Constrained and Unconstrained Optimization
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Lecture - 12
2 Phase Method: Problem Solution

So, let us continue with the earlier class, where we have discussed the algorithm of the 2 phase method and we have shown example. If you remember on that example we have shown that in the phase one whatever solution you are obtaining their artificial variable was not present in the basis; that means, maximum value of z star was 0, and basis artificial variables was not present they are and we obtain the optimal solution using phase 2 method.

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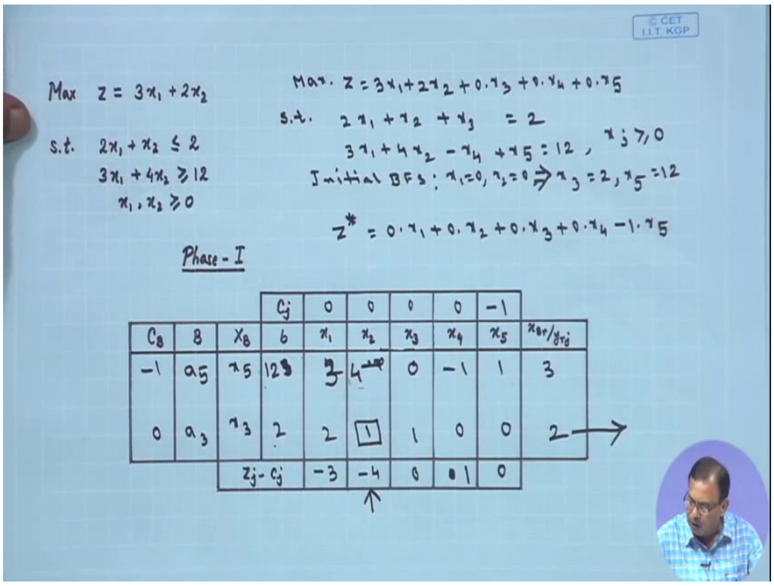

Now let us take up another example of different type, please see this one; Maxima is z equals 3 x 1 plus 2 x 2 subject to 2 x 1 plus x 2 less than equals 1 2 3 x 1 plus 4 x 2 greater than equals 12 and x 1 x 2 greater than equals 0. So, I have to convert this one into standard form, first by using the slack variable, surplus variable and artificial variable since you have one less than equals type constraint and another one greater than equals type constraint.

So, your problem is in standard form I can write down maximize z equals I am writing this later 3 x 1 plus 2 x 2, I will write down the others afterwards subject to 2 x 1 plus x 2 plus x 3 this is equals 2 where x 3 is the slack variable please note this corresponding to this less than equals inequality. And 3 x 1 plus 4 x 2 minus x 4 plus x 5 this equals 12.

So, here your x 4 is the slack variable sorry surplus variable, and x 5 is the artificial variable. So, that here you will add accordingly 0 into these things. 0 into x 3 plus 0 into x 4 plus 0 into x 5. Means again I am repeating this for artificial variable in 2 phase method the coefficient is 0 not like big m method where he was using the coefficient as minus m, and your x j is greater than equals 0 for j equals 1 to 5. This is your problem if you see for this case initial b f's basic feasible solution will be x 3 equals 2; that means, by making x 1 x 2 at the original equations if I make x 1 equals 0 and x 2 equals 0 in that case I will obtain this implies x 3 will be equal to 2, and x 5 will be equals to from this equation x 5 is equal to 12 So that your x 4 automatically becomes 0.

So, the initial basis 2 variables will come that is x 3 and x 5 that is one slack variable and one artificial variable x 5. Now your auxiliary objective function which we are denoting say z star will be equals to 0 into x 1 plus 0 into x 2 plus 0 into x 3 plus 0 into x 4 minus 1 into x 5. As we have told with artificial variable we will the coefficient will be minus 1 where as for all other decision and at slack and surplus variables the value will be 0.

