Modeling Transport Phenomena of Microparticles
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Lecture – 09
Elementary Lubrication Theory

Hello, welcome back. So in the previous lecture we discussed about stokes flow past various particles. For example cylinder and then sphere we have discussed and now we move on to something called lubrication approximation. So most of the biological applications and also fluid machinery so they very much involve this lubrication approximation. So as the word says lubrication you can recall.

So many times we use when there are some thin plates come in contact so then we will patch up with some lubricant okay. So that is a, to avoid the wear okay, friction so that the machinery is not damaged. So understanding the lubrication approximation and the corresponding physics and the dynamics is very much essential to understand such machinery okay. And also at micro level at various biological phenomena such lubrication assumption is very much valid okay.

So let us look at the lubrication approximation. So we generally start with corresponding governing equations which are Navier-stokes but as we have seen suppose you have a unidirectional flow then the complete Navier-stokes equation got simplified to very simple linear equation. And then we could get the solution. Similarly instead of handling full Navier-stokes equations may be under some approximations the system will be reduced and then that will reflect the corresponding physics of the problem.

So same thing may happen here okay, so what is the lubrication approximation and then for regular as well as irregular geometry as how things can work. So we will see in brief okay.

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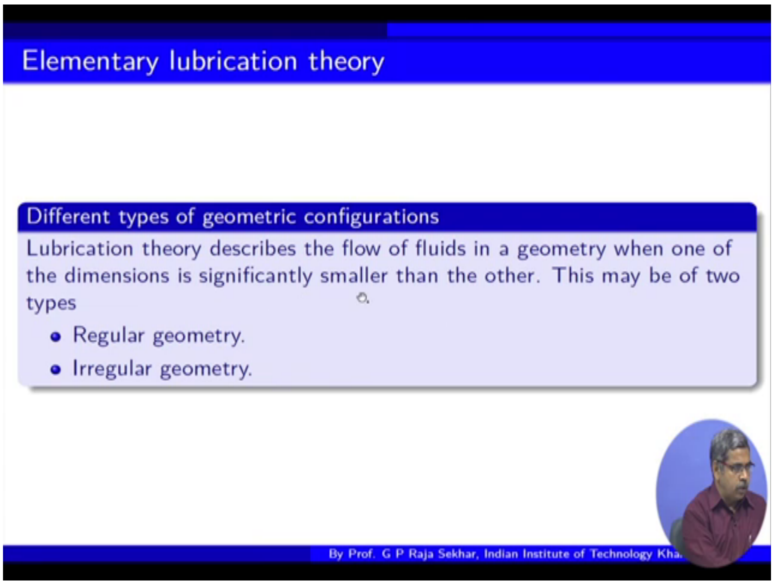

So let us see so lubrication theory describes the flow of fluids in geometry when one of the dimensions is significantly smaller than the other okay. So this is we are talking about regular or irregular geometry. So when we say one of the geometry is smaller than the other so naturally one can bring the concept of aspect ratio okay. So when a large aspect ratio then how the corresponding approximation controls okay.

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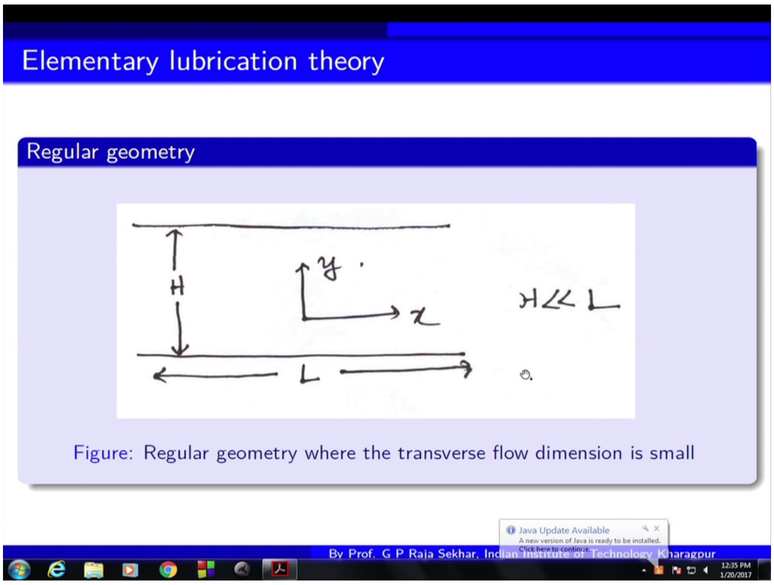

So in case of a regular geometry we are talking about regular in the sense you have say two parallel plates which are separated by a distance H and then let us say length of the plate is L. Then if we introduce the aspect ratio which is H by L so when H is much smaller than L that means compared to the length of the plate if the distance between these two is small so then one can have the corresponding aspect ratio which is much smaller okay.

So this is the aspect ratio when this is much smaller what happens to the corresponding Navier-stokes equations? So that is our idea to understand okay.

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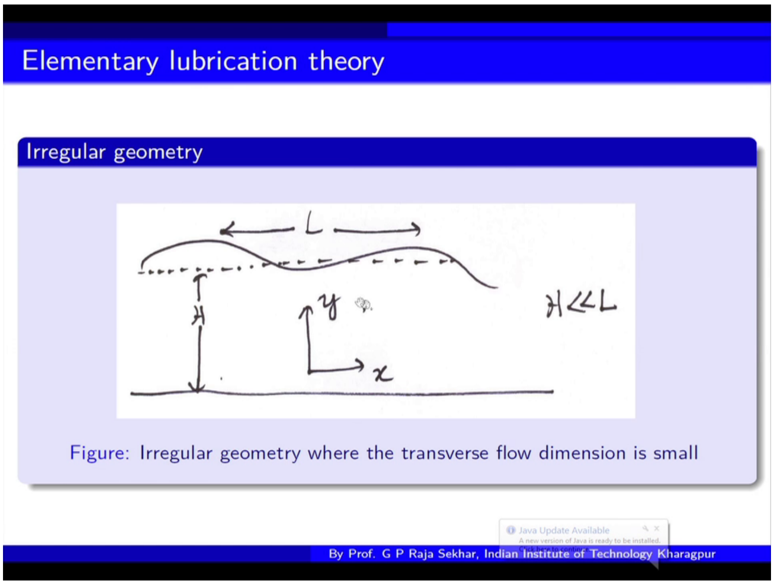

So in case of irregular we have wavy channel okay and then see the mean width is this and then the corresponding wave length is L okay. So then you can introduce the corresponding aspect ratio again in this case H by L okay. So now it matters. So how it matters?

