Constrained and Unconstrained Optimization
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 14
Degeneracy in LPP

So, basically what happens whenever degeneracy occurs, you are moving from one degenerate basic feasible solution to the other basic other degenerate basic feasible solution.

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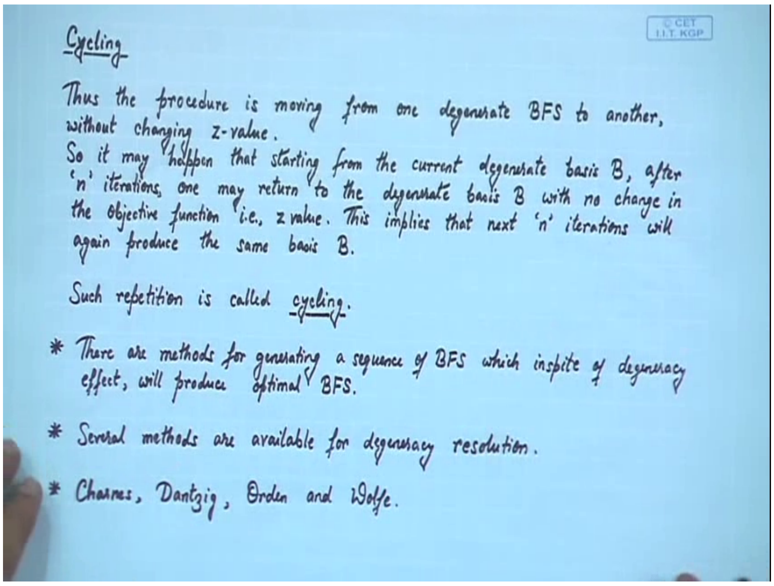

If you see here the procedure is moving from one degenerate BFS to another without changing the z value that is the problem.

So, it may happen that whenever I have started from a current degenerate basis B after n iteration I may return back to the again to the degenerate basis B with no change in the objective function that is the z value. This implies that next iterate n iterations will again produce the same basis B. So, basically there is no improvement and you are just moving from one place to another and ultimately you are coming back to the initial point again, and such repetition we call it as cycling.

So, there are methods for basically generating a sequence of basic feasible solution, which in spite of degeneracy effect will produce the optimal basic feasible solution. So, basically we are saying that whenever degeneracy occurs then usually if you try to follow the normal method in that case there will be no improvement in the objective function. But there are several methods available for degeneracy reduction like Charnes Dantzig Orden and Wolfe various methods are available, but we will discuss only one of them.

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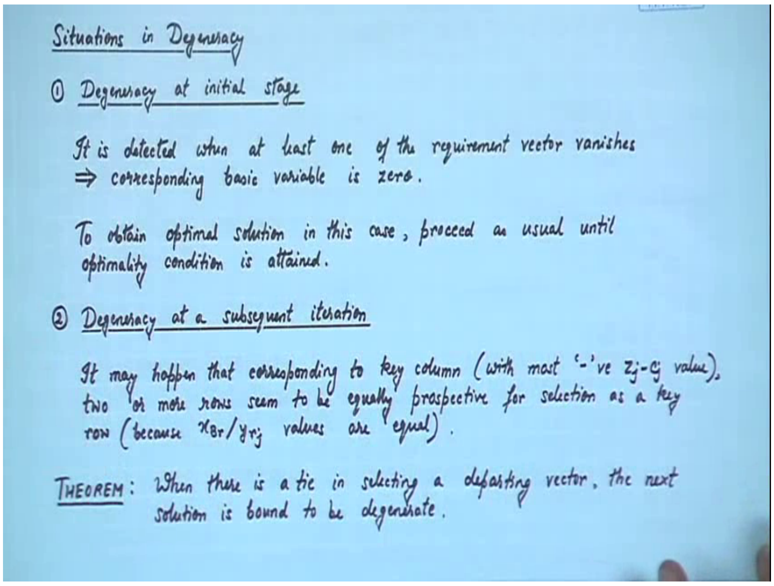

Now, situations in degeneracy. Basically 2 type of situations may occur in the degeneracy, degeneracy occurring at the initial stage itself. That is it is detected when at least one of the requirement vector vanishes. That is corresponding basic variable value is 0. So, initially whenever I have formed I am finding that one of the basic variable is 0 in the initial basis. To obtain the basic solution optimal solution in this case, you proceed normally that is as usual until the optimality condition is obtained. You do not have to do anything extra that is whenever at the initial stage you are finding this one then proceed as usual and it will obtain the required solution.

