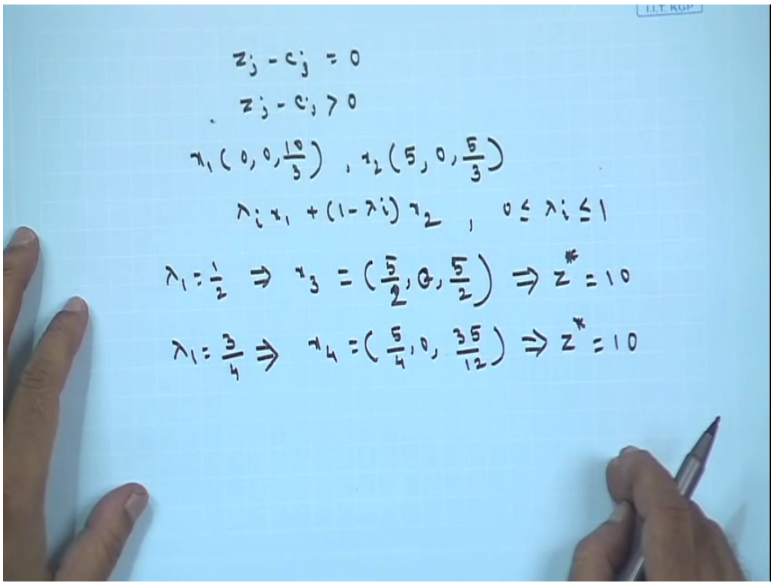
Constrained and Unconstrained Optimization
Prof. Adrijit Goswami
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 13
Special Cases of LPP

In this lecture, we will discuss some special type of LPP problems. Basically we will discuss few things 3 different things. First we will discuss about the alternate solution or infinite number of solutions. Then we will talk about the unbounded solution that is whenever the solution is unbounded. And number 3 we will discuss about the degeneracy whenever degeneracy occurs how to come out from that. These 3 things we will discuss now. The first one is the alternate solution.

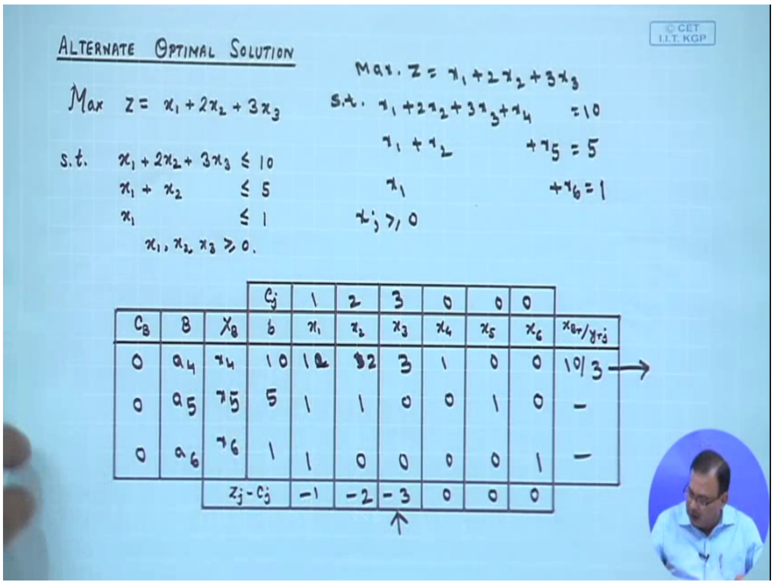
(Refer Slide Time: 01:06)



If you remember, we have told one thing that is Z j minus C j, Z j minus C j if this is equals 0 for basic variables. Z j minus C j should be equals to 0 for basic variables, and Z j minus C j should be greater than 0 for non basic variables. In general for optimality Z j minus C j should be greater than equals 0.

So, Z j minus C j should be 0 for basic variables. And Z j minus C j should be greater than 0 for non basic variables. So, this we will see.

(Refer Slide Time: 01:48)



Let us take one example for alternate optimal solution or the maximum or infinite number of solutions. Let us consider one problem maximize z equals x 1 plus 2 x 2 plus 3 x 3, subject to x 1 plus 2 x 2 plus 3 x 3 less than equals 10. X 1 plus x 2 less than equals 5 and x 1 is less than equals 1.