So, this is the new auxiliary objective function, and I have to now optimize this auxiliary function subject to the original set of constants. So, in this case you are in the basis you have 2 variables that is x 5 and x 3. It is your a 5 and a 3 values are 0 0 0 0 for x 5 only coefficient is minus 1. So, here it will be minus 1 0. So, I am writing I have to write down corresponding to x 5, means the second constraint first I can write the other way also first x 3 then x 5 the problem will remain same.

So, it will be 3 4 minus 1 sorry, 3 4 x 3 is not there. No, your b is 2 sorry I am sorry your b is here 12. So, this is your 12. So, it will be 2 then 12 after that it is 3, then it is 4 x 3 is 0 minus 1 1. So, it is here b value is 2. So, it will be 2 1 1 0 0. If you calculate the Z j minus C j value as usual C B into X B minus C j he will obtain minus 3 minus 4 and 0 0 0 this is 1.

So, it will be 1 and this will be 0. So, Z j minus C j is less than equals 0 therefore, most negative is minus 4. So, entering vector will be this, one most negative Z j minus C j is minus 4. So, entering vector is this. So now, I have to calculate the ratio that is here it is 12 by 4 that is 3 and for this case it is 2 by 1 2. So, you are entering variable departing variable will be x 3. So, your x 3 is going out and x 2 will enter into the basis, and your pivot element is this one intersection of the corresponding low and column of the departing vector and the entering vector.

So, this is already one I do not have to do anything to this row, I have to make this element as one for sorry this element as 0 for manipulation.

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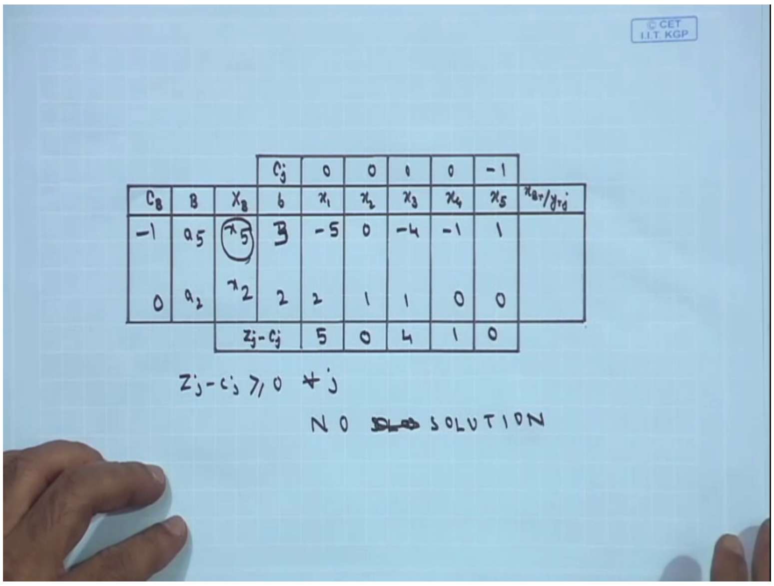

So, for this case, your x 3 will be replaced by the x 2. So, that you will obtain x 5 and x 2. All other things will remain same that is 0 0 0 0 and minus 1, here it is a 5 and a 2. So, that your C B will be minus 1 and 0.

So, second row will remain same that is it will remain 2 2 1 1 0 0. And the first row I have to make 4 as 1. So, I have to make this one as one. So, it will remain 4 minus 5 0 minus 4 minus 1 and 1. So, it will be this in this case, it will not be 4 I think just 1 minute. So, it will be 12 is there 12 divided by 4. So, this value will be 3.

Now calculate the Z j minus C j value Z j minus C j value will be 5 0 4 1 0. So, if you see your Z j minus C j is greater than equals 0 for all j. So, what will be the conclusion? The conclusion is that whether should I go to phase 2 for the optimal solution. This is your basic feasible solution or BFS, but this will not give you the optimal solution, because of the case that your artificial variable x 5 is present in the table. Please see this one your x 5 is present in the basis. This x 5 is present in the basis.

