Modelling Transport Phenomena of Microparticles
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Lecture – 06
Stream Function Formulation of Navier-Stokes Equations

Hello, today we are going to discuss about stream function and then Navier-stokes equation in terms of stream function and stokes flow past a cylinder. So to start with let us introduce what does it mean by stream function. So introducing stream function, okay?

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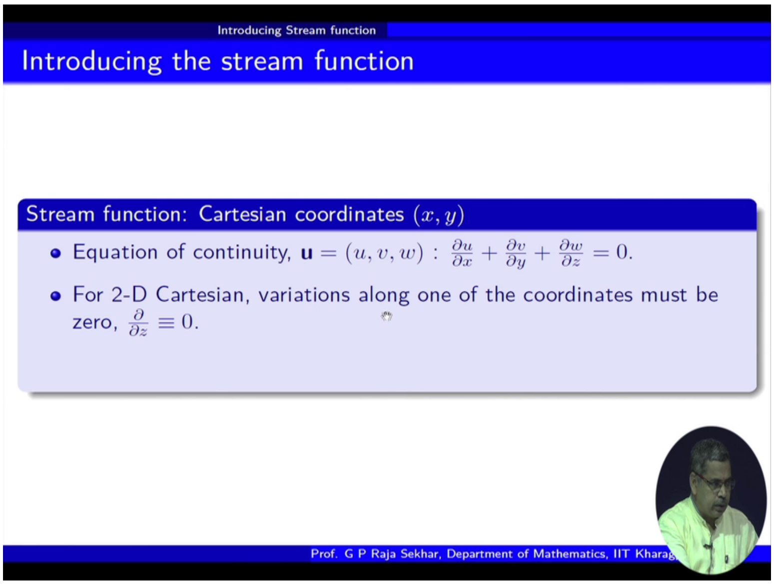

So let us start with equation of continuity. So you have divergence of u0 so if we represent u is in Cartesian u, v, w, so then divergence of u is denoted by this, okay? So once you have constant planes like x equal to constant or y equal to constant z equal to constant so then one you are suppressing one of the variations. So in this case we are projecting 2-D such that the partial derivatives with respect to z are 0.

So once you go for this restriction what happens, so this will be 0 so then the equation of continuity reduces to Dow u by Dow x + Dow u by Dow y = 0. So then one can introduce this such that the equation of continuity satisfies identically.

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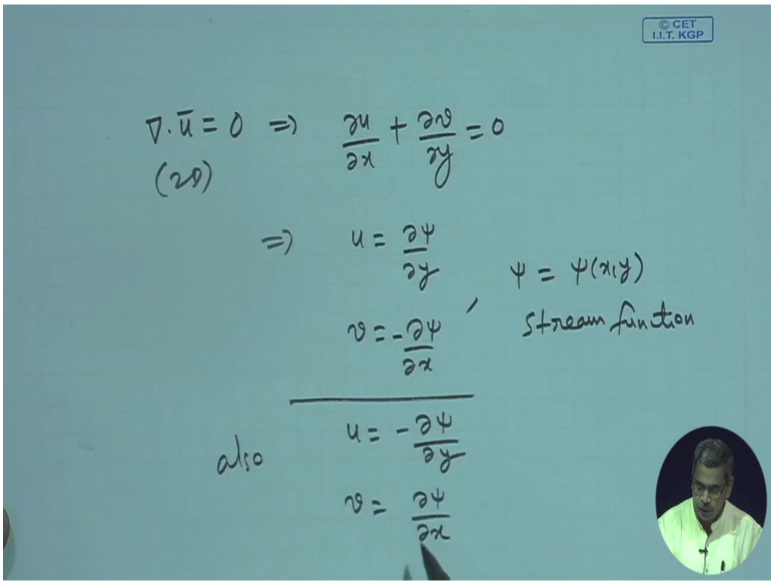

So I would like to show you how this can be done. So we are considering divergence of u0 so this implies Dow u by Dow x + Dow u by Dow y = 0 in 2-dimensions, okay? So here you have a partial derivative with respect to x here we have a partial derivative with respect to y. So one can introduce a function Psi such that, since we have a partial derivative with x here with y and since we have a derivative with respect to y we introduce partial derivative with respect to x then a Psi, Psi and since we need an identity so you can take a negative sign here.