So, first we will write it as in the standard form. As we already we have done the inequalities unless than equals type. So, I have to introduce 3 slack variables, and to make this inequality constraints as equal. So, in standard form we can write down maximize z equals x 1 plus 2 x 2 plus 3 x 3 subject to x 1 plus 2 x 2 plus 3 x 3 plus x 4 this is equals 10. X 1 plus x 2 plus x 5 this is equals 5, and x 1 plus x 6 this is equals 1. And of course, your xi or X j are greater than equals 0 for j equals 1 to 6.

So, at first you are introducing 3 slack variables x 4 x 5 and x 6. And you are making the less than equals type inequality into equality. So, that your initial basis will be x 4 equals 10 x 5 equals 5 and x 6 equals 1 where x 1 x 2 x 3 are 0. So, therefore, in the basis your variables are x 4 x 5 and x 6. So, I am writing here x 4 x 5 and x 6. So, these are a 4 a 5 and a 6. Your C j values are normal that is 1 2 3 and for x 4 x 5 and x 6 the values are 0. So, your C B values will be 0 0 and here 0.

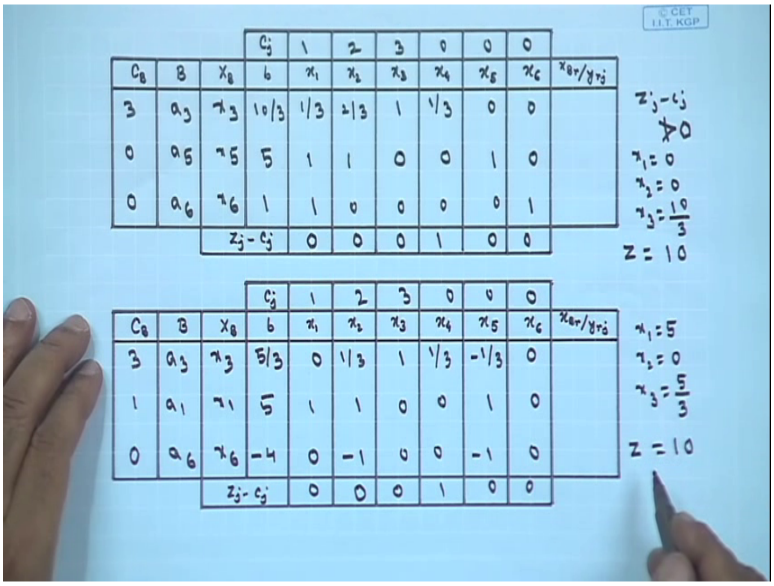
Now write down the rows that is 1 2 3 sorry, b value is 10 5 and 1, 10 5 and 1. So, this will be 1, this will be 2, this is 3 and x 4 is 1. X 5 x 6 are 0. For the second one x 1 x 2 is

1 x 1 x 2 then x 3 x 4 are 0 and this is 1. X 5 is one x 6 is 0 whereas, for the last one x 5 x 1 and x 6 coefficients are one all others are 0s.

So, you calculate now Z j minus C j that is C B into X B minus C j. So, Z j is C B minus X B minus C j that is this coefficient value. If you calculate you will find it will be minus 1 minus 2 minus 3, and these 3 will be 0s are there. So, 0 So Z j minus C j is less than 0 and most negative is this one most negative is this one therefore, your entering variable will be x 3 in the next iteration. So, you have to calculate the ratio b by yrj y means I am talking about the values of the coefficients. So, this will be 10 by 3 and these 2 cases infinity.

So, the question does not arise. So, only 1 minimum that is your departing variable will be x 4. So, in the next column your departing variable will be x 4 an entering variable will be x 3. That is x 4 will be replaced by x 3. So, from here if you see you are entering variables are becoming x 4 will be replaced by x 3.

(Refer Slide Time: 06:25)



So, x 3 x 5 and x 6 are coming here, you are having a 3 a 5 and a 6 one more thing that is in the earlier one your, this is x 3 is the entering variable and x 4 is the outgoing variable departing variable. So, your pivot element is this one that is 3 pivot element is yet 3.