So, according to our algorithm if Z j minus C j greater than equals 0 and basic variable is present in the basis, then there will be no solution there will be no solution. So, this is the second type of problem although Z j minus C j is greater than equals 0, but the basic variable is present artificial variable is present in the basis. Therefore, the problem will have no solution we can check it graphically also for this case.

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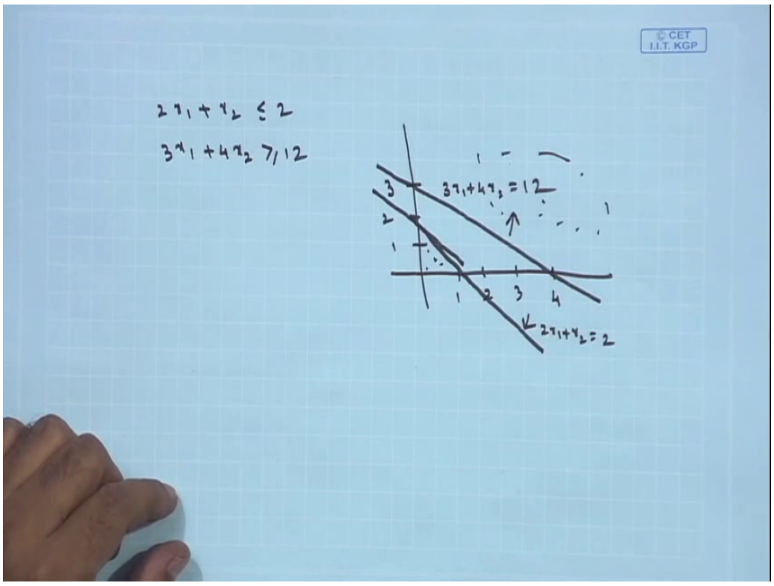

Your problem was if you remember our constraint was 2 x 1 plus x 2 less than equals 2 and 3 x 1 plus 4 x 2 greater than equals 12. If I draw the graph of these things I think there will be 1 2 if I make it 2.

Somewhere 3 somewhere say 4 and in this case 1 2 and 3. So, 2 x 1 plus 1 2; that means, 1 and if x 1 0 x 2 is 2. So, one line will be this one which is 2 x 1 plus x 2 equals 2 it is less than equals 2. So, this side other one is 3 x 1 plus 4 x 2 equals 0 that is x axis will be 4 y axis will be 3. So, the line will be something like this. This is 3 x 1 plus 4 x 2 equals 12 this line and it is greater than equals 2.

So, the direction will be on this side. So, if you see on the positive side for the line 2 x 1 plus x 2 is less than equals 0 means, this area whereas, for 3 x 1 plus 4 x 2 greater than equals 2 means this area. So, basically there is no region which is common to these 2 lines and the coordinate axis. Or in other sense there is no feasible region and if we know that if there is no feasible region, then there will be no solution of the problem and which we have seen it by 2 phase method also.

Without going to the second phase on the first phase itself whenever you have observed that Z j minus C j is greater than equals 0, but the artificial variable is present in the basis. Therefore, we have concluded that there will be no solution for this problem, and this particular with there is no need of going to phase 2 which we confirmed by the graphical method also drawing the constraints on the 2 dimensional plane that there is no feasible region or no area is common to both the lines as well as the positive coordinate axis.

