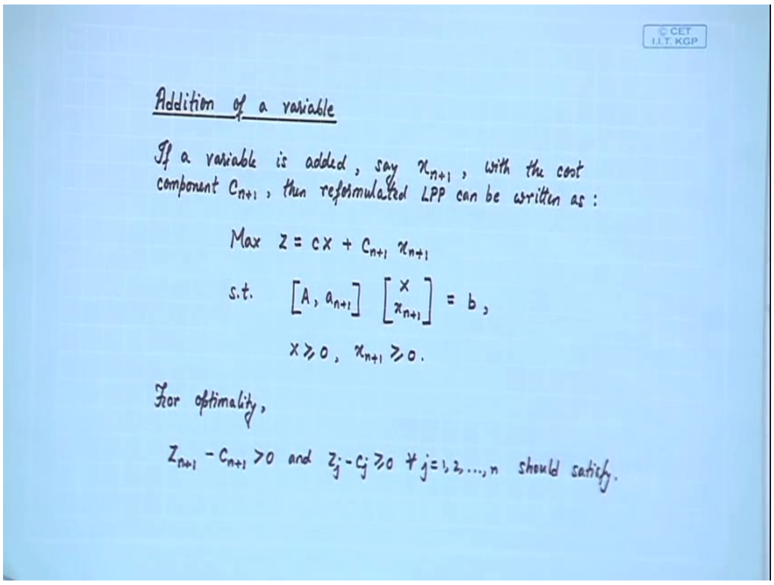


This has to be greater than equals 0, now if you assume this quantity is equals to say theta j.

So, if theta j is greater than 0 in that case delta ark is greater than equals minus zj by cj by theta j, and if your theta j is less than 0 then delta ark will be less than equals minus zj minus cj by theta j. So, from these 2 combine if we combine these 2, then we will obtain maximize maximum of minus zj minus cj by theta j where theta j is greater than 0 less than equals delta ark less than equals minimum of minus zj minus cj by theta j where theta j has to be less than 0. Therefore, to maintain the optimality if the variation in the cost in the coefficient matrix A if it lies in the given range whatever we have given, then the old solution will remain unchanged.

So, basically what we have seen for the coefficient matrix capital A. If the there is a change or variation in the coefficient matrix A, the feasibility and the optimality may change. So, here we have derived one conditions for which the feasibility and optimality will remain unchanged, if the variation lines if the if the various variations lies in the range provided here, then the feasibility condition and optimality condition both will remain unchanged. So, these are the 3 variations in 3 parameters that is c b and the next one was A.

(Refer Slide Time: 19:15)

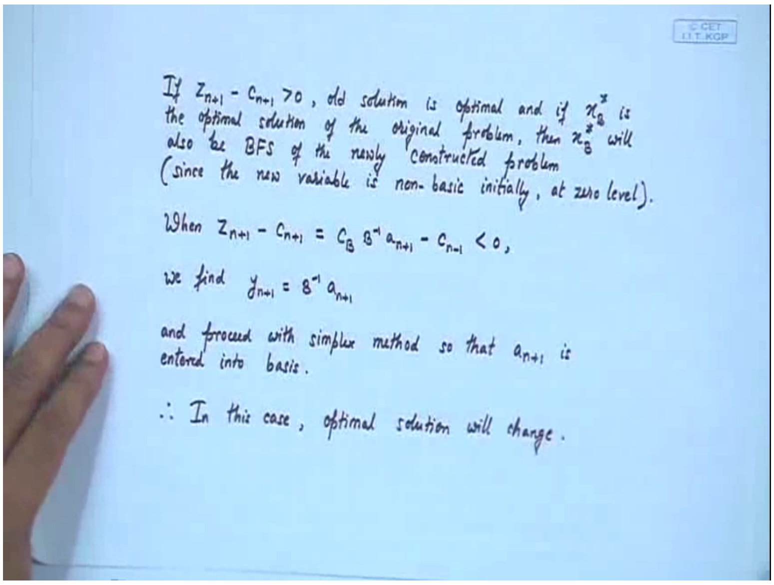


Now, we will go to the other one that is addition of a variable. Suppose a variable is added say xn plus 1, with the cost component cn plus 1. Then the reformulated LPP can be written as maximize z equals cx plus cn plus 1 plus xn plus cn plus 1 into xn plus 1 one variable is added earlier there was n variables x 1 to xn.

Now, I have added xn plus 1. So, cost coefficient is this once I am doing this your a will be changed to one more coefficient will come here. So, in place of a the matrix will be a comma a n plus 1 and your, but decision vector x will be changed to x and xn plus 1 will be there b will be changing; obviously, x greater than equals 0 and xn also plus 1 greater than equals 0. Now for optimality your zn plus 1 minus cn plus 1 should be greater than 0 and zj minus cj is already satisfied.