So, that I will make one and all other elements of that column will be 0. Your C j values are same 1 2 3 0 0 0. So, that now x 3 has entered. So, it will be 3 0 and 0. Here it will be

I am just a directly writing 10 by 3, 1 by 3 by 3 it will be 1, 1 by 3 0 and 0. Here it will be 5 1 1 corresponding to this element will be 0 0 1 and 0. Here it is one b value is a 1 then this one will be 1 0 0 0 0 and 1.

So, this will be the next level and here if you see in this case your Z j minus C j value will be 3 into 1 by 3 minus 1. So, this will become 0, this is also becoming 2 minus 2. So, this is also 0, this is 3 minus 3. So, this is becoming 0, after that it is one it is 0 and it is 0. So, in this case if you see all Z j minus C j is greater than equals 0. So, that is no need of calculation. So, Z j minus C j greater than equals 0, but Z j if you note here. Z j minus C j this one is not greater than equals 0 not greater than 0 for non basic variables x 1 and x 2 please note this one I will come to this point that Z j minus C j is equals to 0, not greater than 0 for the non basic variables x 1 and x 2.

So, in this case your optimal solution will be in the basis you have only one decision variable x 3. So, in the basis your solution will be x 1 0 x 2 0 and x 3 is 10 by 3. Sorry x 3 is 10 by 3. So, therefore, your x 3 will be 10 by 3 and value of z or z max maximum value of z you can obtain from here as z. So, at first you are obtaining the solution for Z j is greater than equals 0. Z j minus C j is greater than equals 0, but we also note one thing that Z j minus C j are 0 for 0 for 2 non basic variables x 1 and x 2. Does we can make 2 different tables again by entering one as x 1 and entering another table I can make by x 2 and I can get some other solution; that means, we want to say here is that, if I change from here x 3 x 1 and x 6. The values will be a 3 a 1 and a 6 your C j is same 1 2 3 0 0 and 0.

So, that here it will be 3 1 and 0. So, your b values are you are just making it 5 by 3. 0 because you have entered here 5 by 3 0 1 by 3 1 1 by 3 minus 1 by 3 and 0. This column will be 5 1 1 0 0 1 0 this will be may not changed. And this one will be x 6 corresponding to x 6 it will be 0 minus 1 0 0 minus 1 and 0.

So, from here by replacing x 5 by x 1 and the reason is your x 1 and x 2 are non basic variable in the optimum table to their value is 0. Therefore, if I replace in the basis one variable by either by x 1 or by x 2 I will get other solution that we are checking by replacing this, if we calculate again Z j minus C j value you will find that all these are 0 0 0 1 0 0; that means, Z j minus C j is greater than equals 0 therefore, you can tell that other solution is x 1 in the basis now x 1 and x 3 are present. So, x 1 is 5 x 2 is 0 and x 3 equals 5 by 3. They are satisfying the constraints. And if you calculate the z value here again in that case again z value is 10.

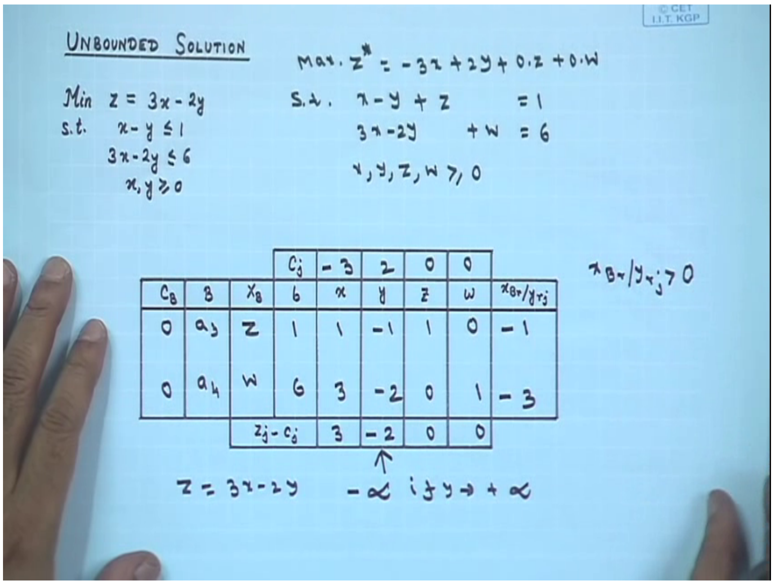
So, optimum value of the objective function remains same it is unaltered that is it is value determines 10, but the feasible basic feasible solution changes or the values of the decision variables changes. So, this one table I have shown where by replacing the non basic variable x by some basic variable, I am getting alternative solution. Similarly if I replace some variable of the basis by that another non basic vector x 2 then I will obtain another solution. And since I have obtained 2 solutions now any linear combination of these solutions also will be a solution.