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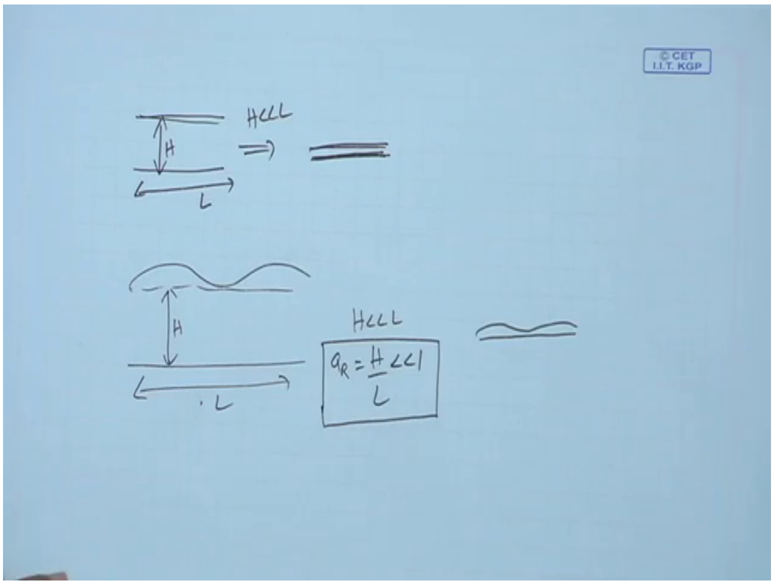

So you have so this is H and this is L. So if H by L is small so we boil down to something like very narrow scenario. So this is a lubrication or sometimes a capillary type okay. Similarly so if we have such scenario so when we have corresponding H by L small okay that is H by L you can define some aspect ratio this is much. So then what happens we are talking about very thin okay?

So most of the capillaries for example biological tissues, arteries etc., so they have such configurations, so it is very much essential to understand the physics under this approximation. So when we say H by L is much smaller so naturally what we are trying to do is probably we are trying to neglect the higher order terms okay.

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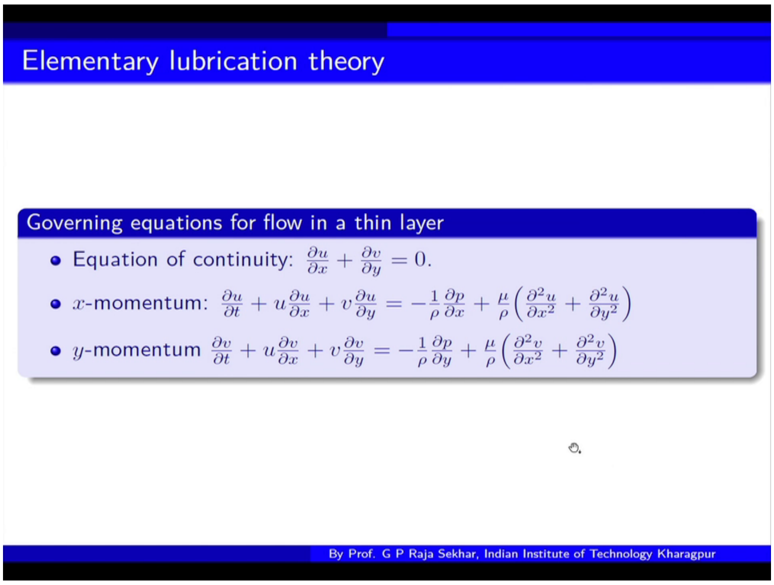

So we consider the corresponding governing equations. So to start with we consider the equation of continuity and then we would like to fix the length scales, various length scales. So this is a full Navier-stokes equations we have considered this is x momentum and y momentum in component form. So we have discussed already how to write from the vector equation okay.

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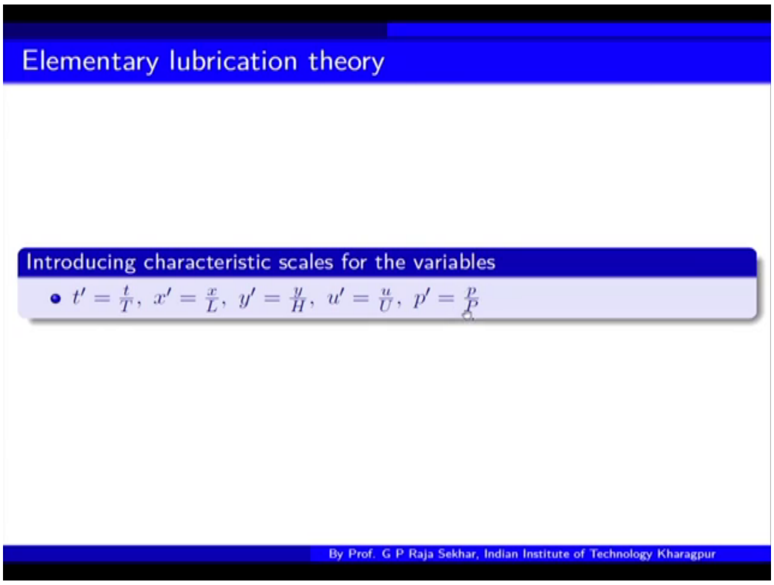

Now we would like to fix the corresponding non-dimensional length scales okay. So for T we are introducing characteristic time and for x since we are considering let us say rectangular geometry parallel plates which are separated by distance H, so along the direction of the flow we have L and perpendicular we are using the corresponding distance between the plates. And for u velocity assumes you have a unidirectional flow kind of or in general flow along x-direction.