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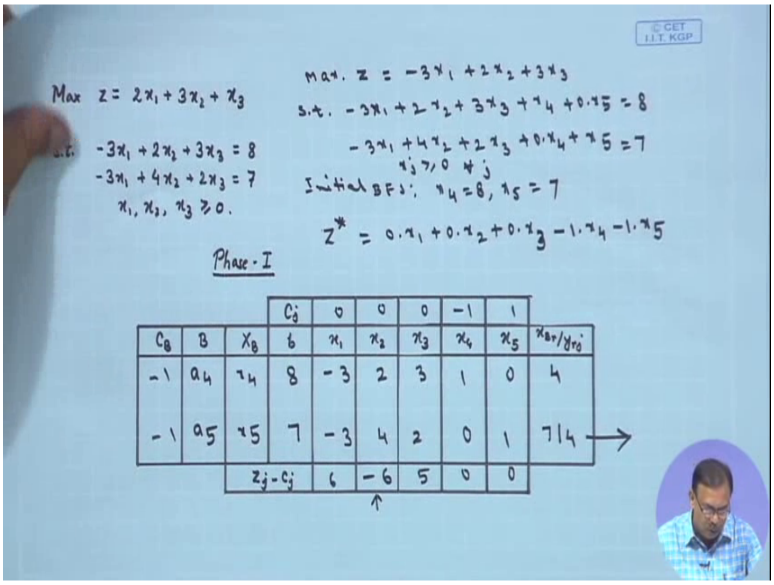

Now let us take the other example some other example of other type say it is z equals 2 x 1 plus 3 x 2 plus x 3, subject to these 2 equality constants. So, here you do not have the less than equals or greater than equals constraint, but you have the equality constants for both. So, in standard form first I have to write down maximize z equals subject 2 minus 3 x 1 plus 2 x 2 plus 3 x 3. As I was telling earlier whenever you have the equality constraint if required I have to use the artificial variable.

And please note that if I do not use the artificial variable from the given constants these 2 equality constants. I cannot obtain the initial solution initial basis or basic feasible solution. Because if I make all the variables x 1 x 2 x 3 equals 0, that is not satisfying my condition. So, we are adding some artificial variable for both. So, therefore, it will be plus x 4, and for the second 1 minus 3 x 1 plus 4 x 2 plus 2 x 3. Here it will be 0 into x 4 plus x 5 is the artificial variable. So, for this case 0 into x 5, this is equals 8 this is equals 7 X j greater than equals 0 for all j.

And maximize z equals minus 3 x 1 plus 2 x 2 plus 3 x 3. So, for this case please note this one initial BFS, initial BFS is equal to x 4 is 8 and your x 5 this is equals to 7 initially you are obtaining this one and that is the only reason we have to incorporate artificial variables for this particular problem in both the equality constants. So, sometimes if required in the equality constraints, we have to add the we have to add the artificial variable depending upon the requirement of the problem.

So now, what I have to do I have to clear the auxiliary objective function, and I will try to go for the simplex method for the auxiliary objective function subject to the original set of constraint constraints of the standard equation whatever we have written down. So, what is the auxiliary equation? Auxiliary equation in this case z star will become your variables decision variables are x 1 x 2 x 3. So, therefore, 0 coefficient will be assigned where as for the other 2 cases it will be minus 1 into x 4 minus 1 into x 5, because it is 4 and x 5 are the artificial variables.

So now, in the phase one let us go for the simplex method, your X B is x 4 and x 5. So, it is a 4 and a 5 here it is 0 0 0 minus 1 and minus 1. Your C B is minus 1 minus 1 your b value for both is 8 and 7. So, let me write down this equation coefficients of the respective variables from the constants. So, minus 3 2 3 minus 3 2 3 then it is one it is 0, for the next one it is minus 3 4 2 0 and it is one. Value of z j if I calculate using the formula it will obtain 6 minus 6 5 0 0.

So, therefore, you have a negative most negative is minus 6. So, corresponding to these column variable is x 2. So, your entering vector is x 2. For this case and I have to calculate the ratio now the ratio for this case 8 by 2 4 for this case it is 7 by 4. So, 7 5 4 is minimum. So, your departing variable for this case will be x 5. So, therefore, in the next simplex table of phase one, x 5 will be replaced by the variable x 2.

