Constrained and Unconstrained Optimization

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Lecture - 11
2 Phase Method: Introduction

So, in the last class, if you remember we completed the big M. Method in big M method basically we are handling the greater than equals type constants, by the introduction of surplus variable and artificial variable. Wherein the cost coefficient again if you remember then in the cost coefficient what we are doing for the artificial variable. We were attaching one high penalty that is minus M, and we have made one assumption that value of M is sufficiently large.

Now, sufficiently large means we wanted to say that M minus some quantity if we subtract from M if we subtract some quantity then M will always be greater than that quantity. But whenever we are implementing it in computer because when number of variables number of constants are much more; then manually we cannot do it. So, if we cannot do it manually we have to write down programs for implementation of this. That it becomes a issue what should be the possible value of this capital M. So, that is one problem of implementation of the big M method. To overcome that particular method we now today discuss another method which we call as 2 phase method.

Basically in 2 phase method we are trying to do one thing. That is do we will try to remove the artificial variable from the basis that is one type, and if we have the redundant constants we will come to that there are certain effects if we have redundant constants that is also being taken care of in 2 phase method.

So, basically it has 2 particular steps 1, 1 step is if you have redundant constants, then how to handle the redundant constants, because it leads to certain thing. And the other one is to handle the artificial variables here you will find that the artificial variables are not associated with some very large coefficient value, but a negative only one value that is minus 1 will be the coefficient of the artificial variable and that is also not in the original objective function.

Basically we are doing it in 2 phases in the first phase we are trying to find out the initial basic feasible solution, and by doing this we are creating one auxiliary objective function by considering all the elements or sorry, for all the very well variables of the auxiliary objective function as 0. Except the artificial variables and for the artificial variables we have taken the coefficient as minus 1.

Then we are running the first phase of the 2 phase method, using the normal simplex approach whatever we learned earlier. Then once zj minus cj greater than equals 0 and it satisfies some other criteria then we go to the phase 2 for the optimal solution of the initial basic feasible solution whatever we have derived from phase 1.

Now, let us see; what are the phase 1 and phase 2 of the system.

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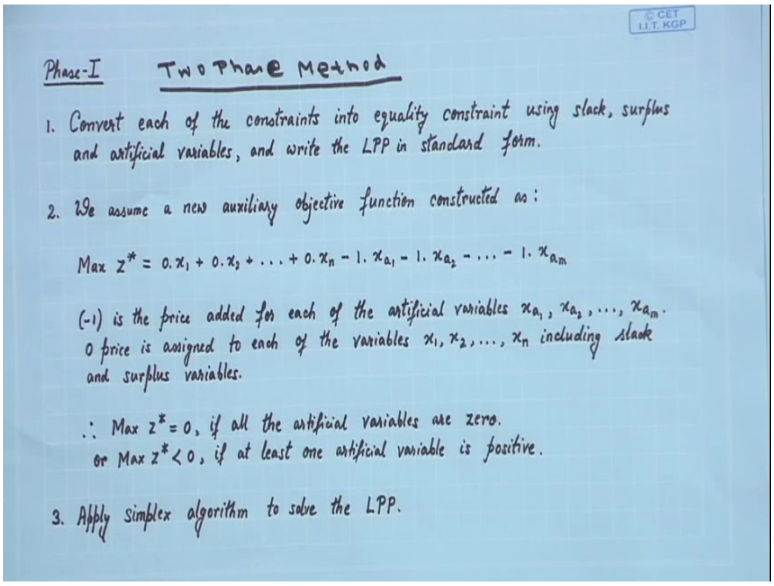

So, we are now trying to do the 2 phase method. In 2 phase method as I told the first phase is phase 1 here you see we have written it point wise what steps you have to follow. Convert each of the constraints into equality constraints, using slack surplus and artificial variables. And write the LPP in standard format. Please note this one the step one already we have done for simplex approach or for the big M approach. That is you have been given the LPP in canonical form. From the canonical form you are converting the LPP into the standard form by using converting each constraint into equality constraint with the help of slack variable, surplus variable and artificial variables.

Step 2 is we assume a new auxiliary objective function. Here we create this auxiliary objective function auxiliary objective function if you see it is maximize z star equals as I told for all the variables 0 into x 1 0 into x 2 like this way 0 into xn, minus 1 into x a 1 minus 1 into x a 2 minus 1 into x n. Where minus 1 is the price added for each of the artificial variables x 1 x 2 x m.