So we are using a characteristic velocity and for pressure we are not fixing the corresponding non-dimensional group a priori. The idea is we will non-dimensionalize and then use the lubrication approximation and then we will see how the pressure can be estimated in terms of various dimensional quantities. So hence, we just declare a non-dimensional characteristic pressure and assume that that is normalizing p so that we get a non dimensional pressure okay.

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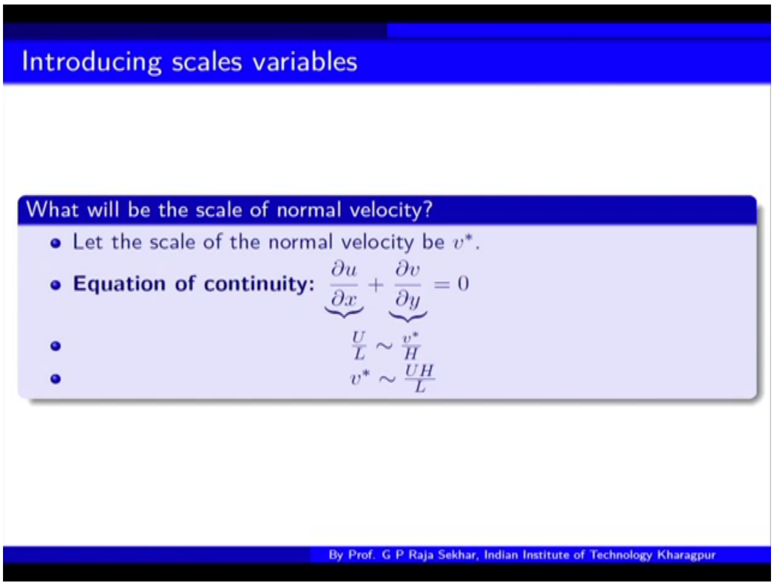

So now let us consider the equation of continuity. So we have various length scales which we have fixed already.

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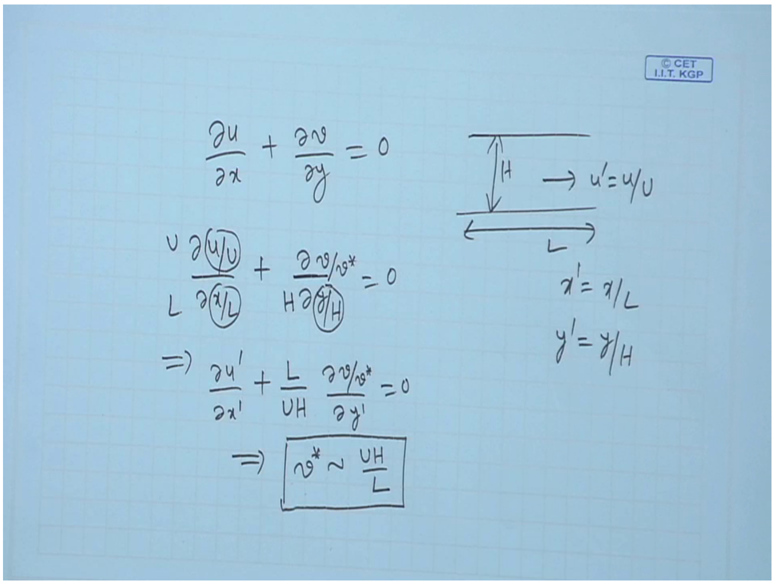

So we are considering equation of continuity okay. And we have the length scale and for u we have used. Now our aim is to estimate the corresponding non-dimensional group for v okay. So if we do that and here for x we have x/L and for y we have y/H okay. So if we do that okay so here we get the H. So let us call some v star which we are trying to estimate okay. So this is non-dimensional u non dimensional x non dimensional y right.

So then plus we get L/UH okay. So this indicates the non dimensional grouping for v star should be okay. So this is the non-dimensional grouping for velocity to be non dimensionalized. So here so the lubrication approximation before we ensure the lubrication approximation what we have seen so along the flow direction you have a different scaling and the vertical you have a different scaling okay and we would like to see the corresponding contribution.

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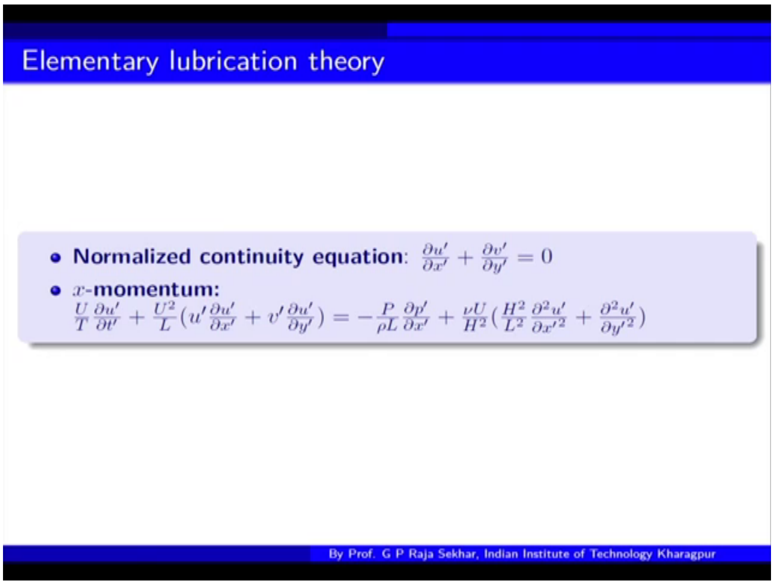

So we have the normalized continuity equation okay. So now we try to non-dimensionalize the x momentum equation okay. So I will show for x so then now y we can do it.