So Psi = Psi is function of xy and this is called stream function, okay. So alternatively one can also define this negative sign. So this definition also will give an identity, okay, so only thing the corresponding velocity direction is controlled by the sign. So if it is left to right the other way is right to left, okay, otherwise both the definitions are valid.

So what is the physical significance of such a stream function, okay? So as you know suppose you have a uniform flow so then what we assume constant with a constant magnitude so then now if you take along x-axis maybe you have a lines like this. So what are they? These are called stream lines, right? So stream function can be used to represent the streamlines, okay? So one can use this and then try to plot the streamlines.

How I will explain with a couple of examples, okay. But so let us say if you switch over to plane polar coordinates so then this is the equation of continuity.

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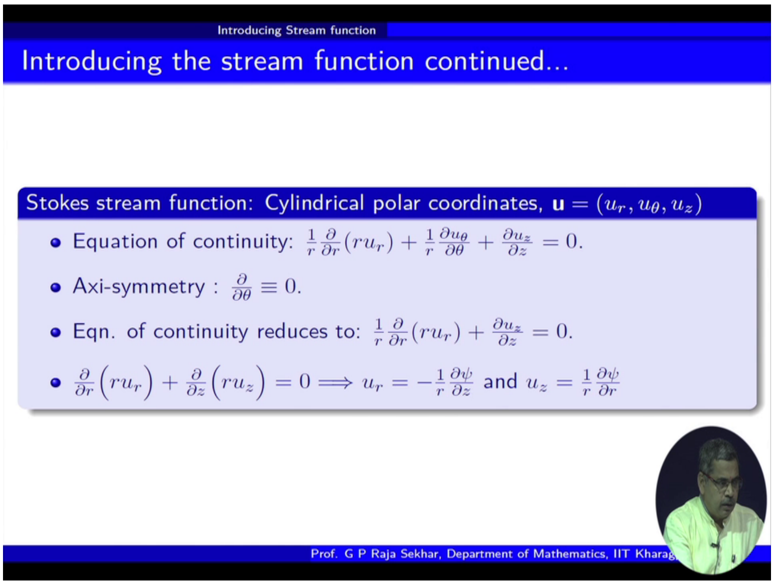

So 2-D plane polar coordinates we have this equation of continuity; that is divergence of u0. So then here also we can do a similar trick. If you see here we have the r derivatives and here we have Theta derivatives of the corresponding velocity components and the notation that we are using is velocity vector is represented by the components ur and u Theta namely radial velocity component and tangential velocity component, okay.

So we are, please do not misunderstand these subscripts with partial derivatives so these are to indicate the components okay. So now we can consider this equation of continuity and introduce as I indicated you have an r derivative here and Theta derivative here. So for Psi we are introducing Theta derivative and for uTheta we are introduced in r derivative in terms of Psi and you can see if you substitute ur and u Theta as defined by this, we get an identity. So that means this is the relation for plane polar coordinates, okay.

So similar thing one can try for other coordinates but there is a big restriction. So what is the restriction? So the restriction is: so far we discussed divergence of u equals to zero.

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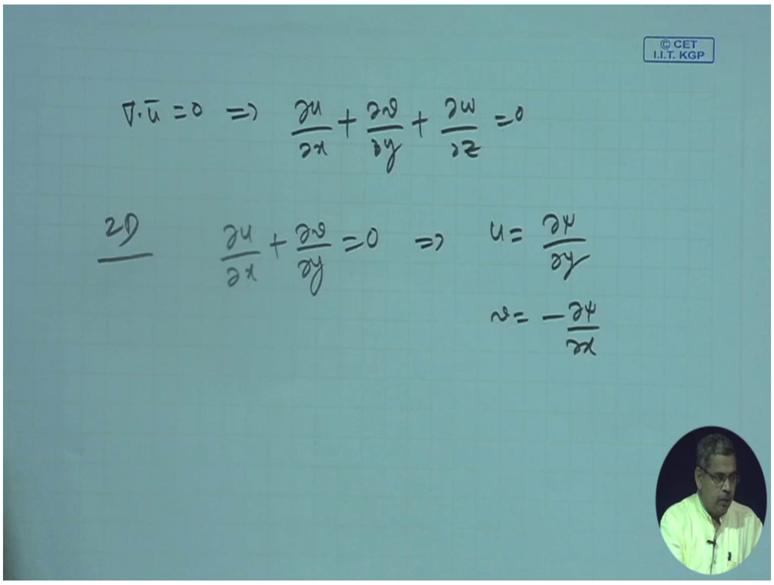

This implies Dow u by Dow x + Dow v by Dow y = 0. But restricting to 2-D we have and then we could introduce since we have x derivative we introduced a y derivative if we have a y with a negative x derivative and we have introduced u equals to Dow Psi by Dow y v is okay. Now the question is can we introduce for a general three dimensional equation of continuity such a function? So unfortunately we cannot okay.