So, please note that if I have m artificial variables x a one x a 2 x am. Then in the absurdity of this objective function we are adding a price minus 1 by writing minus 1 into x a 1 minus 1 into x a 2 like this way minus 1 into x am. And 0 price is assigned to each of the variables x 1 x 2 xn. This x 1 x 2 x n contains the decision variables as well as slack and surplus variables. So, please note this one if I have the decision variables, and slack and surplus variables as n variables. Then those n variables x 1 x 2 x n the coefficients in the auxiliary objective function we are giving us 0.

So, the new objective function we have created where all the variables except the artificial variables we are giving the quick coefficient we are assigning as 0 whereas, for artificial variables we are assigning the value as minus 1. So, maximum value of z star if you see it will be 0, whenever value of all artificial variables will be 0, which actually we want that we want to make all artificial variables as 0. So, whenever maximum of z star equals 0 all the artificial variables will be 0. Or you it may happen that value of maximum of z star is less than 0 this will be possible only when is at least one artificial variable is positive.

So, if at least one artificial variable value is positive then maximum value of z star will be less than 0. From this z star. So, please note these 2 things. Step 3 is apply the simplex algorithm to solve the problem. That is the simplex algorithm whatever we have used earlier classes you use that particular algorithm to solve this problem, maximize z star equals this new auxiliary objective function subject to the original constraints.

Now certain cases may arise, what are the cases?

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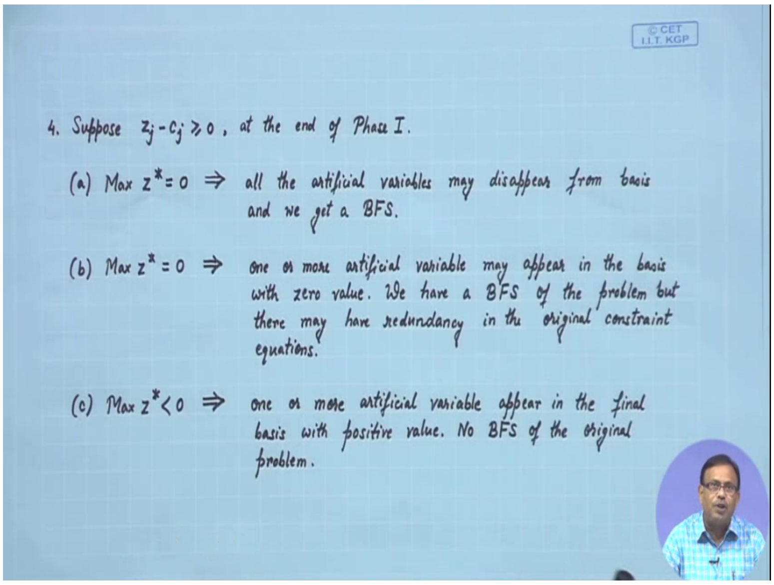

Suppose your zj minus cj you know it zj minus cj is the result of the reduced cost value. So, if zj minus cj is greater than equals 0, you got after some iterations on the simplex table at the end of phase 1. So, whenever zj minus cj will be greater than equals 0, then value of z star can be 0 or it may be less than 0.

Now, what does it mean? Now whenever maximum of z star equals 0, it means that all artificial variables may disappear from basis. All artificial variables may disappear from basis, because it will be 0 only when your artificial values of the artificial variables are 0. So, from this maximum of z star equals this objective function it is clear. And in that case if the values of all artificial variables disappear from the basis then we get a basic feasible solution.

The next case may come on z star equals 0. One or more artificial variable may appear in the basis with 0 variable. Because in the first case we have told z star equals 0 means all value of the all artificial variables must be 0. Then only z star will be 0, but your value of all artificial variables are 0, but one artificial variable still is there in the basis that is still it is a basis variable. If that means, we have a basic feasible solution of the problem, but there may have redundancy on the original constraint equations; that means, whatever constraints we have used there may have redundant constants means if we would not have written one constant then also my problem would not have affected.

So, please note this one, that if maximum value of z star is 0 and if one of more artificial variable appear in the basis with 0 value appear in the basis or it becomes basic variable. In that case you will obtain the basic feasible solution, but there may have redundancy in the constraints. And redundancy causes certain other problems which we will discuss again later stage.

The third case that is c part would be maximize z star less than 0. So, whenever z star is less than 0; that means, all the artificial value of the all the artificial variables are not 0. Sum or one or more artificial variable will appear in the final basis with the positive value. So, please note this 1, 1 or more artificial variable must appear in the final basis with positive value, if value of maximum value of z star is less than 0. And in that case there will be no basic feasible solution of the original problem.