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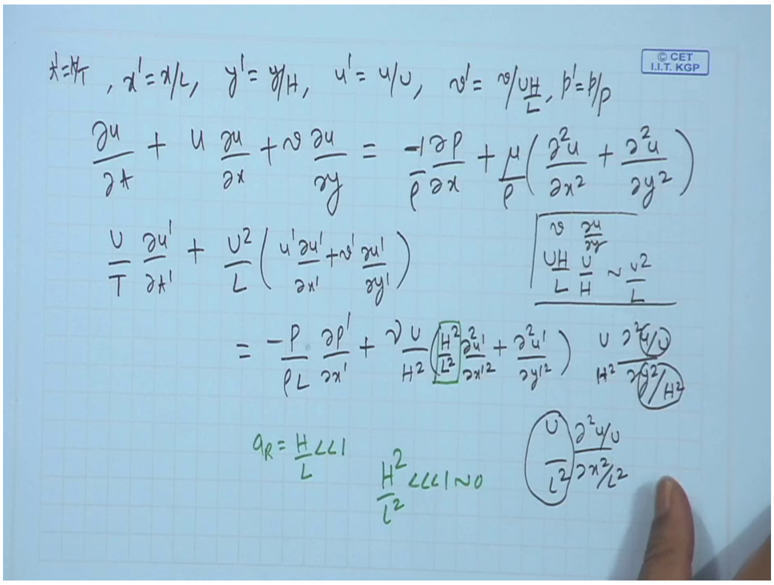

So we have the Navier-stokes equation okay. And just now we have obtained for v, v is okay. So this is the scaling. Of course we have T so we can introduce. So correspondingly if we do so there is a u so it is very straightforward, we get this. Then here one U and another u and we have a length here okay. So this is very straightforward we get U power 2 /L but here we have to check because v has a different scaling okay.

So v is producing UH/L this is for v and then we have dow u by dow y. So that will produce U/H. So this is again we are getting u power 2 L. So therefore we get so this is the scenario and for p we have corresponding normalizing is done by P and then we have L okay. So we are because Rho I have brought it to the other side okay. So what we get is we get the -P/ Rho L. Now here we get different scalings because of x scaling and y scaling which are different so we get different scaling.

So Mu/Rho is a new okay. If you consider we get u and here we get H power 2 okay. Here u and y power 2 so y power 2 is corresponding to that of what we get is. So we have for U we get this then corresponding to y power 2 we have to divide by H power 2 so you get H power 2 okay. So this is the term which we get scaling for this term and similarly for this if you do and here L power 2 so U/L power 2 okay.

But we would like to take U/H power 2 common. So this comes straight away so therefore we get here u'/y power 2. But for x if you see we have U/L power 2 coming out but we would like to take H power 2 common. So therefore so we get this okay. So the aspect ratio exclusively as a non dimensional parameter is appearing in this. That means if at all one would like to go for a lubrication approximation that is you call it the aspect ratio H/L.

And if you go for this assumption then what we are trying to do is definitely we would neglect is much smaller than 1 okay. So therefore we are almost trying to neglect. So in which case the corresponding lubrication approximation is going to reflect and it is going to suppress the corresponding actual variations okay. So any case so this is a non domestication we have done and the similar non dimensionalizeation.

So that what is given exactly here and so this is a simplified version okay. So once we have this, the pre factor that we have MuU/H power 2 so we have divided by that. So once we divide we get this okay.

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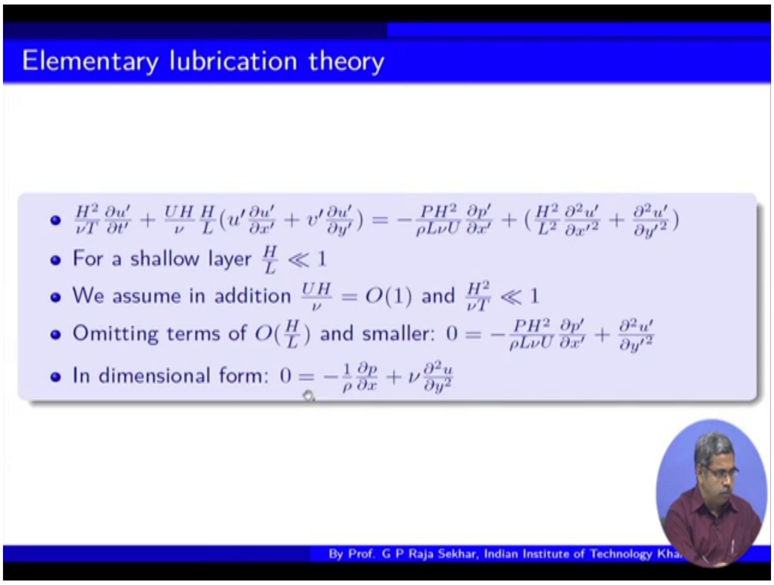

So if you pay attention this is nothing but Reynolds number. We have done the non-dimensionalization. So this is nothing but the corresponding Reynolds number and H/L is the aspect ratio then here H power 2/ v is a diffusion time. So exactly you are getting the non-dimensionalization of T, T has to be of order H power 2/v okay.

So that is so for shallow layer that is the lubrication approximation H/L is much smaller than 1. So therefore we assume this is a smaller. Now our aim is to study the approximation under small aspect ratio. Now if go for small aspect ratio you see for this term you have a Reynolds number. Now if Reynolds number is growing even if you control H/L much smaller, so the total contribution of the term will be depending on Reynolds number and the corresponding growth of Reynolds number.

So our aim is strictly to consider the lubrication approximation. So in order to control that we are assuming the Reynolds number is of order 1 so that we capture the H/L much less than 1 okay. And we would like to have the steady behavior so we are assuming the corresponding diffusion time scales are very small. So we are neglecting that okay. So once we have such a small parameter identified so we are approximating this with an assumption that consider only terms of order H/L and then H/L power 2 onwards.