So you can see whatever may be the combinations you would not be able to capture this in terms of one single scalar function okay. So similarly when it is 2-D plane polar coordinates, so we have only two velocity components and then we have introduced the corresponding stream function in polar coordinates. Now the natural question is can we do it in higher dimensions? So let us consider cylindrical polar coordinates okay.

So here we are representing the velocity components ur, uTheta, uz and the equation of continuity is given by this. So can we introduce stream function okay? We have three components. Can we represent these three components in terms of a single scalar? So the answer is no. So then in which simplified situation we can introduce. For plane polar coordinates if you introduce axisymmetry. We discussed this before.

So that means variations with respect to Theta are not there then reduced equation of continuity is given by this which is in terms of a radial and z components okay. So now again if you observe we have an r derivative and we have z derivative so therefore one can introduce stream function as follows okay. So one can introduce stream function as follows ur and uz. So this so this is the corresponding stream function in a cylindrical polar co-ordinates okay.

So now finally we have a spherical polar coordinates.

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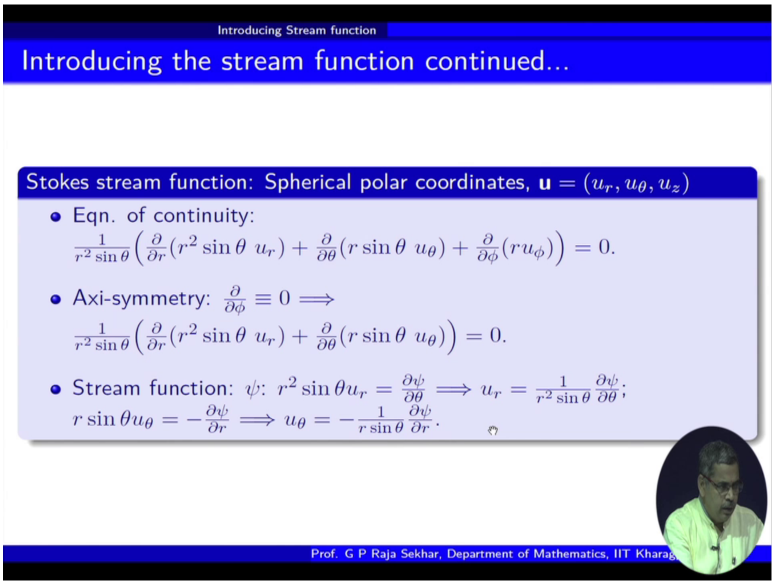

So again if you see so this is uϕ this should be uPhi. So ur, uTheta and uPhi. So this is equation of continuity. Then access symmetry in terms of spherical polar coordinates is all the flow quantities are independent of the azimuthal angle Phi so then reduced equation of continuity is this and one can define.

So this whole quantity with Theta derivative and this whole quantity with r derivative we are introducing. That is what we are doing. You see this is with Theta derivative and this quantity this quantity with r derivative, so once you introduce this, that will define ur and once you introduce this that will define uTheta. So this is the corresponding stream function okay.

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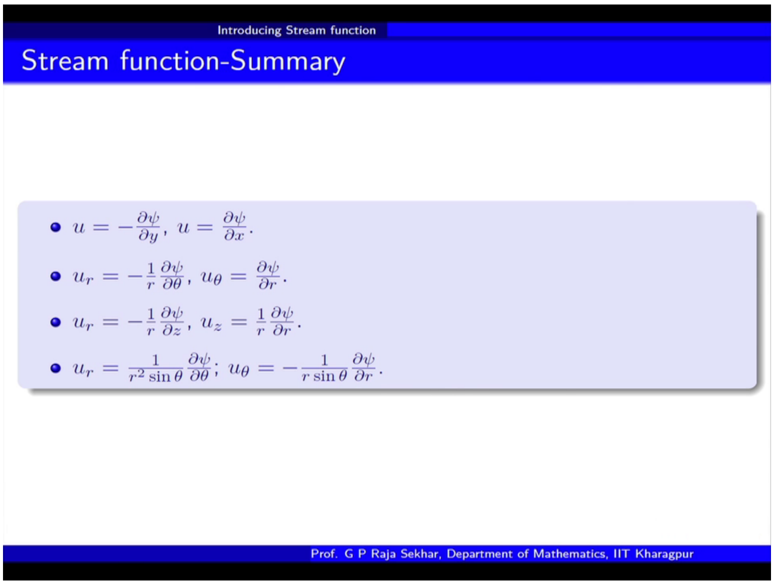