So, but the of value of the optimum value of the objective function will remain same; that means, we want to say that suppose your x 1, here is this point x 1 is this value I am taking 0 0 10 by 3. 0 x 1 is 0 0 and 10 by 3, and x 2 I will take 5 0 and 5 by 3, say because already we have seen that x 2 is 5 0 and 5 by 3. So, since these 2 are the solutions. So, any convex combination of these 2 will also be a solution of this one. That is any convex combination means lambda I into x 1 plus 1 minus lambda I into x 2, that will also be a solution when lambda lies I lies between 0 to 1. Suppose we choose lambda I one equals half. In that case I can obtain another point x 3 by using this combinations I can get 5 by 2 0. And sorry 5 by 2 0 and 5 by 2. This is also another optimal solution and the value of z max at this point will be will remain the same that is z star.

What if I choose lambda 1 equals 3 by 4 in that case I can obtain another solution as 5 by 4 0 and 30 5 by 12. And in this case also optimum value of z remains same that is 10. Therefore, for this particular case what is happening if you see here, we have the optimum solution, but the number of optimum solutions are infinite. So, if I have to find out by finding 2 solutions from there I can generate optimum number of solutions by changing the value of lambda where lambda lies between 0 to 1.

So, this is one case where you can obtain infinite number of optimum solutions. The second point comes second case comes is the unbounded solution. The second point is unbounded solution.

(Refer Slide Time: 15:50)



Let us see what is unbounded solution. In this case let us take a problem a minimization problem we have taken. Z equals 3 x minus 2 y subject to x minus y less than equals 1 3 x minus 2 y less than equal 6, since it is minimization problem.

So, at first I have to write it in the standard form and I have to make it maximize. Z star will be equals to minus 3 x plus 2 y, because you know that maximum of z equals mean of minus z. Here we have 2 less than equals type constants. So, we have to add 2 slack variables over here to make the equations as inequalities. So, subject to x minus 1 x minus y plus. So, I am taking z this is equals 1. Another one is 3 x minus 2 y plus w this is equals 6, say where x y z and w all are greater than equals 0.

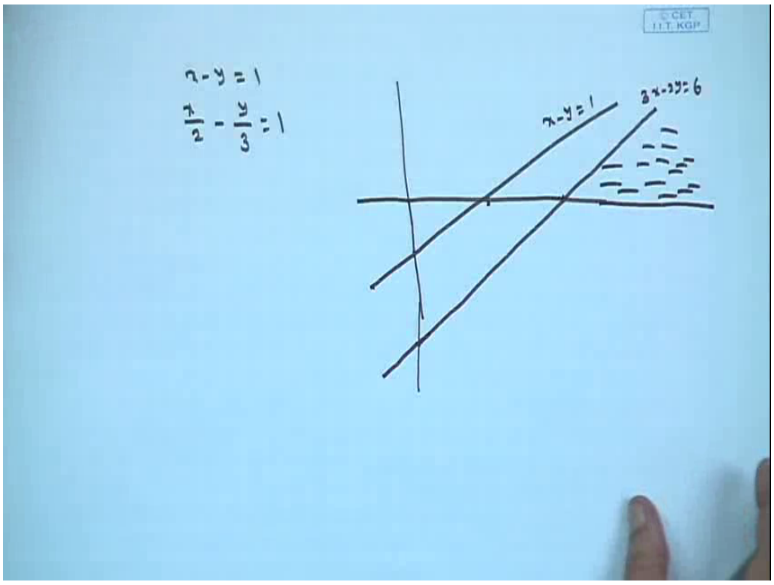
So, all are greater than equals 0, but therefore, what is happening in the standard form? Since it was minimization problem I am converting it into maximization problem that max z star subject to this 2 equations, where inequalities are replaced by the equalities by adding the slack variables. Now let us apply the simplex algorithm to check what is happening over here. So, in the basis the variables will be z and w the initial basic solution feasible solution will be z equals 1 w equals 6 and value of x and y are 0. So, in the basis your z will come and w will come. So, I am writing this a 3 and a 4. Your C j values are minus 3 2 coefficient object and coefficient of w are 0. So, you can write down 0 into z plus 0 into w in the quadratic in the objective function.