H/L and square everything is neglected. So that means we are considering constant terms and then since this is much smaller we are omitting H/L and then the higher order terms. So this is gone because of the steady assumption then since H/L is a small and this is restricted to order

1, so this term is gone and we have the corresponding right hand side. We have got the, if you see one can get the corresponding non-dimensional group for P okay.

So the remaining whatever is left out that is exactly the non-dimensional grouping for P okay. And since H by L is small this is also gone. So we get the corresponding x momentum equation under lubrication approximation okay. So in dimensional form we get this. Now this is this is a the x momentum and similar analysis we do it for y momentum.

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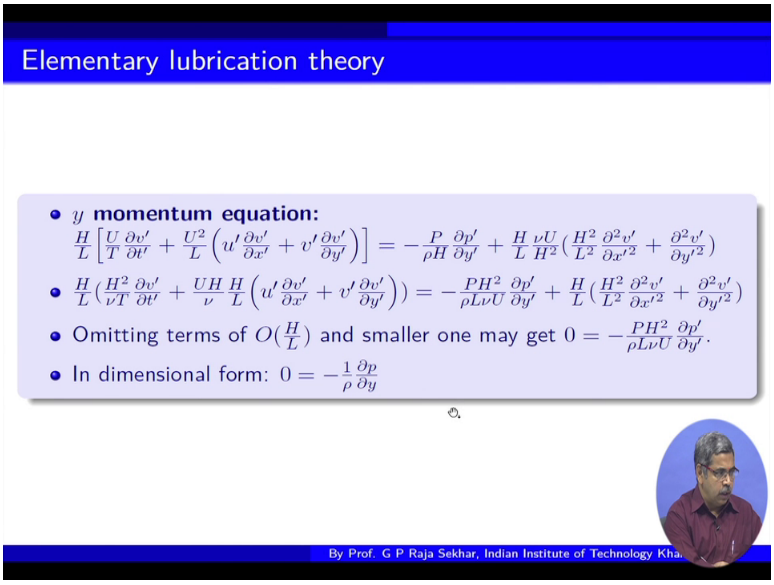

So I have shown for x you can do it for the y momentum and then under lubrication approximation you would see the corresponding pressure gradient vanishes. So that is the equivalent.

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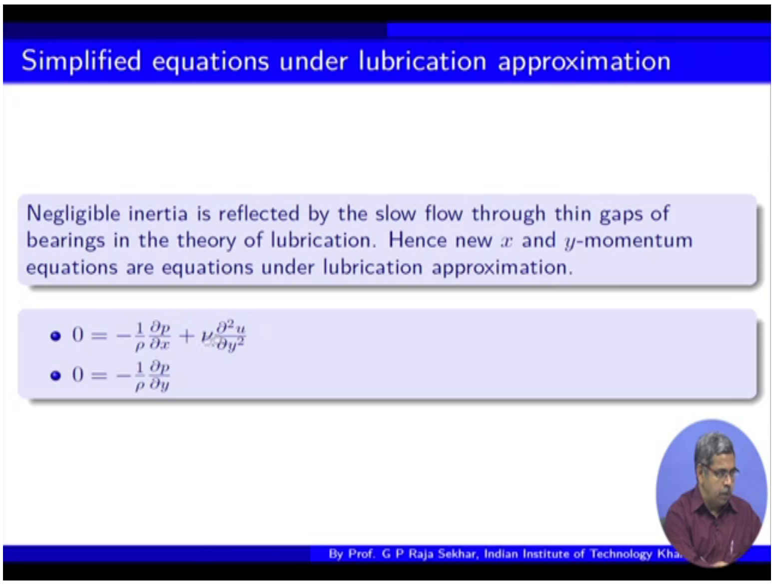

So we have the corresponding x momentum and y momentum under lubrication assumption okay. So what is happening here is the inertia which we have a neglected okay that is reflected by slow flow through these gaps okay. So that is reflected in this sense okay. Now we would like to get how the solution behaviour of such lubrication approximation can be estimated okay.

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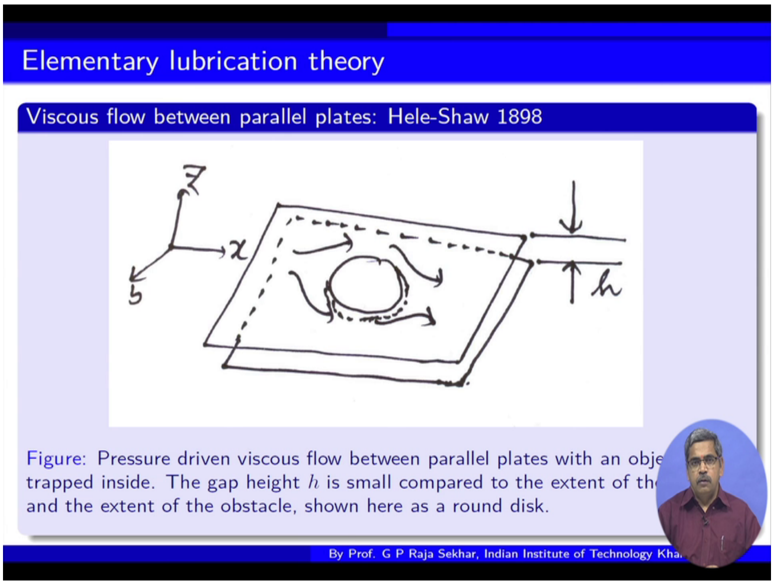

So before we do that so there are a lot of applications okay as I mentioned so one application is a flow between now thin plates with some objects included. So this is a popularly known as a Hele-Shaw flow. So here it is like a very thin plates are coming in close contact and there is an object placed. So this can be solid object and sometimes viscous drops there are several applications of this problem and very popular.

And in order to solve the flow since the gap width is small so you can expect the vertical variations are not normal okay. Like you need not consider full variations. So what is the corresponding the vertical variation is happening that is reflected in the y momentum equation which is simply the corresponding pressure gradient is 0 okay. So that is the power of lubrication approximation okay.