So in summary what we have in Cartesian the definition of the stream function is this, in plane polar this is the definition and in cylindrical this is the definition and in spherical this is the definition okay. So natural question is what is the advantage of introducing stream function? So what we are saying is if you in an arbitrary 3-dimension, definitely as of now we have to use only the primitive variables that is velocity components and pressure.

But if you are in 2-D maybe one can introduce auxiliary variable that is a stream function okay. So what we have seen so far is one can introduce stream function in neither 2-D or 3-D access symmetric okay. So the vector equations can be reduced in terms of the scalar equation which is by virtue of defining the stream function. Stream function being scalar so all the vector equations can be converted into corresponding scalar equation. So we will see before we solve any physical problem using this approach okay.

So but before that so once you have corresponding vector quantities using this relation we should be able to integrate and obtain the corresponding stream function. So how do we do it?

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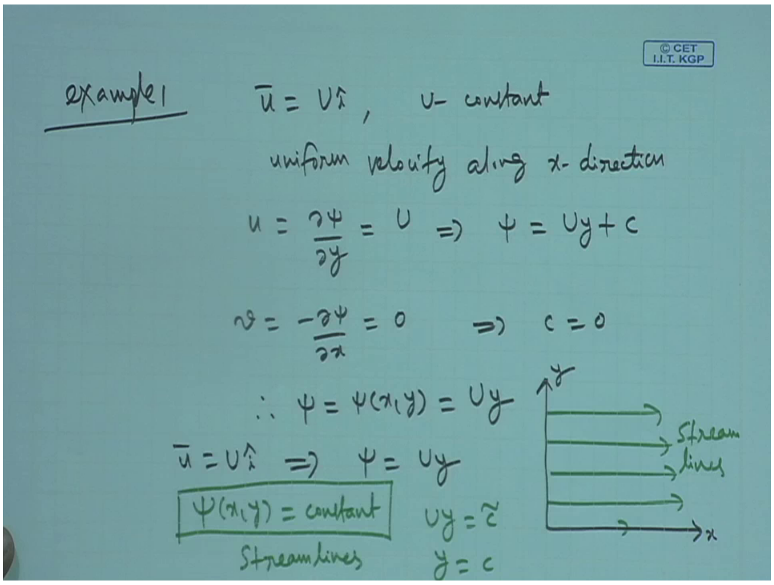

So let us take an example. So this is example to convert stream function. So let us say the velocity is so this is the unit vector and u is constant. Which means this is supposed to be uniform velocity along x-direction.

Now using the stream function we have u is Dow Psi by Dow y and v is -Dow Psi by Dow x. But from this what we have is u component is U and v component is 0 okay because there is no z component. Therefore be 0. So from this what we get is u is function of y alone right? So then from here one can get Psi is uy + a constant and we have Dow Psi by Dow x is 0. So if we use this implies c is zero so therefore, what we have is u times y.

So corresponding to the uniform velocity ui we have the stream function y okay. So once you have the stream function we can now represent the streamlines okay. We can represent the streamlines suppose using the contours. So what are the streamlines? Psi(xy) equals to constant. So they represent streamlines. So these are the: so in this case what we have is uy = some u~ or y = some constant.

So you can see so including the y = 0. So these are the streamlines okay. So once we have vector form velocity we could get the corresponding scalar and then now one can get the corresponding streamlines okay. So this is very useful tool so but with the restriction. What is the restriction? We can introduce stream function in 2-dimensions or 3D axisymetric okay. So now let us see how Naiver stokes equations can be simplified.

So there will be a bit of algebra but some I would show and then some you can work it out okay.