But the second one is degeneracy at subsequent iteration, it may happen that corresponding to the key column that is where value of Z j minus C j is most negative. 2 or more rows sum to be equal to be equal prospective for selection as a key row. That is if you remember in the earlier examples if I take a table. Whenever I am taking a table you are calculating the ratios.

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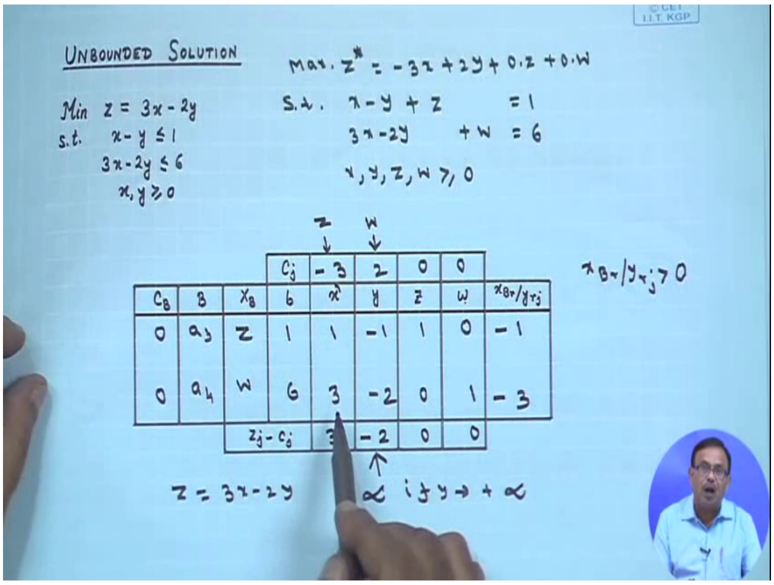

Now, if it may happen that the value of these ratios are same in more than one entry, that is the because of the reason xbr by yrj values are equal. So, degeneracy occurs number one if you can find it initially, where the value of the basic variable is 0. Another one comes whenever you are calculating the xbr by yrj ratio that value is same in more than one element on that ratio column, then also the degeneracy occurs. There is a theorem with respect to this, when there is a tie in selecting a departing vector that is a tie. So, tie will occur whenever the values of the ratio are same the next solution is bound to degenerate. So, whenever there will be a tie in selecting a depart variable then in the next generation their solution will be degenerate. So, I have to find out the method by which I can remove this degeneracy. So, that degeneracy does not occur.

So, if I have degeneracy as we have told earlier then there will be no improvement in the solution you will go on doing it results. Therefore, whenever the ratios of one or one or more than one elements of the ratio column are same degeneracy occurs. So now, let us see how we can improve degeneracy or eliminate overcome degeneracy, Charne's perturbation method to overcome degeneracy.

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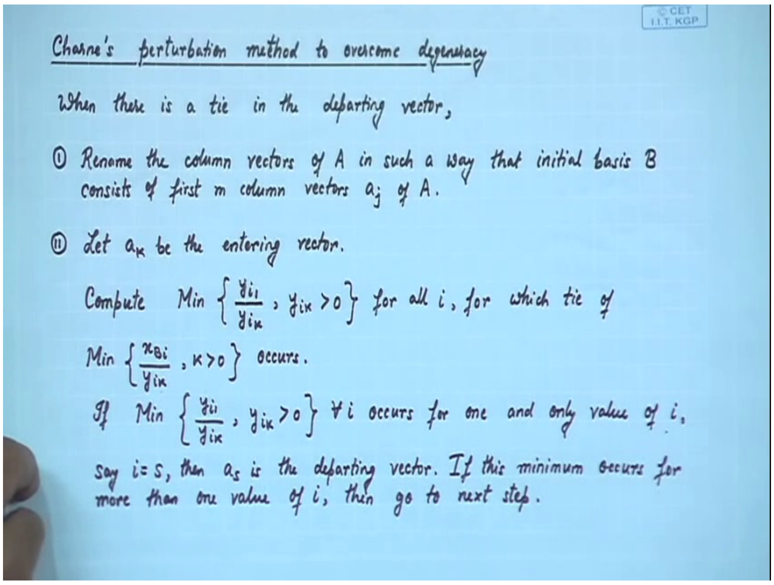

When there is a tie in the departing vector, number one rename the column vectors of a in such a way that initial basis B consists of first m columns of vectors a j of A. Or in other sense if I consider this thing. Here your basis variables are z and w. So now, you consider this as z basically we are saying you consider first column as z and this w you consider as this one.