So this is a again coming to the Hele-Shaw flow. So we have a plate two plates located at a distance H and then it is some kind of object.

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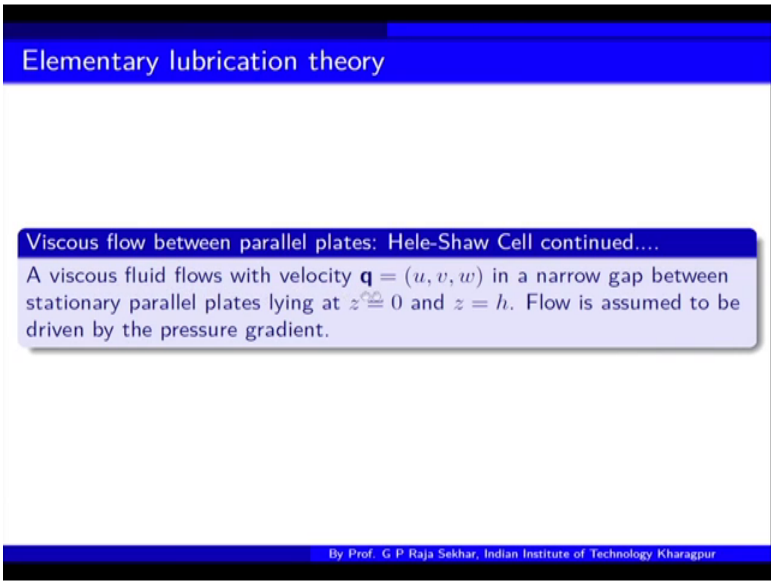

So viscous fluid flow with the, this velocity in a narrow gap and then the plates are located at z = 0 at h and the flow is assumed to be driven by pressure gradient. Then if we go for the corresponding lubrication approximation, in the earlier case here we considered only x and y momentum equations okay in 2-dimensions. But here in this application what we are showing we have u, v, w, okay.

So correspondingly if you go for lubrication approximation what happens? So the aspect ratio is the competition with with the H along x and H along y. So these two aspect ratios they will come into play right. So and we are assuming that H by the corresponding length scale which we use it for x and y the same length scale so that is small. So therefore similar to the 2-D case that we have done.

If you follow the lubrication approximation what we get is the corresponding x and the y momentum, you get contribution of the velocity, whereas for z you will get the corresponding pressure gradient is 0 okay.

So that is what we get so if you go for the lubrication approximation you see that x momentum, you have a pressure gradient which is balanced by the corresponding variations of velocity along z direction.

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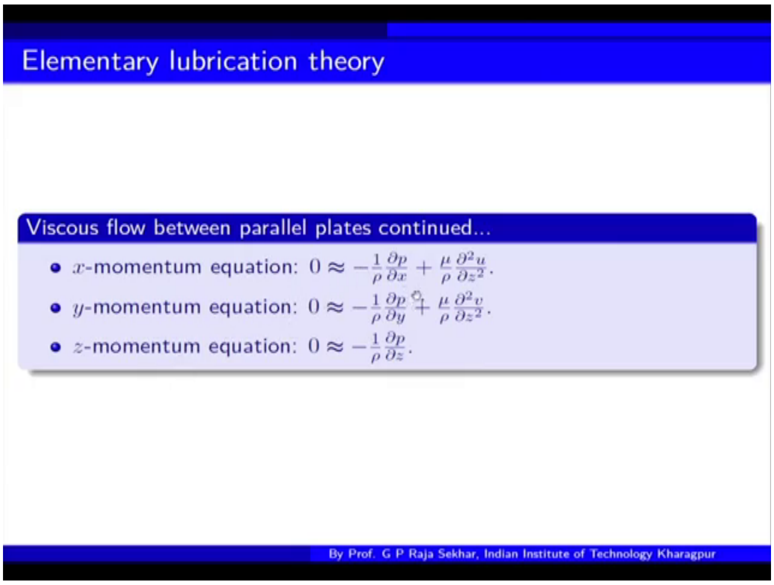

Then similarly y momentum variations of v along the directions are balanced by the corresponding pressure gradient. And in case of z momentum you have the corresponding pressure gradient vanishes okay.

So now we would like to attempt solution exploiting the fact that Dow p/ Dow z is 0 okay. So since we have z derivatives here so and the fact is Dow p/ Dow z is 0 we can integrate this two times with respect to z. Because this is the independent of z so we can integrate each of these equations to time with respect to z and get to the following solution okay.

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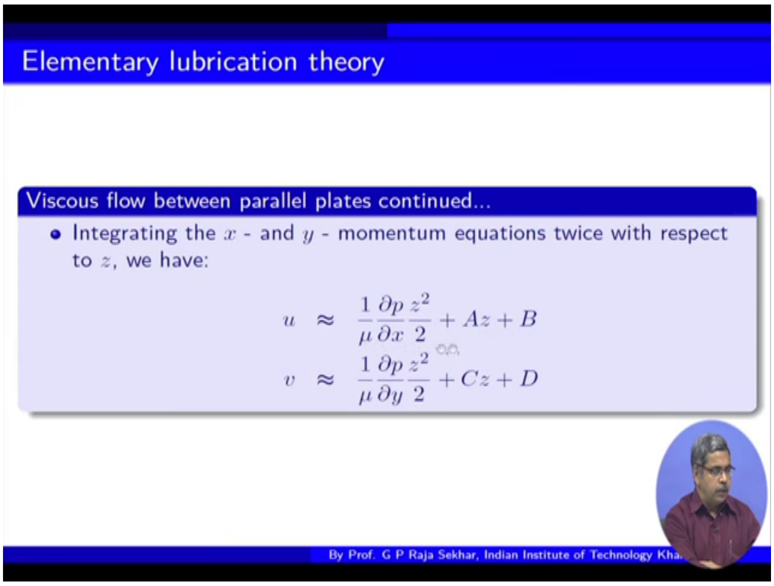

So this involves each of the velocity components involves two arbitrary coefficients. We could do this because of the corresponding lubrication approximation okay. So once we have this we can eliminate these arbitrary coefficients using the boundary conditions which are typically we are using no-slip at both the plates at y = 0 and y = h. We are using no-slip and that will determine the coefficients.