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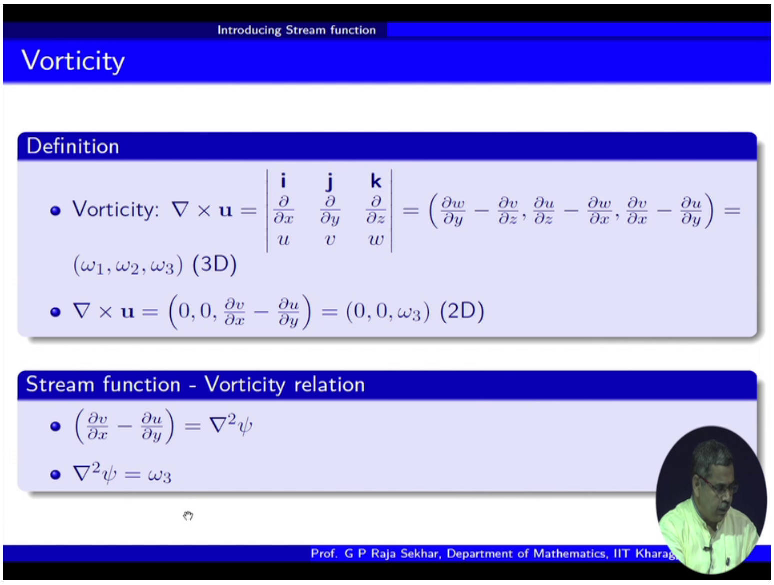

So another important quantity is Curl of u in Cartesian if you take so this one can now check very easily and since we have three non zero components assuming u, v, w, do not give so general sense. Therefore non zero we are assuming so Omega 1, Omega 2, Omega 3. So these are the components each of them you are calling and this is called Vorticity. So vorticity basically indicates the rotation okay.

So it indicates if for vorticity is non zero means so you have a rotation flow happening okay. So that is the vorticity. And similarly if you have only 2-dimensions then you will see for any general u, v, w, so you have only one component. So that is given by this. So we call it only Omega 3 okay. So we have two scalars one is stream function and curl u can be represented in 2-D in another scalar whereas in 3-D Omega is also a vector okay.

So now there is a relation in 2-D that is called if you simplify. See v is Dow Psi by Dow x, u is a Dow Psi by Dow y, so we get Laplacian. So this is and if you see 2-D this is exactly the third component of the vorticity okay, third component of vorticity which means Del square Psi is equals to the third component of vorticity okay.

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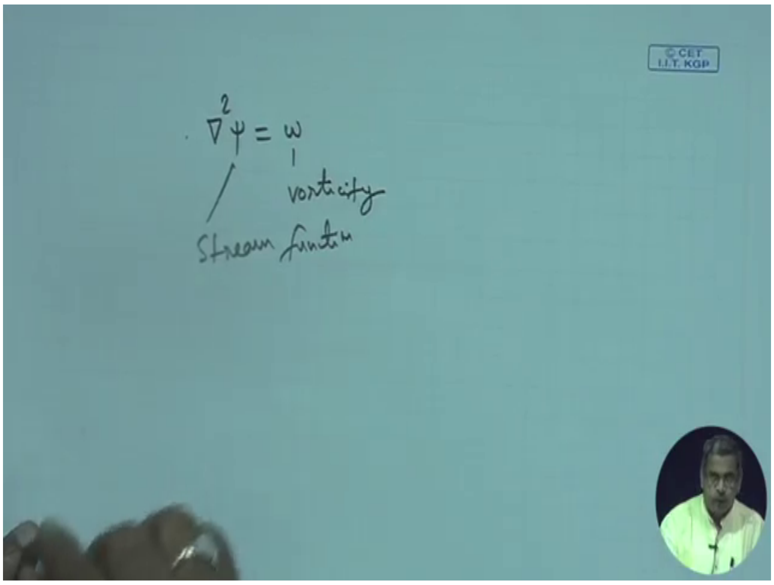

So we are defining stream function vorticity relation. So that is Del square Psi = Omega assuming this is Omega 3 and Psi is stream function okay. So this is very useful tool to convert the corresponding vector equations in terms of stream function vorticity formulation which we are going to do now. So that is the next section Navier-Stokes equation in terms of stream function.

So let us consider Navier-Stokes equation of a viscous incompressible flow and we have ignored the external forces okay. Now we restrict the dimension to 2-D because in 2-D we have stream function and vorticity okay. So what is our aim? Our aim is to get rid of primitive variables and convert these equations to equivalent equations in terms of stream function vorticity okay.