So, basically here we want to say is that whenever there is a tie you consider the first column as the first element of the basis, and the second column for the second one of the basis which I have written here that rename the column vectors of a in such a way that, initial basis B consists of first m columns of vectors a j of A.

Now, let ak be the entering vector I am giving the example of this, here your y is the entering vector here which I am saying as a k. Compute minimum of yi 1 by yik where yik greater than 0. Minimum of yi one and yik. So, please note that here if this is the case then yi 1 by yik yi one yik whenever you are going to calculate, there, we will consider this as the first column for the basis variable this as the second column for the basis variable. So now, you compute minimum of yi 1 by yik, where yik greater than 0 for all i. For which your tie occurred that is minimum of x bi by yik greater than 0 occurred. If minimum of yi 1 by yik yik greater than 0. For all i occurs for only n 1 1 value of y, that is whenever again you are finding out the minimum and after finding the minimum if I can find the value of i for which and which is unique where the value is minimum.

Which is say i equal says then you will tell a s will be the departing vector. That is for which I the minimum occurred by changing these columns of the changing the columns of the simplex table, that will be the departing vector. And if this minimum occurs for more than one value of i then I have to go to the next step. That is still there is a tie. So, I have to go to the next step.

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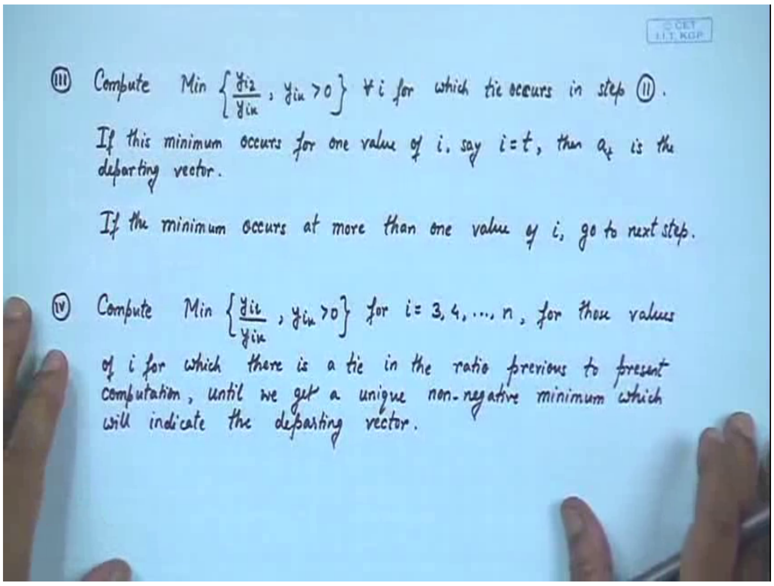

Step 3 is compute minimum yi 2 by yik. That is the next one you take and y in the first case you have taken minimum of y i 1 that is the first column and this here you will take y I 2 by yik, for all i for which tie occurs in step 2. And if this minimum occurs for one value of i say i equals t then your at is the departing vector. So, please note this one then a t is the departing vector. If the minimum occurs at more than one value of i then I have to go to the next step. So, compute like this way minimum of y il by y i k for i equals 3 4 n like this means still you are getting at least 1 minimum you have to continue the process. For these values of i for which there is a tie in the ratio a previous to present computation until we get a unique non negative minimum, which will indicate the departing vector. So, basically I have to repeat the process until I am obtaining a unique minimum.