Your zj minus cj is already satisfied for the old problem, Along with that for optimality your zn plus 1 minus cn plus 1 also should be greater than 0.

(Refer Slide Time: 20:38)

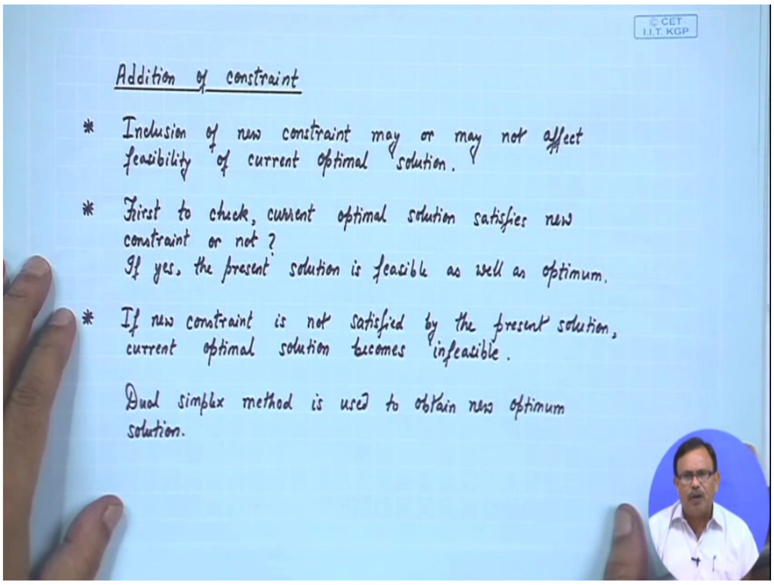


Therefore if your zn plus 1 minus cn plus 1 is greater than 0, old solution is optimal because there is no change automatically this is also satisfying the optimality condition new variable. And the B star is the optimal solution of the original problem, then B star will also be the basic feasible solution of the new problem. So, you see if zn plus 1 minus cn plus 1 greater than 0, then old solution will remain the optimal solution of the new problem also.

This is because since the new variable is non basic initially at the 0 level. Now say when zn plus 1 minus cn plus 1 equals to C B into B inverse a n plus 1 minus cn minus 1. So, zn plus 1 I am writing this C B B inverse an minus cn plus 1. If it is less than 0 then we have to find out yn plus 1 equals B inverse an plus 1 and in that case I have to proceed with simplex method. So, that an plus 1 entered into the basic or in other sense if zn plus 1 minus cn plus 1 less than equals 0, then I have to reformulate the problem please note this one we have to reformulate the problem and we have to use the simplex algorithm to obtain the solution.

So, if I have added a new variable xn plus 1 and if the coefficient of the new variable xn plus 1 is cn plus 1, and if I find zn plus 1 minus cn plus 1 is greater than 0, then the optimal solution of the old problem will remain the solution optimal solution of the new problem. But if zn plus 1 minus cn plus 1 is less than 0; in that case I have to reformulate the problem I have to generate the new initial basic feasible solution, and I have to use the simplex method to solve the problem. And the last one is if I want to add some constraints that is addition of some constraint.

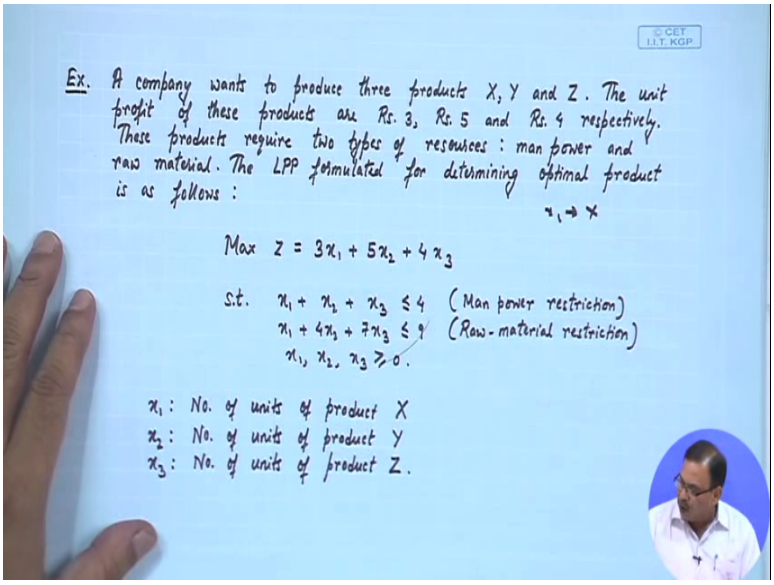
(Refer Slide Time: 23:07)



Now, whenever you are including some new constraint, the new constraint may or may not affect the feasibility of the current optimal solution, please note this one. That the inclusion of the new constraint may or may not affect the feasibility of the current optimal solution. First to check current optimal solution satisfies the new constraint or not. So, basically at first you are checking whatever optimal solution you have obtained for the old problem. After adding the new constraint whether that optimal solution is satisfying the new constraint or not. If the new constraint is not satisfied if the new constraint is satisfying the current optimal solution Then the present solution is feasible and that solution will give you the optimum solution means in that case you do not have to compute.