So that means, from phase 1 itself we can conclude that there is no solution of the original problem, I do not have to go for the interactive shapes in the phase 2 for getting the optimal solution. So, basically we are reducing here the number of iterative steps before concluding that the original problem has no solution. So, note these things at the end of phase 1 maximum value of z star can be 0 it may be less than 0. So, z star equals 0 it has 2 implication one implication is all the artificial variables will disappear from the basis and we will get the optimum solution, if it is there.

And it may happen that although z star is 0, but one or more variable are appearing in the basis one or more artificial variable is appearing on the basis with 0 value. This means that some redundancy is there in your constraints whereas, z star less than 0 means there is no solution of the original problem, because z star less than 0 means one or more variables will appear in the basis or final basis, where the value of that artificial variable is positive value that is greater than 0.

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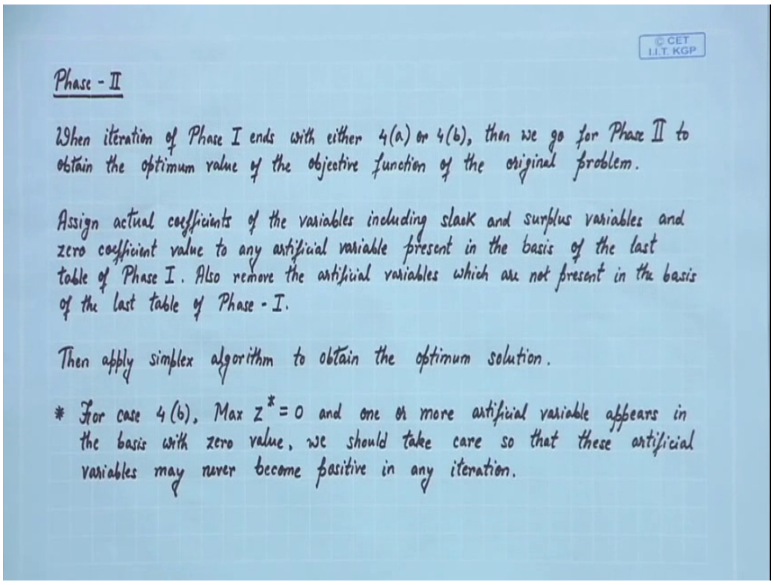

Now in phase 2 what we do? When iteration of phase 1 and phase 2, I ends then either 4 a or 4 b will come because 4 c will not come. We will not whenever we know that original problem has no solution we will not go for phase 2. So, for phase 1 ends we will go for there will be either 2 cases and 4 a or 4 b, and in that case only we will go for phase 2 to obtain the optimum value of the objective function of the original problem.

Now, you assign actual coefficients of the variables including slack and surplus variables and 0 coefficient value to artificial variables present in the basis of the last table of phase 1. So, as I have told at the beginning itself at first we have created one auxiliary object function, where the coefficients of the variables were not same of the original problem. But we assign 0 values to all decision variables and slack and surplus variables and we have assign minus 1 coefficient to the artificial variables.

But now whenever we are going to the phase 2 we are assigning actual coefficient values of the variables for all the decision variables slack variables and surplus variable. And please note it if any artificial variable is present then we are assigning 0 value as a coefficient. For that as a for that artificial variable; that means, in this phase 2 phase method we are very clearly telling what will be the coefficient of artificial variable in the objective function which we defined as M a very large number in the bigger method which is the basic difference between these 2.

Next is also remove the artificial variables which are not present in the basis of the last table of phase 1; that means, columns of the artificial variables will be removed which are not present in the basis. So, any artificial variable will be present in the table only if it is present in the basis of the last simplex table of phase 1. And then you apply the simplex algorithm as use as usual to obtain the optimum solution.

In case of 4 b that is whenever your; you may have some a or 4 b says that one or more artificial variables may appear in the basis with 0 values. So, in that case add one or more artificial variables appear in the 0 value we should take care; so, that these artificial variables may never become positive in any iteration. Or in other sense their values should remain 0 always they should not take any positive value that we should take care.

So, this is the basic idea of the 2 phase method in phase 1, you are creating one auxiliary function by giving the 0 value to all the variables including decision variables slack variables and surplus variables and minus 1 as a coefficient for the artificial variables. And you will stop whenever your zj minus cj is greater than equals 0.