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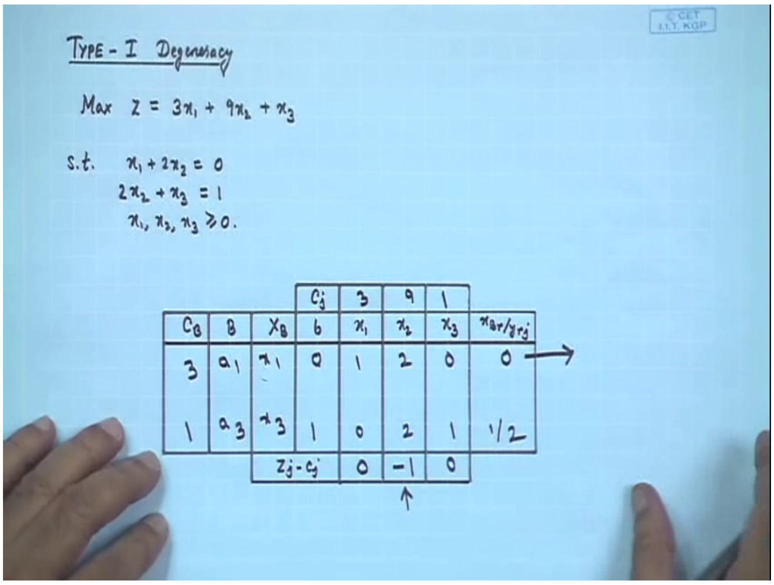

Let us explain this one with example. First one let us take it type one degeneracy that is maximize z equals 3 x 1 plus 9 x 2 plus x 3, subject to x 1 plus 2 x 2 equals 0 and 2 x 1 plus x 3 equals 1 and x 1 x 2 x 3 equals to 0.

Now, here if you see all these variables the constraints are equality constraints. So, as such I do not require any slack or surplus variables here, but earlier we have told if there is a equality to form a basis if required you can add artificial variable also. But from here I can obtain if I make x 2 as 0 x 1 will be 0 and x 3 will be 1. So, that solution basis you will get your basis will be x 1 0 and x 3 equals 1 and which will form one a unit vector 1 0 0 1.

So, for this case I do not need any artificial variable over here. And the basis will be in the basis the basic variables will be x 1 and x 3. So, I am writing here x 1 and x 3. So, therefore, this will be a 1 and a 3, values are 3 9 and one coefficients of x 1 x 2 and x 3 are this. So, here it would be 3 and a 3 is 1 B value is 0 1. So, 1 2 0, 0 2 this is 1 0 2 this is x 1 was absent; so 0 2 1.

Now, if you calculate Z j minus C j value you will find 0 here it is 6 plus 2 8 minus 9. So, minus 1 and this value is 1 minus 1, therefore, your entering vector will be x 2. Your entering vector is x 2 and what will be the this one what will be this case the ratio the first ratio is 0 by 2 that is 0 second ratio is half. So, your departing variable is in this case is x 1. So, x 1 will go out and x 2 will enter, but if you have noted one thing in the basis, for the basic variable the value is as 0 value, in B column in the basic variable is 0 value. So, this is type one degeneracy whatever we have told. Initially we have seen that there is a basis and in the basis value of the basic variable is taking as 0. So, degeneracy occurs, but in this case you do not have to bother anything just proceed as usual to solve the normal problem. That is the you do the next iteration of the simplex algorithm because you know what is the entering vector here and you know; what is the outgoing vector here.

So, for this case x 1 is the outgoing vector and x 2 is the entering vector. So, x 1 will be replaced by x 2 in the next table. So, if I form let us form the next table.

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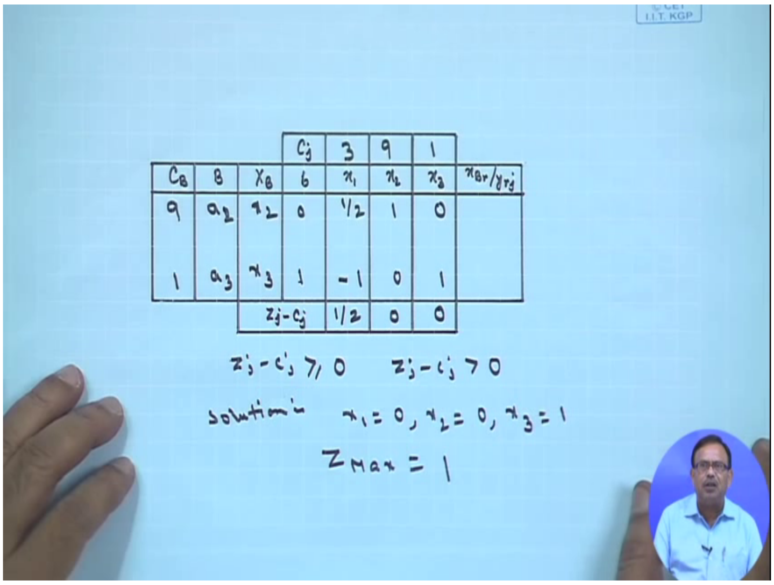