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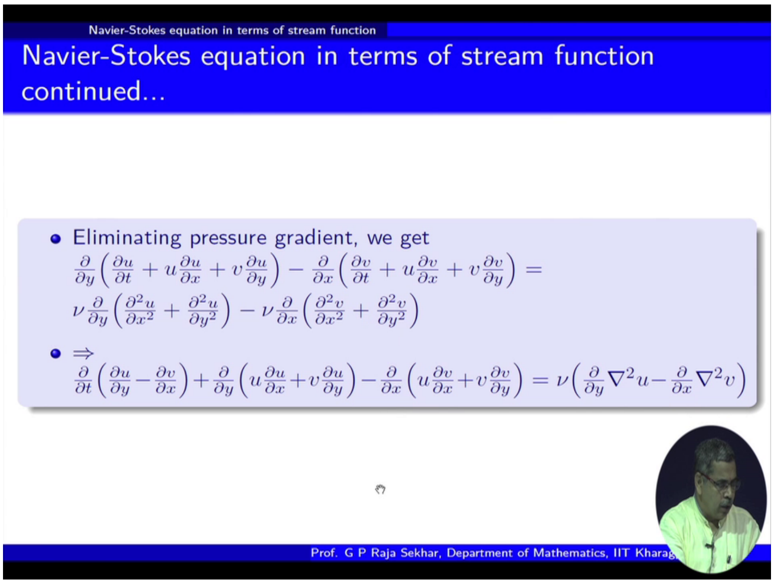

So these are the component form. Then one can eliminate you see this is a pressure is having partial derivative with respect to x and here y. So our aim is to eliminate pressure so one has to differentiate this with respect to y and differentiate this with respect to x and subtract. So we do that so once we do you see this with respect to y minus the second component with respect to x. So this is what is done.

This whole thing with respect to y minus this thing with respect to x okay we are subtracting right. So that is what is done so it is a simple algebra. Now let us group some terms so that we can clear identify. So what we are grouping so this time derivative we have taken out. So that will be Dow u by Dow y and here Dow u by Dow x then y derivative of the entire term, then similarly x derivative of this entire term.

Then so these two are given then the right hand side you can see this is a Laplacian. This is Laplacian, so there as it is. So now we have our stream function definition. So we introduce the corresponding definition in the above equation. All that we have done is a simply substitute corresponding u and v as it is. We do not do much. You can see Del square u becomes this Del square v becomes this. So this is not much we have done okay.

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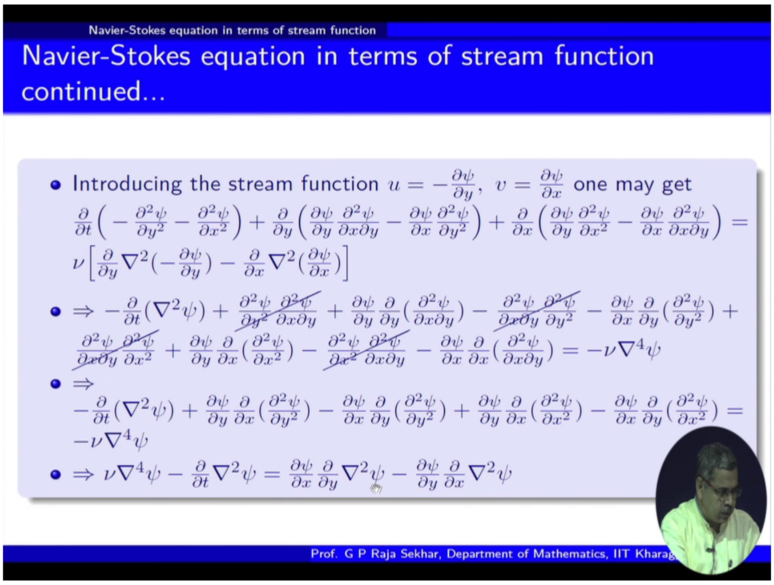

Now you see if you expand little bit so some terms get cancelled okay. So in this produces two terms, this is a mixed derivative and Dow by Dow y operating on this okay. So if you do the complete expansion so some terms get cancelled. You see this term and this term are similar with negative sign and this term this term okay. So some cancellation occurs and we get. So this is a simple algebra so it is not very difficult.