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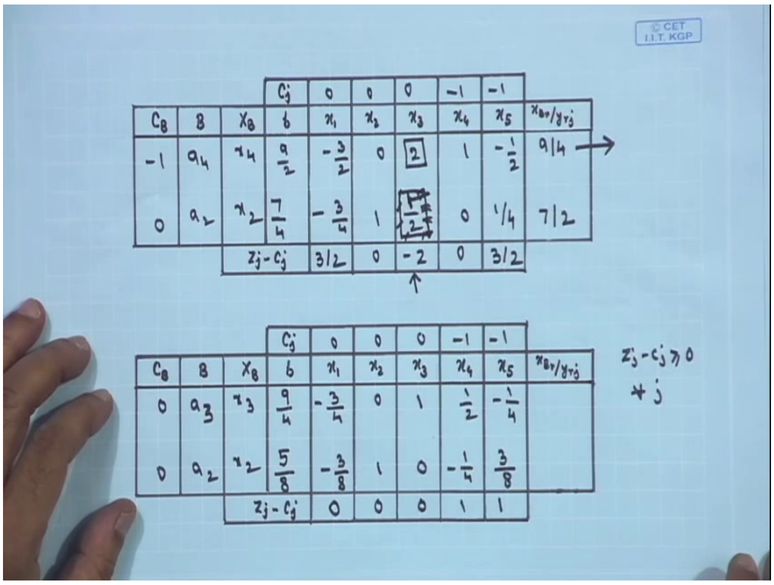

So, it will obtain the table like this, x 4 will be as it is you will have x 2. So, here you are writing a 4 and a 2, here it is 0 0 0 minus 1 and minus 1 x 4 coefficient is minus 1. For x 2 it is 0 sorry, in the in this case your pivot element was 4; that means, I have to make this element as 1 and all other elements of this column as 0 by manipulation of the rows. So, if I write down directly I will obtain 9 by 2 minus 3 by 2 0 2 1 minus half. For this case we will obtain 7 by 4 minus 3 by 4 1 half 0 and 1 by 4 from here if you calculate the Z j minus C j value Z j minus C j value will be 3 by 2 0 minus 2 0 and 3 by 2.

So, for this case it is becoming again Z j minus C j value is not greater than equals 0 1 negative is still there. So, I have to continue the iteration. So, in this case your entering variable will be x 3. So, corresponding to this I have to calculate the ratio it will be 9 by 4 b by x b whereas, for the second one it will be 7 by 2 9 by 4 7 by 2. So, minimum is this one. So, in the second iteration your departing variable is x 4. And your entering variable will be x 3; that means, x 3 will be replaced by x 4. So, here it will come x 3 x 2 a 3 a 2 values are cjs are remains remaining same.

So, C B will be 0 and 0 in this case. So, here again if you see your pivot element is this one, your pivot element is half. So, therefore, I have to make this element as one and other elements as 0. So, I am writing this row at first. So, this row will become I am sorry not this, these row not this one sorry. Intersection is this point this column and this row. So, pivot element is 2 not this one. So, I have to make this element as one and all other elements as 0. So, that you will divide by 2 we will obtain in this case 9 by 4 minus 3 by 4 0 1 half minus 1 by 4, to make it 0 you will have 5 by 8 minus 3 by 8 1 0 minus 1 by 4 and 0 3 by 8.

So, by making this one as 0, we are making this one this as one and this element as 0. We are performing the row operations on these 2. Now calculate the Z j minus C j value Z j minus C j value for this case 0, for this case 0 for this case it is 0, for this case it is one and for this case it is one. So now, your z j minus C j is greater than equals 0 for all j that is x 1 x 2 x 3 and x 4. Now in this case what is happening all z g by minus C j greater than equals 0 and no artificial variable is present in the basis.

So, there will be basic feasible solution and we have to go for the go for the phase 2 with the original objective function, that is original coefficients of the decision variables and other variables. So, let us go to the phase 2 of this now. On the phase 2 if you see this table will remain as it is, please note this one. The last table that is this table all the values will remain as it is the only change will come on the cjs. That is this coefficients now will be the proper one, and also note another thing. We had 2 artificial variables in our problem x 4 and x 5. And the variables artificial variables x 4 and x 5 are not present in the basis. Therefore, we will remove also these 2 columns since artificial variable is not present over here.