Now, in the next table your x B will be then x 2 and x 3 because x 1 is replaced by x 2 you are having a 2 and a 3 c B values remains the same that is 3 9 and 1. So, x 2 is 9 this is one in the earlier one you are pivot element is 2, this is the pivot element therefore, your this I have to make as one and all other elements of this column have to make us 0. So, I will make this that is B value I am directly writing 0 half 1 and 0. This will be 1 minus 1 0 and 1. So, this one is one entering vector and this column is 0. Z j minus C j value if you calculate it will become half this will be 9 minus 1 0 this will be 1 minus 1 0.

So, here if you see your Z j minus C j is greater than equals 0 for all j. And for the non basic variable it is Z j minus C j is greater than 0, it is here. That is this value for non basic variable x 1 Z j minus C j is greater than 0; whereas, for basic variables x 2 and x 3 Z j minus C j is greater than 0. So, you will obtain the unique optimal solution in the basis you have only 2 ds and variables x 1 and x 2. So, your solution is x 1 equals 0 x 2 equals from here it is 0 and x 3 is equals to 1 and z max that is maximum value of z from here it is one this into 0 plus this into this.

So, 1 into 1; so, value of z max is one because z will be z is c B into B. So, z max is equals to 1. So, basically if you see although we had the g degeneracy which we obtained in the initial table itself, we have checked there that the value of a basis variable basic variable is 0. So, in this case we do not have to bother anything we proceed as usual and we will obtain the solution.

Now, let us see the type 2 degeneracy in the type 2 degeneracy what happens? In the type 2 degeneracy Let us take an example.

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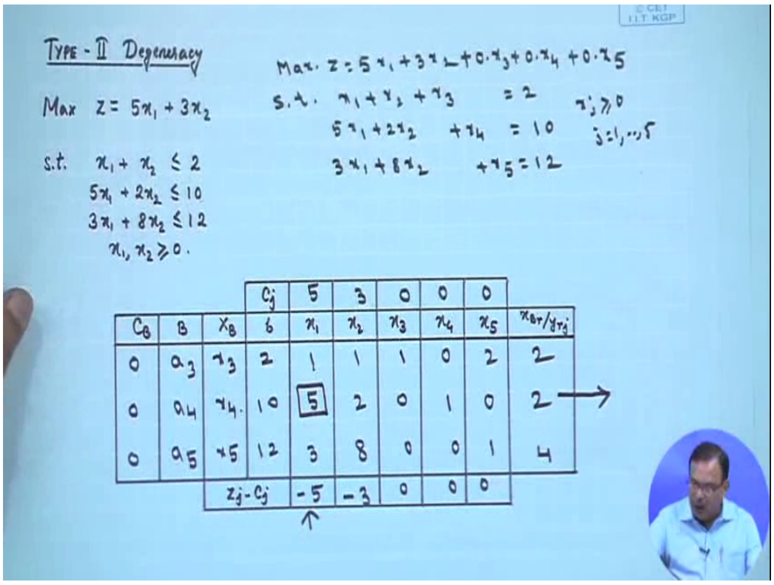

Maximize z equals 5 x 1 plus 3 x 2 subject to x 1 plus x 2 less than equals to 5 x 1 plus 2 x 2 less than equals 10, 3 x 1 plus 8 x 2 less than equals 12. So, we have 3 inequalities of less than equals type. So, you have to introduce 3 slack variables. So, that in standard form we can write down maximize z equals 5 x 1 plus 3 x 2, subject to x 1 plus x 2 plus x 3 this is equals 2, then 5 x 1 plus 2 x 2 plus x 4 this is equals 10. And 3 x 1 plus 8 x 2 plus x 5 this is equals 12. So, x 3 x 4 and x 5 these are the basis sorry, these are the slack variables we which we have introduced, and xj greater than equals 0 x j equals 1 to 5.