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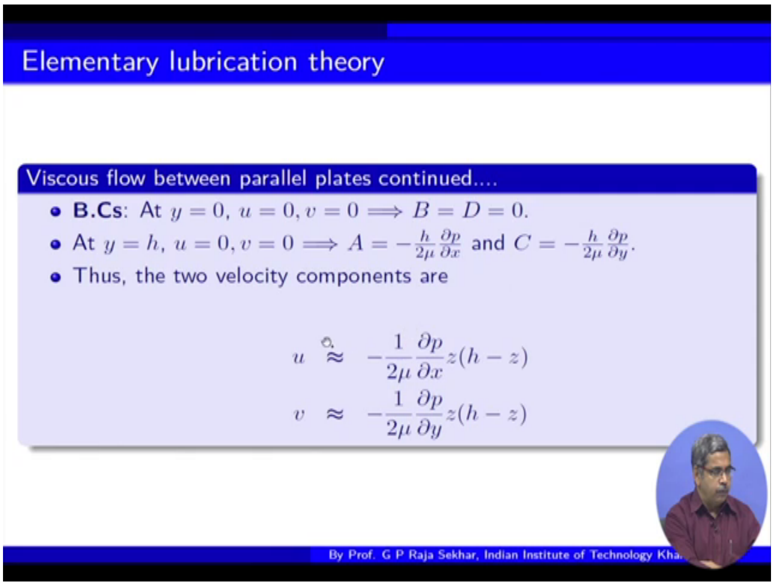

One of the coefficient here will be zero and one of the coefficient here will be zero and the other non-zero coefficients A and C are determined in terms of the corresponding pressure gradients okay. So we have the complete velocity structure okay.

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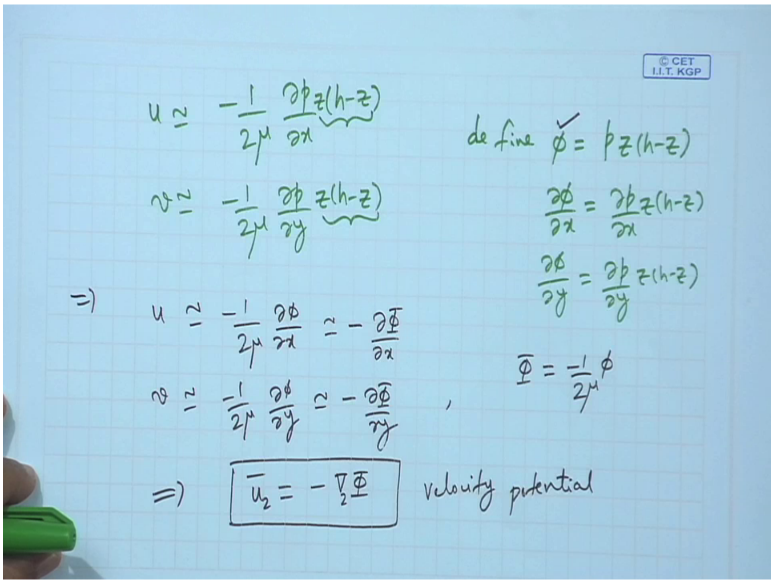

So now if you closely observe the velocity structure, what we have U behave like so we have z(h- z) and v okay. So this is function of z and this quantity is independent of Dow p/ Dow z is 0. So p is independent of z okay. So what we can do so this is a common. So we can introduce say define some Phi which is pz okay then we can realize x exactly and this is exactly okay.

So once we have this structure what we are trying to do is we are trying to define a function Phi such that Dow Phi / Dow x okay. And we can even absorb this in this so we can define minus where we have absorbed okay so this relation. Which means we got the corresponding vector; of course in 2-D. This is gradient of this is again 2-D. So we got a velocity potential okay.

So what we got is the corresponding lubrication approximation. If you do so then the corresponding two dimensional velocity components can be expressed in terms of a velocity potential okay. So that means even though you have highly viscous flow what we got is there are no rotations and it is captured by only the potential flow okay. So that is a very interesting observation.

So now once we have the corresponding Phi the advantage is, we can get the solution in terms of Phi right? So how do we get the solution in terms of Phi? So this I have explained so we got the corresponding velocity potential which is given by this right.

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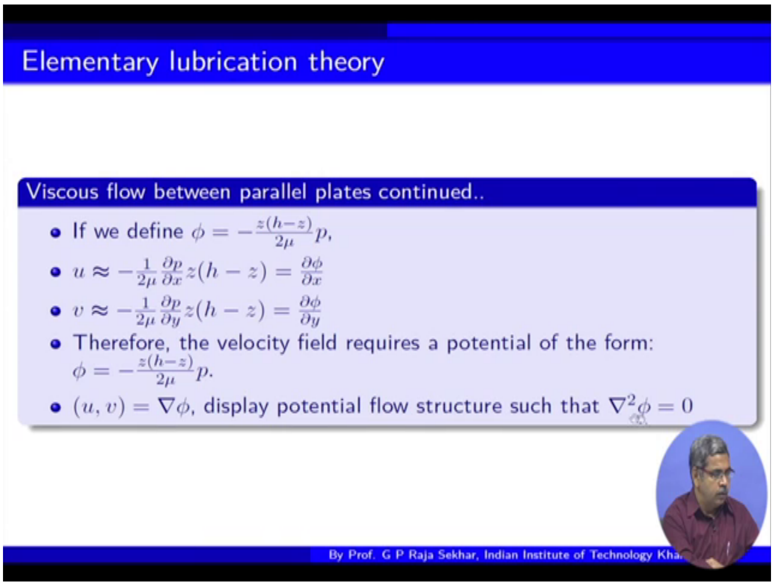

So now our aim is to get the solution we have to get the equation satisfied by Phi okay. So that is nothing but once we have the divergence free then now we get Phi is harmonic okay. Which means under lubrication approximation it is enough to solve harmonic function okay instead of solving the complete component equations that is a great achievement okay.