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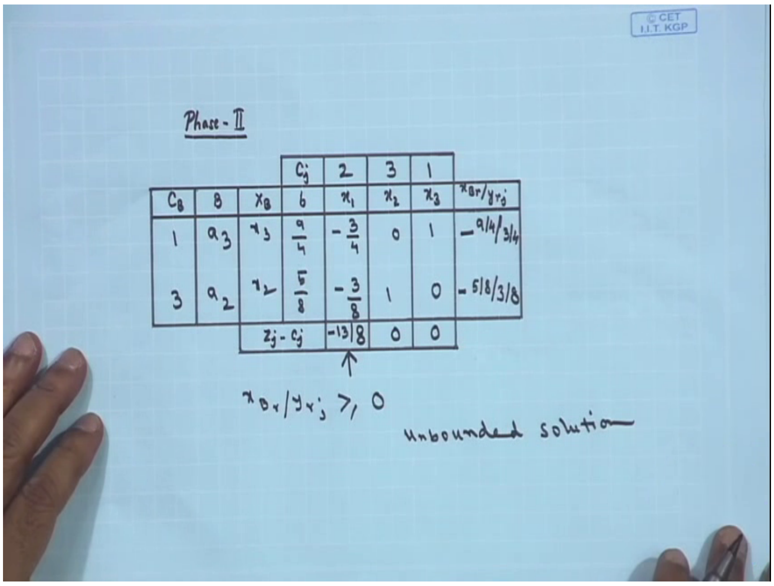

So, whenever we are going for this, we are having x 3 and x 2. And here it is a 3 and a 2 whereas, the C j values will be above your original problem was 2 3 and 1. So, it is 2 3 and one will come as a coefficient for the variables x 1 x 2 and x 3. So, we are writing here 2 3 and 1. So, you are obtaining this as 1 and these as 3. So, correspondingly the values will remain same that is 9 by 4 minus 3 by 4 0 and 1.

And the values for x 4 and x 5 if you see we have removed the values 4 x 4 sorry, the columns for the artificial variables x 4 and x 5 from here. Since the artificial variables x 4 and x 5 are not present in the basis. So, the next one is 5 by 8, minus 3 by 8 next one is 1 and it is 0. If I calculate the Z j minus C j value, now Z j minus C j value Z j minus C j value will be one that is minus 3 by 4 minus 9 by 8 minus 2.

So, it will become minus 13 by 8 this one will be 0 3 minus 3 it is 0 for this case one this is 0. So, 1 minus 1 0. So, here if you see your Z j minus C j is not greater than equals 0. That is one value which is negative. So, in this case your x 1 will be the entering vector your x 1 will be the entering vector. Therefore, I have to calculate the ratio whenever you are trying to calculate the ratio b by X B. That is this value is minus of this 9, 9 by 4 divided by 3 by 4. And for this one it is minus of 5 by 8 divided by 3 by 8. Please note this one. In this case both the ratio values are negative.

But if you remember we have this condition always earlier, that is this value should be greater than equals 0. This should be 0 or positive we cannot take negative values, and whenever this takes the ratio takes negative values, both are negative here and if the both are negative; that means, I cannot take the negative values here. Since I cannot take the negative values; that means, I cannot enter or I cannot depart any variable here. So, please see the situation where I have an entering vector, but what will be the departing vector that is not clear. That is x 3 can be departed or x 2 also can be departed therefore, in this type of cases whenever your ratio is negative or infinity in that case if both all the ratios are negative or infinity you will have unbounded solution; that means, no optimal solution always will be obtaining the unbounded solution.

So, this is another case where if the ratio is less than equals 0 negative, then the solution will not be present there and you will obtain the unbounded solution. So, you see you can obtain the unique solution, you can obtain infinite number of solution, you may obtain no solution. And that from 2 phase method without calculation after phase one you can tell, or the last example which we have explained that is the unbounded solution. This unbounded solution also we have discussed graphically now for 2 phase method also we have shown this one. So, I hope it is clear the 2 phase method. In the next lectures we will discuss some special cases of linear programming problem.