But if the new constraint is not satisfied by the present solution then your current optimal solution will become infeasible. So, in that case you can we can use the dual simplex or some other problems to obtain the optimal solution, that we will see. So, in other sense whenever I am adding a constraint, in that case if the new constraint is satisfied by the optimal solution by the feasible solution, then the old solution will remain the solution of the new problem itself. But if the new constraint is not satisfying the present solution in that case yours current solution or solution of the whole problem will not be feasible. So, I have to reformulate the problem and I have to do this. So now, let us take one example and see how it effects the other parts of the problem.

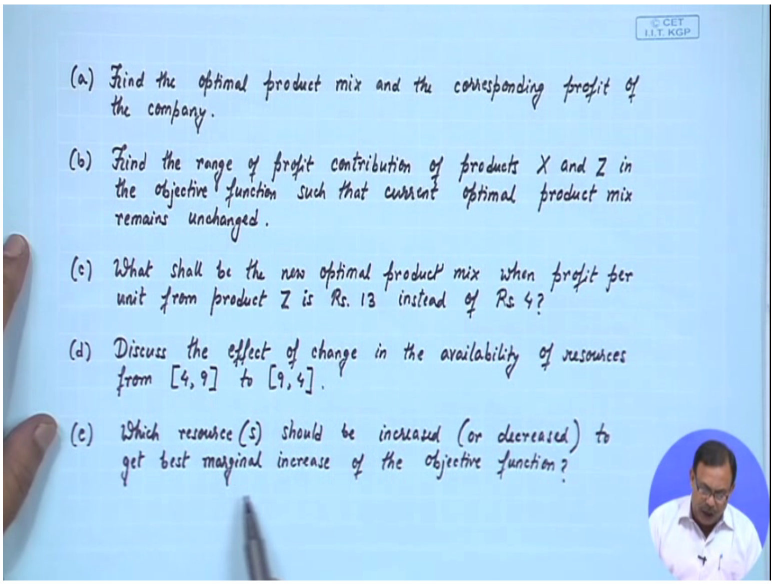
(Refer Slide Time: 25:23)



Just I am presenting the problem first we will discuss, the solution of this problem in the next class; obviously, just I am discussing the problem. A company wants to produce 3 products x y and z. The unit profit of these products are rupees 3 rupees 5 and rupees 4 respectively. So, unit price of each of the products for product x it is rupees 3 for product y it is rupees 5 and for product z it is rupees 4. So, basically I am producing 3 products and the quantity of products are x y and z respectively. So, my objective is to maximize which one 3 x plus 5 y plus 4 z or x 1 x 2 x 3 if I am producing x 1 quantity of x if I am producing here.

I have written if I am producing x 1 is the number of units of product of x, x 2 is the number of units of product of y and x 3 is the number of units of product of z. So, in this case my problem will be I have to maximize 3 into x 1s because unit profit for product one is 3. Similarly 3 x 1 plus 5 into x 2 plus 4 into x 3 I have to maximize. These products as 2 type of requires the 2 type of resources manpower and raw material. What is the manpower restriction? I am saying x 1 plus x 2 plus x 3 less than equals, 4 and raw material restriction is x 1 plus 4 x 2 plus 7 x 3 less than equals 9.

(Refer Slide Time: 27:18)



So, my problem is this one, what I want to find out we will solve 5 different problems for this. First one we will find out the optimal solution of the LPP.

Next we will see the range of profit contribution of x and z in the objective function, such that the current optimal product mix remains unchanged. That is how much variation I can make in x and z So that current solution will remain the solution even if I have made changes in x and z then number c what shall be the new optimal product mix when profit per unit for product is rupees 13. That is whenever I am changing a coefficient then discuss the effect of change of availability of resource from 4 9 to 9 4, that is whenever you are changing the matrix B. What happens? And which resources should be increased or decreased to get best marginal increase of the objective function? Or in other sense the theory which we have discussed in this lecture and the earlier lecture how to implement those theories that we will see in the next class.