So, your C B is 0 and 0 b values are 1 and 6. So, it is 1 minus 1 1 0. Next one is 3 minus 2 0 and 1. So now, we calculate the Z j minus C j, that is C B into X B minus C j. So, the values you will obtain as 3 minus 2 0 and 0. So, in this case your entering vector is basically y your entering vector will be y. And it will be replaced by which one it will be replaced by the ratio b by X j so, but in both case if you see first one is 1 by minus 1, that is minus 1 and in this case also it is 6 by minus 2 that is negative minus 3.

So, both are minus 1 and minus 3. So but as per theory which we have told earlier X B r by yrj must be here this ratio should be greater than 0. So, the computationally this should be greater than 0. It cannot be negative and if this X B r by Y r y br X B r by yrj value is less than 0 then there will be unbounded solution. What it means actually; that means, if we start increasing the value of y then the value of the objective function z will go on increasing indefinitely. Because there is no change or no other variable is coming departing from here. If you see z equals 3 x minus 2 y this is your objective function and this will decrease to minus infinity if you make y approaches plus infinity. If y increases z equals 3 x minus 2 y will approach to minus infinity; that means, as you increase the value of y value of the objective function will go on decreasing.

So, there is no minimum value of this. Therefore, for this case the solution is unbounded. You will obtain unbounded solution means there is no optimum solution you can obtain. And the reason is also quite clear from here that whenever y is approaching plus infinity in that case the objective function decreases to minus infinity; that means, if I go on increasing the value of y z will decrease like anything. So, I do not know where it will finish therefore, the solution will become unbounded.

(Refer Slide Time: 21:46)

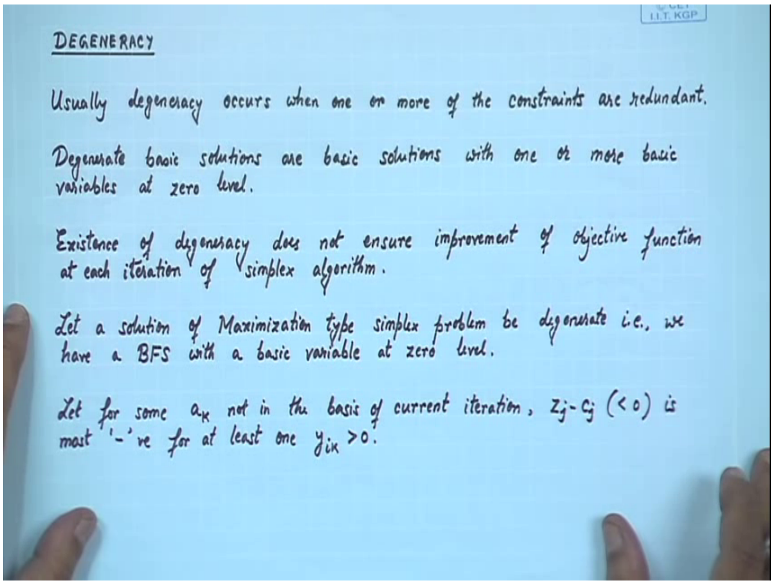


And this will be quite clear whenever if I try to think about the graphical case. Your problems are x minus y equals 1 the constants if I say and x by 2 minus y by 3 this is equals to also 1.

So, if you try to draw something like this over here in this case, then if this is the point 1 0 0 minus 1. So, one line will go like this. And there will be another one that is 2 0 and this is the line x minus y equals 1 and this will be another 1 0 minus 3. So, there will be another point something like this, which is the line 3 x minus 2 y equals 6. So, if you try to see the feasible region from here, feasible region will be on the positive axis that is positive x axis y axis and the lines bounded by these. So, this is here.