So now the equation satisfied by Phi can be obtained directly by taking divergence or other words we can also integrate the corresponding mass conservation across this okay. So that is what we have done so then we conclude that p is harmonic or the corresponding potential is harmonic okay. So this gives additional flexibility in case of lubrication approximation. You need not solve the complete the component equations.

It is enough to solve one velocity potential which is a scalar equation. Once you get the solution we get the corresponding velocity and then we can integrate for getting the pressure. So there are several applications as I indicated and typically when it comes to lubrication approximation the first and foremost example comes to anybody's mind is, the bearings.

So the bearing structure involves two metals which are in close contact but they have to produce a nice frictionless rotating system right. So these are typically used in various rotating equipments okay. In motor vehicles and the bicycles etc. So there the corresponding lubrication approximation will be used okay.

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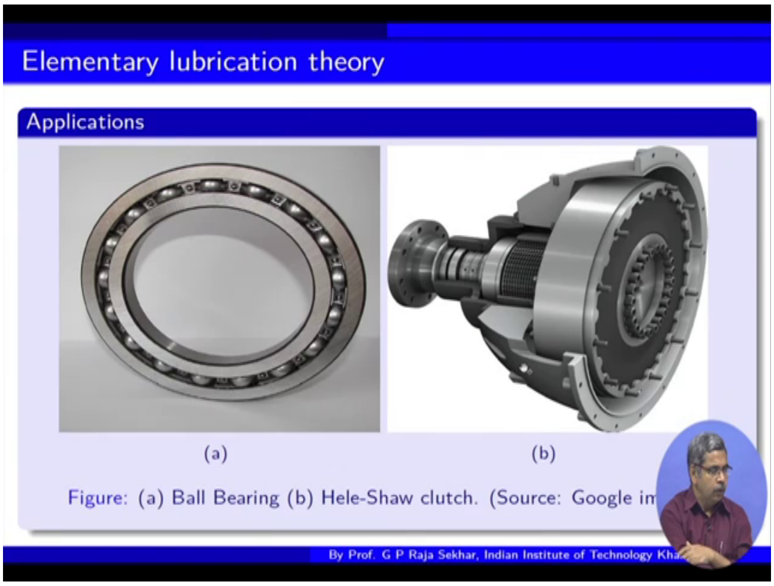

So you can see so this compared to the length scales the width is very small. So one has to go for corresponding lubrication approximation okay. And another several applications are there but we would like to show one of our contributions using the lubrication approximation.

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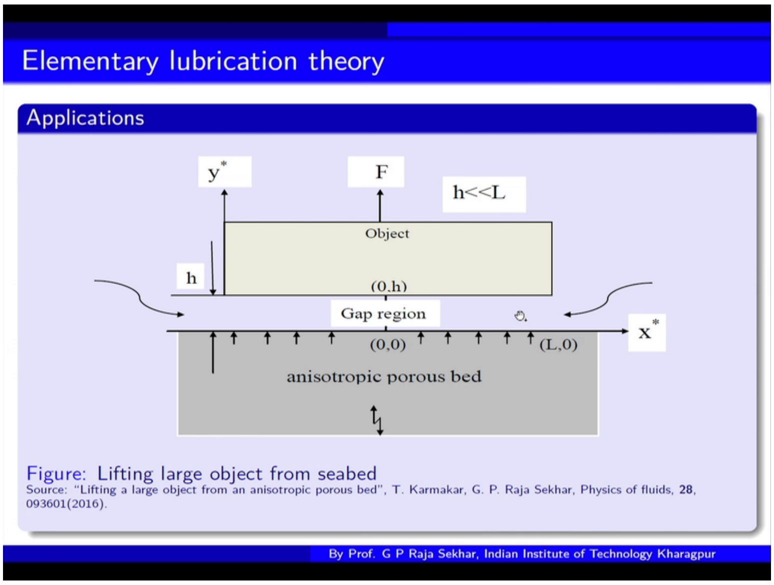

Figure: Lifting large object from seabed
Source: "Lifting a large object from an anisotropic porous bed", T. Karmakar, G. P. Raja Sekhar, Physics of fluids, **28**, 093601(2016).

The problem is as follows. So when a large objects are sunken in sea, so how do we lift it okay. So this is a, the bottom of the sea and this is the interface where you have a flow that is the liquid above and then this is a porous bed okay. So please ignore about this what is anisotropic and etc. So porous bed that is a seabed which consists of sand and then on top of it we have a water okay. Then when object is sunken what happens?

If you visualize the object will be slightly floating okay due to the corresponding bond effects right. So once it is floating, our aim is to lift it right. So how do we do it? This mechanism simply we do not go and then execute with some crane okay. So you have to estimate what kind of object and then what will be the forces with which you lift so that the object is not damaged etc right.

So now in this if you visualize there will be a thin gap where the object is floating and then if you start lifting so initially flow is coming from the bottom and then once you try to lift so within this gap you have a lubrication approximation. So once you start lifting so then the flow comes from the side and then you have overcome the corresponding lubrication force okay.

So with this idea this particular problem has been solved that has appeared in this article. Since we are talking about lubrication approximation to make you realize that there are various real-life applications we have given this okay. So you can see here H is a gap between the bottom of the object of course here we are assuming flat and then again flat bed and the corresponding L is the length of the object so H/L is small.

So we are in exactly similar scenario as water just now we have seen okay so with this I am sure you get an idea of how the lubrication approximation works and that will ensure that you can simply solve a scalar equation okay. So we see these applications of lubrication approximation in the coming lectures. Thank you!