One can follow almost have given almost every step okay. So then the simplified remains this, you can see how this is bi-Laplacian okay. So you have a Laplacian and it is Cartesian coordinate so they commute. So Laplacian comes out and this goes in and we have a second derivative with a negative sign of y power 2 and similarly here we have a second derivative of x with respect to x and minus sign. So therefore we get minus Del 4 Psi okay.

So this is what we have. Now this can be put it in a compact form further you see. So this term is brought to the left hand side and the remaining terms they are put in a compact form okay. This there is a symmetry Psi with x and Laplacian of Psi with y, so Psi with y Laplacian with x. This is exactly nothing but the definition of the Jacobian.

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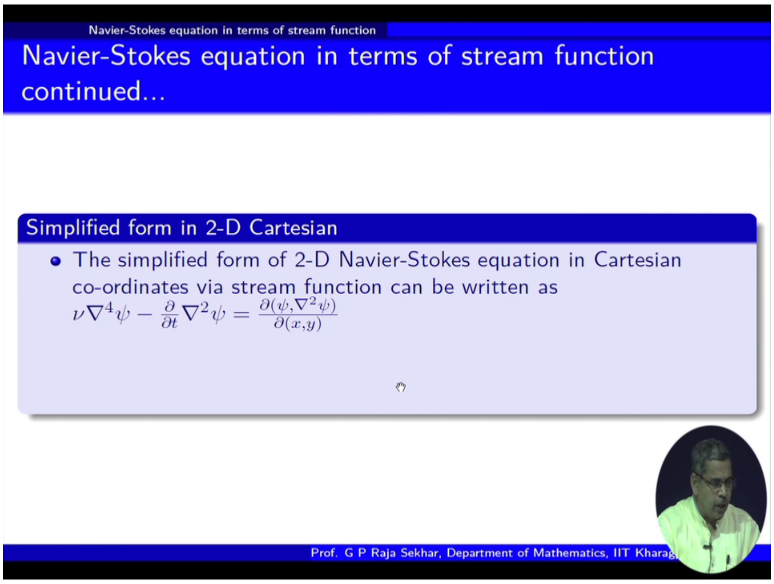

Hence we got a simplified Navier-Stokes equation okay. So now the equation as it stands what we have achieved is really we have converted the vector Navier-stokes equation in 2-D via component form to a one single scalar equation. Ofcourse it is a non-linear retaining the non-linear structure of the Navier-stokes equation okay. All that we have achieved is we have converted into a scalar equation.

So one has to solve the corresponding non-linear scalar equation okay, so this will be further simplified depending on a steady case or if it is stokes flow. So correspondingly if it is a steady stokes flow what we have seen if it is steady?

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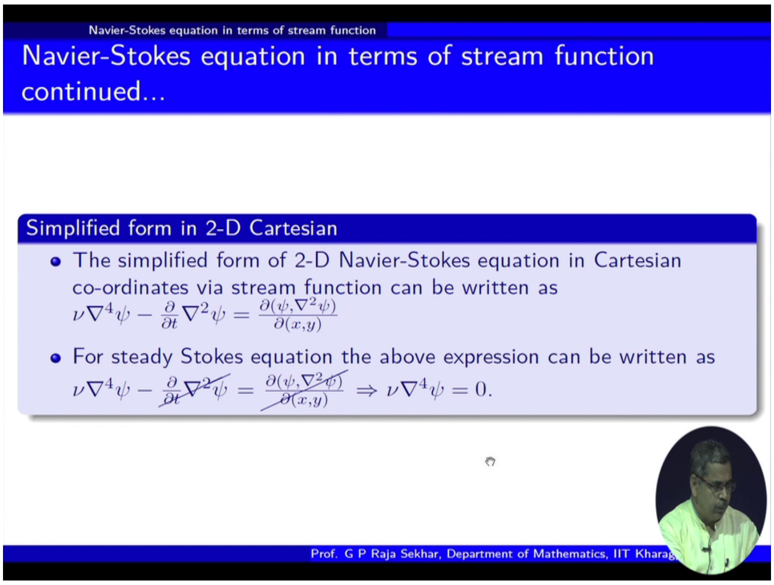

There would not be any time dependency, so therefore this has to go and then if it is a stokes flow inertial term is not there. If you see this is due to the viscous term. That is how the kinematic viscosity is multiplied and this is due to the inertial term. That is how the non-linearity is sitting in this. So now if it is a steady stokes equation then we are neglecting time dependency and we are throwing away the non-linear terms, inertial terms.

So then we get simply stream function satisfies by harmonic equation okay, so that is one.