So, initial basic feasible solution will be if I make x 1 and x 2 as 0 then x 3 will be equals to 2 x 4 equals 10 and x 5 equals 12. So, initial basic feasible solution is x 3 equals 2 x 4 equals 10 and x 5 equals 12. So, in the basis you are having 3 variables x 3, x 4 and x 5. Here I am writing a 3 a 4 and a 5; obviously, in the objective function here the coefficient of artificial variables will be 0 only plus 0 into x 4 plus 0 into x 5. So, your objective function is z equals 5 x 1 plus 3 x 2 plus 0 into x 3 plus 0 into x 4 plus 0 into x 5. So, the C j values are 5 3 these, these and this. Your c B values x 3 x 4 x 5. So, these are 0s B values are 2 10 and 12 these are 1 1 1 0 2. This is 5 2 0 1 0 5 2 0 1 0 next one is 3 8 0 0 and 1.

So, again you calculate Z j minus C j if you calculate Z j minus C j this will be c B into B all are 0 minus 5 this will be minus 3 0 0 0. So, most negative is this column that is your x 1 will be the entering vector it is fine your x 1 will be the entering vector in this case. So now, you have to calculate the ratio now what will be the ratio ratio is 2 by 1, which is true then it is 10 by 5, this is 2 and 12 by 3 this is 4.

So, the question comes now here if you see the ratio. In more than one element has the same value in the ratio column in both case it is 2. So, there is a tie and whenever there is a tie if you go for next iteration by choosing any one of them then degeneracy will occur. From the theorem we can tell that much. So, whatever we have told from there I can conclude that now I have to use that method computational method what we have told. According to the solution as we have told the next iteration must be will give me de degenerate.

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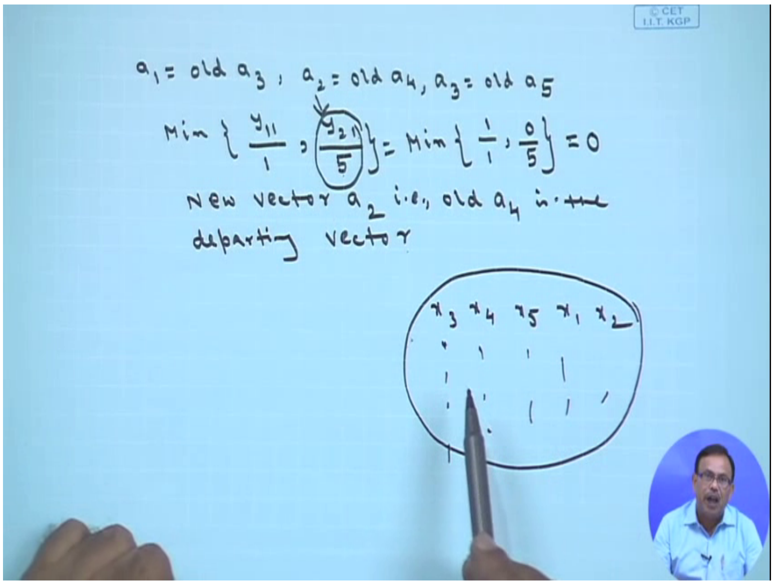

So, let us assume that I am keeping it inside your a 1 equals old a 3. So, a 2 equals old a 4, and a 3 equals old a 5. Means as we have told what I have to do? I have to recompute the ratios again, but before that I have to consider the first column as the first variable in the basis. That is now for x 3 I will consider this is the column, for x 4 I will consider this is the column and for x 5 I will consider this is the column. Means first 3 columns will now represent the 3 basic variables x 3 x 4 and x 5. And that is the reason we are writing this one a 1 equals old a 3 a 2 equals old a 4 and a 3 equals old a 5.

Now, I have to find out minimum of this 1 minimum of first I have to take yik i 1 then if you remember in the earlier degeneracy examples it was yi 1 by yik, yi 1 by yik. Where your value of i in kth order or k greater than 0 minimum occurs your. Entering vector is a k. So, it is minimum of yi 1 by yik. So, I will write down is minimum of y 1 1 divided by what was that yi 1 by y i k was there. So, I will tell that minimum of this one, that is yi 1 by y i k next one will be y 2 1 by y I k. So, what is yik yi k will be the entering vector for this one that is I have to take the element one for the first case and for the second case I have to take the element 5, in this case. So, it is y 1 and it is 5; that means, y y 1 1 1 by y 1 1 and this is y 2 1, this is your 5.