So, it is not feasible region itself is not bounded. So, please note this thing feasible region is not bounded means the solution will be unbounded therefore, in the ratio that is X B r by yrj if it is less than 0 always we will obtain the unbounded solution.

(Refer Slide Time: 23:20)

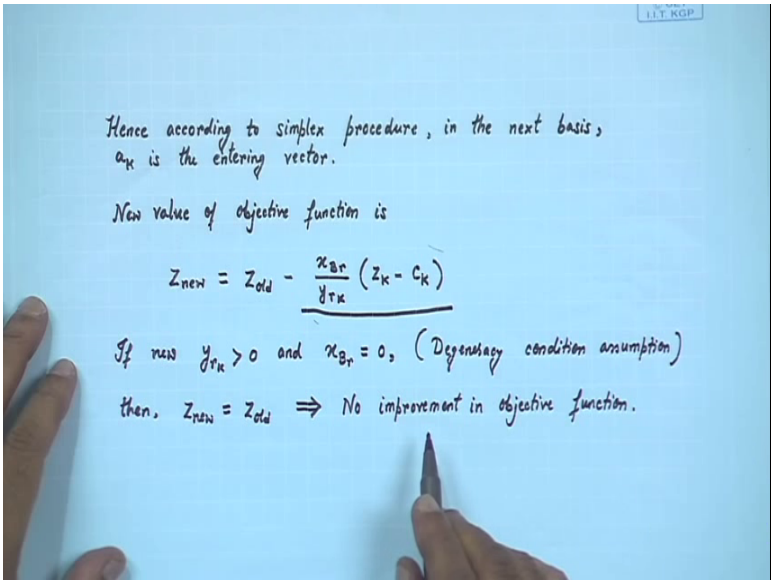


So, this we have to remember. Our third point is the degeneracy. Usually degeneracy occurs when one or more of the constraints are redundant. Here redundancy means we want to say that degeneracy occurs means constraints are redundant means if I do not write down a particular constraints then also the nature of the problem will be unaffected or the solution will remain unaffected.

So, please note that usually degeneracy occurs when one or more of the of the constraints are redundant. Degenerate basic solutions are basic solutions with one or more basic variables at 0 level. So, if the problem or if I have redundant constraints, in that case the basic solution which you will obtain you will find that the value of one or more basic variables is equals to 0. Existence of degeneracy does not ensure improvement of the objective function at each iteration of the simplex algorithm, please note this point. If degeneracy exists then there is no guarantee it does not ensure that objective function value will improve at each iteration of the simplex algorithm; that means, it may happen that you are going on doing iteration and instead of improving you are coming back to the same solution afterwards.

Suppose you have a solution of maximization type, simplex problem be degenerate. That is we have a basic feasible solution where a basic variable is at 0 level. We are assuming that for some k not in the basis iteration your Z j minus C j is less than 0 is most negative, for at least one yik greater than 0. So, for some ak which is not in the basis in this current iteration Z j minus C j is less than 0, and which is most negative for at least one ik greater than 0 or in other sense your ak.

(Refer Slide Time: 25:59)



Or the vector xk will enter into the basis, Since Z j minus C j is most negative for that value. Therefore, hence according to simplex procedure in the next basis ak is the entering vector. Now the value of the objective function will be which one z new equals z old minus X B r by yrk into zk minus ck. Note this one old value this is the iterative value will be changing z old equal minus X B r by yrk into zk minus ck. Now if yrk is greater than 0 and X B r is equals to 0, X B r means we are saying that the element which is in the basis it is 0. Because we have assumed one thing that is whenever degeneracy occurs in the basis or in the basic feasible solution in the feasible solution you have one basis basic vector whose value is 0.

So, therefore, your yrk is greater than 0 it has to be greater than 0 and X B r is 0. So, whenever X B r is 0; that means, this term if you see this term this term vanishes. So, therefore, z new is equals to z old. Or in the other sense no improvement in the objective function. Therefore, please note this thing that whenever degeneracy occurs usually degeneracy occurs due to the redundant constraints. And whenever degeneracy occurs it may happen that in iteration after iteration you are not getting any improvement in the objective function. So, this is one thing that it does not degeneracy does not ensure that there will be any improvement in the value of the objective function in the iterations of the next iterations of the simplex algorithm. So, in the next class we will see other property that recycling of the degeneracy.