So, when zj minus cj greater than equals 0; that means, value of z star will be either 0 or it will be less than 0, which we have discussed 3 different cases 4 a, 4 b and 4 c. In the case of 4 c there will be no solution since artificial variable is present in the basis with the positive solution. So, you can obtain the basic feasible solution for the case of 4 a and 4 b only. In phase 2 you are actually assigning the original coefficients of the decision variables and artificial variables and slack sorry decision variables slack variables and surplus variables; whereas, if artificial variable is present in the basis then we are assigning 0 coefficient to that particular artificial variable.

Now, let us explain this algorithm by some examples, let us take some cases.

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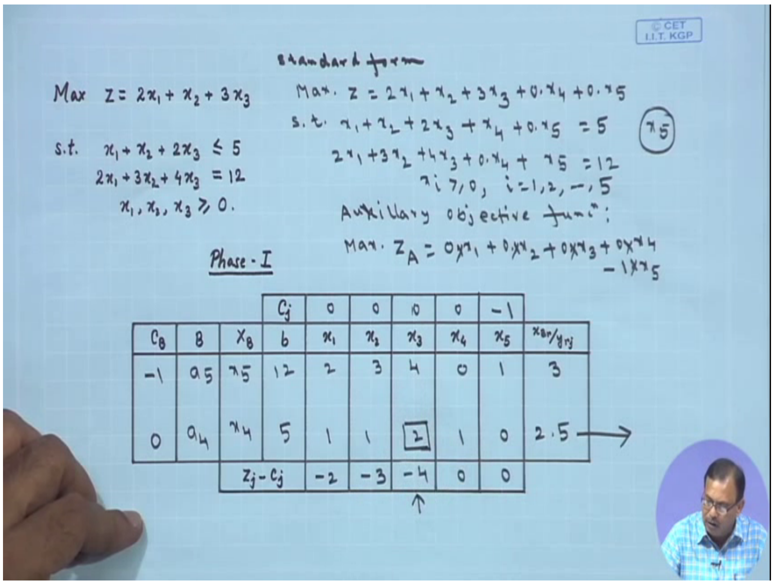

Let us take the first example I I want to maximize 2 x 1 plus x 2 plus 3 x 3, subject to x 1 plus x 2 plus 2 x 3, less than equals 5 and 2 x 1 plus 3 x 2 plus 4 x 3 which is equals to 3. And of course, you have the non negativity restriction that is x 1 x 2 x 3 greater than equals 0.

So, first thing what I will do I will write it in the standard form. I will write this one in the standard form in standard form it will be maximize z I am not discussing this, equals subject to x 1 plus x 2 plus 2 x 3 plus it is less than equals. So, you will add once or slack variable here. And this is equals 5 whereas, 2 x 1 plus 3 x 2 plus 4 x 3 plus 0 into x 4 to get the initial basic feasible solution, as we have discussed earlier. You have to add one artificial variable over here.

So, plus x 5 this is equals to 12. And here it is plus 0 into x 5. So, that you are writing your problem as 2 x 1 plus x 2 plus 3 x 3 plus 0 into x 4 plus 0 into x 5. Please note that here your x 5 is the artificial variable just to get the initial basic solution for this particular set of constraint. And although x 5 is the artificial variable, but you have added 1 0 coefficient only. This is the difference first difference between the big M method and the your 2 phase method, because in big M method you added a penalty of minus M with the artificial variable x 5.

Now, this is your problem where your x i is greater than equals 0. i equals 1 to 5. So, your problem standard problem is this from here now you will create one auxiliary objective function. So, your auxiliary objective function will be, your auxiliary objective function now you are writing as we have told your auxiliary objective function will be say za or z star whatever name you can give. Here you have only one artificial variable x 5 you have 3 decision variables x 1 x 2 x 3 and one slack variable x 4.

So, according to the algorithm all the decision variable and the slack variable will be associated with the 0 coefficient; whereas, the artificial variable will be associated with the coefficient minus 1. That is za equals you can write down 0 into x 1 plus 0 into x 2 plus 0 into x 3 plus 0 into x 4 plus minus 1 will be added with the as a coefficient of the artificial variable.

So now your new auxiliary objective function is maximized za equals 0 into x 1 0 into x 2 actually this is these are into x 3 and minus 1 into x 5. So, your problem is maximize za subject to these 2 given constraints using the simplex algorithm. So, what will be your initial solutions your initial solutions will be your variables are x 4 and x 5 and the variables from here you can obtain x 4 equals 5 and x 5 will be equals to 12 this is the initial basis in this case, because I have to make all other variables as 0. So, you write down here x 5 and x 4 So that this will be also a 5 and a 4.