Now, what is yy 1 1? Now what is your y 1 1? Y 1 1 is your x 3 that is your this and 0. So, therefore, minimum of it will be equals to 1 by 1 comma 0 by 5. So, this I am getting from the next 2 variables. Because this has come here now because we have told we have interchanged x 3 x 4 x 5 now coming at the first 3 columns. So, since x 3 x 4 x 5 are the first 3 columns. So, y 1 1 will be one and then y 2 1 this will be 0. So, we are taking 1 by one comma 0 by 5, and the value is this one there therefore, the new vector the new vector a 2 it is occurring for y 2 1 by this one old value.

So, new vector that is basically old a 4 will be the departing vector, old a 4 is the departing vector. So, is it clear little bit let me explain once more since we have the ratio in 2 elements are same there is a tie. So, once there is a tie, what I have to do? I have to compute the value of the new ratio for that reason what I will do I am writing a 1 equals old a 3 a 2 equals old a 4 a 3 equals old a 5 and this table I will visualize that x 3 x 4 x 5 has come first; then x 1 then x 2. I may write down a new table basically, where the columns will come as x 3 x 4 x 5 then x 1, then x 2 this I am visualizing. So, that the elements of these will be now changing.

Now, now the minimum is this is the column yi i will take the first one. So, y 1 1 divided by y 1 1, old value of y 1 1 is 1. So, I have made it one then it is y 2 1 by this one this value here it is 5. So, it is 1 by 5 now what is yy 1? This y 1 1 I will take from this new table. So, y 1 1 means basically the value of x 3 that is one. So, I am writing 1 by one and in the second column it is 0. So, it is 0 by 5 and it is equals to 0.

So, therefore, the minimum is occurring for the vector this one this ratio is this which corresponds to this vector a 2. So, new vector a 2 that is old vector a 4 will be the departing variable. So now, we are now a 4 is the departing variable. So, since a 4 is the departing variable, your pivot element is this one. So, like this way you are just finding out whenever there is a tie, by this method if you select which the vector will be the departing variable.

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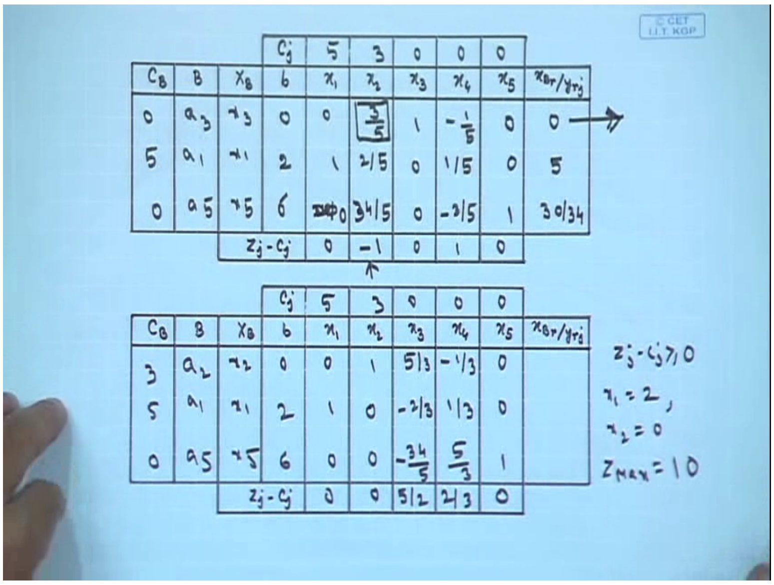

So now let us go to the next column in the next column what happens your x 4 will be replaced by x 1. So, here it will come x 3 x 1 and x 5. This is your a 3 a 1 a 5 your C j values are 5 3 1 2 and 3. So, you here it will be 0 5 a 1 is 5 it is 0. Your B these values will become now 0 0 3 by 5 1 minus 1 by 5 and 0. The second row will be 2 1 2 by 5 0 1 by 5 and 0 third one will be 0 34 by 5, sorry this will be 6 this will be 0 next one is 34 by 5 this is 34 by 5 0 minus 3 by 5 and one. If you calculate Z j minus C j it will become 0 minus 1 0 1 0. So, that your entering vector is x 2 now. So, I have to calculate the ratio it will find that 0 5 30 by 34. So, the minimum is this one therefore, your departing vector is x 3. And this is your pivot element and there is no degeneracy till now, sorry that is a degeneracy because one of the basic variable is 0 over here.