Value of cj is not the original one, but for the new objective function new objective means auxiliary objective function. So, it will be for all variables it will be 0 and minus 1 only for this objective function. So, cb here it is minus 1 and 0 your b value is 12 and 5. So, corresponding to x 5 you are writing this one that is 2 3 4 0 1 and corresponding to x 4 it is first one. So, it will be 1 1 2 1 0 1 1 2 1 0. So, once you are obtaining this you can calculate the value of zj minus cj. As usual and you can check whether zj minus cj greater than equal 0 or not and he will repeat the process.

So, zj minus cj value if you calculate then cb xb minus cj. So, it is equals to minus 2, I am not explaining minus 4 0 and 0. So, the most negative is this one that is this column. So, your entering vector will be x 3. So, your entering vector is x 3. So, you have to calculate the ratio that is b by x b. So, ratio in this case 12 by 4 that is 3 and in this case 5 divided by 2 that is 2.5. So, your outgoing vector is x 4. In this case your outgoing vector is this one that is x 4.

So, in this particular tradition your entering vector will be x 3 and it will be replaced by the outgoing vector x 4. And your pivot element is the intersection of these 2 row and column. So, this one; that means, I have to make this column as one and this column as all other columns as 0, all other elements of this column as 0 as usual.

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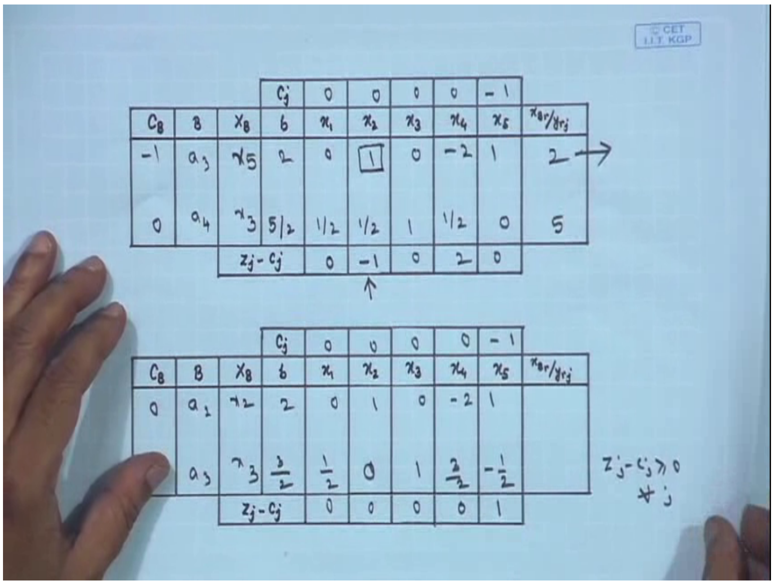

			c_j	0	0	0	0	-1	
C_B	B	X_B	b	x_1	x_2	x_3	x_4	x_5	x_{Br}/y_{rj}
-1	a_3	x_5	2	0	$\boxed{1}$	0	-2	1	2 →
0	a_4	x_3	5/2	1/2	1/2	1	1/2	0	5
			$z_j - c_j$	0	-1	0	2	0	

			c_j	0	0	0	0	-1	
C_B	B	X_B	b	x_1	x_2	x_3	x_4	x_5	x_{Br}/y_{rj}
0	a_2	x_2	2	0	1	0	-2	1	
	a_3	x_3	$\frac{3}{2}$	$\frac{1}{2}$	0	1	$\frac{3}{2}$	$-\frac{1}{2}$	
			$z_j - c_j$	0	0	0	0	1	

$z_j - c_j \geq 0 \ \forall j$

So, the next level whatever you will obtain is this one, 0 0 0 0 minus 1. And here it will come as x 5 and your entering vector is x 3. So, x 4 will be replaced by this. So, that it is a 3 and a 4. Your cp values will be minus 1 0. So, I am directly writing this I have to make this as one and this as 0. Accordingly I have to perform operations on the row. So, it will be 5 by 2 half half, 1 half 0. And the other row will become 2 0 1 0 1 0 minus 2 and 1. And your zj minus cj becomes if I calculate it will be 0 minus 1 0 2 0. So, if you see here still your zj minus cj is not greater than equals 0, we have a negative. So, this will be x 2 will be entering. So now, I calculate the ratio 2 by 1 2 and 5 divided by 1 by 2. So, it will be 5. So, outgoing vector will be x 5. And your pivot element will be one. So, this is already one I have to make it 0.

So, in the basis now x 3 will be replaced by x 2. So, it will be now x 2 x 3. So, 0 0 0 0 minus 1 your b is a 2 a 3 your cb will be in that case 0 0 x 2 x 3 is 0. So, this values will be 2 0 1 0 minus 2 and 1. And this one you will obtain as 3 by 2 half 0 1. So, this is 0 1 3 by 2 and minus half.

So now calculate the zj minus cj using the normal formula, you will obtain 0 0 0 0 and 1. Now in this case your zj minus cj is greater than equals 0 for all j. So, what we are finding now in this case, zj minus cj is greater than equals 0. And if you see the basis the variables are x 2 and x 3. No artificial variable x 5 is present in the basis. So, therefore, we must obtain some solution for this and we will go to the phase 2.

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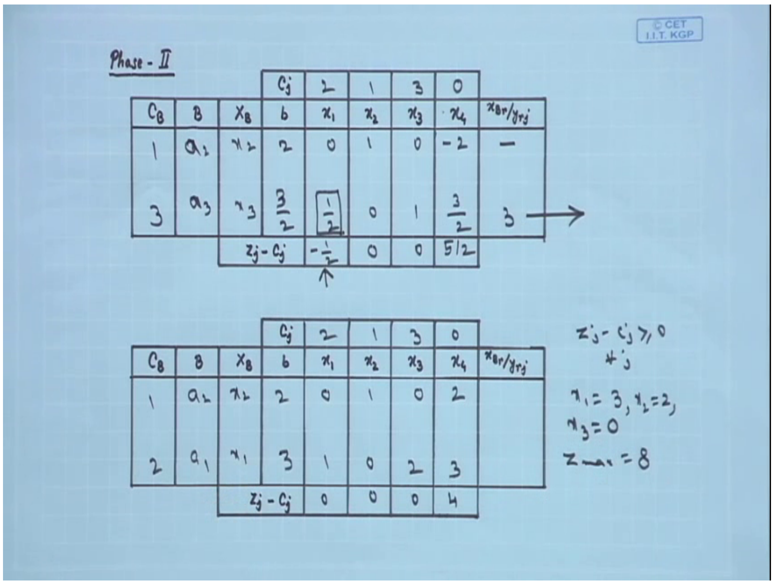

In the phase 2, again we will start with this table only with the earlier table only that is with this table. The only problem will be in this case is this cj values will be replaced by the original coefficients of the variables x 1 x 2 x 3 x 4. And also you see in the last table of phase 1, this variable x 5 is not present in the basis. Therefore, x 5 will be removed this column will be removed from the basis as we have discussed in the algorithm.

So, that it will obtain your x 2 and x 3. These values now will be 2 1 3 0. If you see the original problem, sorry this will be b is a 2 a 3 this will be x 2 x 3. So, cb values will be 1 and 3. I am writing these 2 0 1 0 minus 2. This will be zj minus cj. So, sorry this is 2 this will be 3 by 2 half 0 1 3 by 2. Whatever was there in the earlier table if I now calculate zj minus cj I will find that it is coming 5 by 2. Zj minus cj less than 0 1 negative is there. So, your entering vector will be x 1.

So, since the hectoring vector is x 1. So, calculate the ratio by b by xb. This is 2 by 0. So, you will not get infinite value and for this case it is 3. Since only one number is there. So, outgoing vector will be now x 3 and this will be your pivot element. So, that x 3 will be replaced by x 1 now. So, this is a 2 a one your cj values are remaining same. So, here it will be 1 2. So, this I will make one this is 0 already. So, the table you will obtain 0 1 0

2 and in this case it will be 3 1 0 2 and 3. Zj minus cj if you calculate 0 0 0 4. So, your zj minus cj is greater than equals 0 for all j.

So, we obtain the optimal solution and the optimal solution is unique. Because zj minus cj equals 0 for basic variables x 1 x 2 and it is greater than 0 for the greater than equals 0 for the non basic variables. So, optimal solution in this case it will obtain as x 1 equals 3 from here, x 2 equals 2 and your x 3 is not present here. So, value of x 3 will be 0 and z max will be 8. So, this is the optimal solution unique optimal solution of this particular problem. So, I hope the method is clear. In the next class we will discuss 2 different problems on this 2 phase method.