So, you are calculating this in the next table your x 3 is going out it will be depressed by x 2. So, x 2 x 1 x 5 5 3 0 0 0, and c B is a 2 a 1 a 5. These values are 3 5 and 0. So, pivot element will be one all other elements of that column will be 0, I am writing directly 0 0 1 5 by 3 minus 1 by 3 and 0. This will be 2 1 0 2 1 0 minus 2 by 3 1 by 3 0 6 0 0 minus 34 by 5 5 by 3 and this is. So, if you calculate now it will be Z j minus C j is 0 0 5 by 3 2 by 3 and this is 0.

So, therefore, Z j minus C j is greater than equals 0 for this case. And Z j minus C j Z j minus C j is greater than 0 for non basic variables, that is at the these 2 cases x 3 and xc 4 therefore, you will obtain the unique optimal solution your optimal solution is x 1 equals 2 x x 2 equals 0 and your z max from here it will be 5 into 10 that is 10.

So, you are obtaining the optimum solution. So, whenever there is a tie we have to find out the solution. Now quickly let us see one thing that is if there is a time why we are told that redundant constraints.

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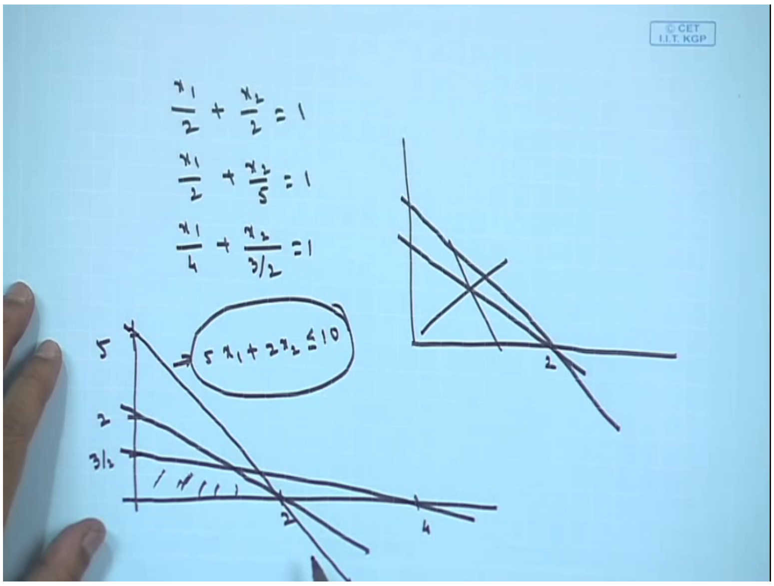

From the earlier problem I can write down the constraints like this. X 1 by 2 this x 1 by 2 plus x 2 by 5 equals 1 x 1 by 4 plus x 2 by 3 by 2, this is equals 1. If I draw something like this there will be a line which will go through this is a point 2. There will be another line which will go to and 2 like this. I am sorry and this is no I have not drawn it correctly, this is not correct there will be a point 2 here there will be a point 4 here, here I need a point 3 by 2 2 and there will be a point 5.

So, that will be 3 by 2 and 4. So, one line will be like this there will be one point which goes through 2 2. So, this will be 2 2 and there will be another one which goes through 5 2. So, 5 and 2 so, if I draw like this another point will be like this. So, this is the feasible region. So, this one this is your 5 x 1 plus 2 x 2 is less than equal 10. And this one is redundant because this is if I do not write down this then also your feasible region will remain same the characteristics of the problem remains same.

So, this constraint is the redundant constraint, and due to this redundant constraint only your degeneracy occurred in the problem. So, whenever there are redundant constraints there will be degeneracy, or degeneracy can occur in 2 ways one it can be that is a 0 value of the basic variable in the initial step itself. Do not bother proceed as usual the other way degeneracy occurs whenever there is a tie and if there is a tie to reserve the tie follow the procedure which we have discussed just now.