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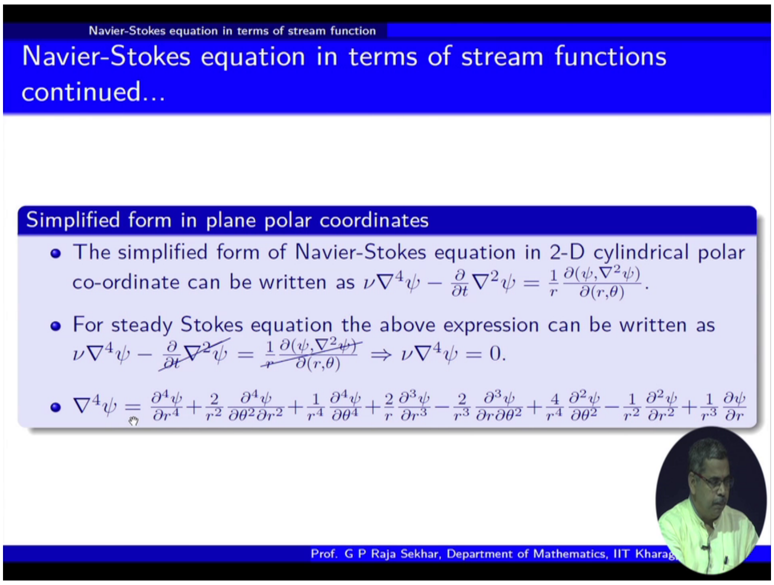

So now similar approach can be done in polar coordinates. Well the corresponding algebra that we have done for Cartesian is slightly manageable. We can straight away substitute and then do this kind of manipulations. A similar thing one has to do in plane polar coordinates so that involves little bit of calculations. But it can be done okay. So even if we do that we are going to get the same.

As long as you are in 2-D so the final equation is going to be the same when it comes to Stokes flow because you do not see the non-linear terms. But once you have non-linear terms then the Jacobian will be in Cartesian it will be in terms of Cartesian and in polar coordinates it will be in terms of polar coordinates as you can see here okay. So even in this case studies limiting both are same okay.

So this is about Navier-stokes in terms of stream function okay. Now our job is if somebody wants to solve stokes flow past some object using say stream function then we need to find the solution of this right? It appears complicated but we can handle. Why we can handle? This is a linear operator so that is the biggest advantage that we have okay. So let us see now next section is we would like to use this scalar function.

And then solve some practical problem okay. So the first problem that we are discussing is stokes flow past a circular cylinder. So when you say a circular cylinder we are not talking about flow past a cylinder like this okay. So we are taking about vertical, the flow across the cylinder so that anytime we see a 2-D projection from above. So this is a 2d problem, so let us consider stokes equation okay.

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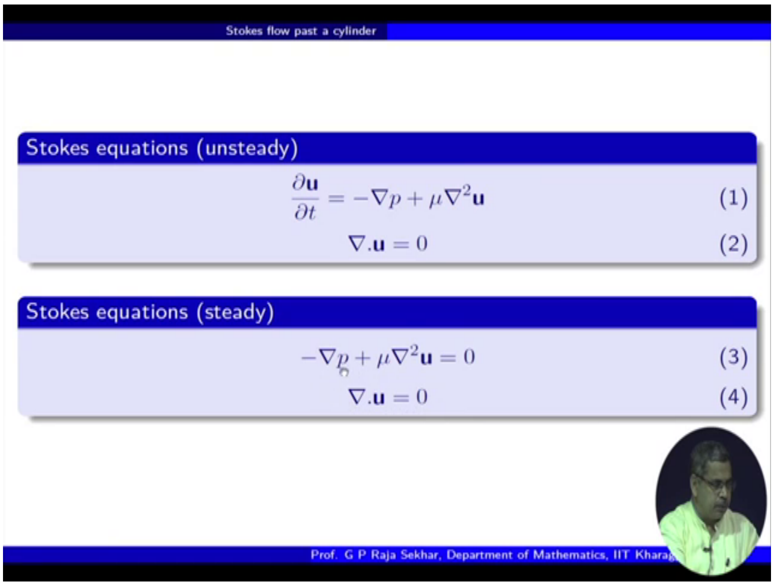

So this is conservation of mass and then momentum balance you have seen unsteady okay. Then steady case we have viscous terms and pressure terms and this is equation of continuity.

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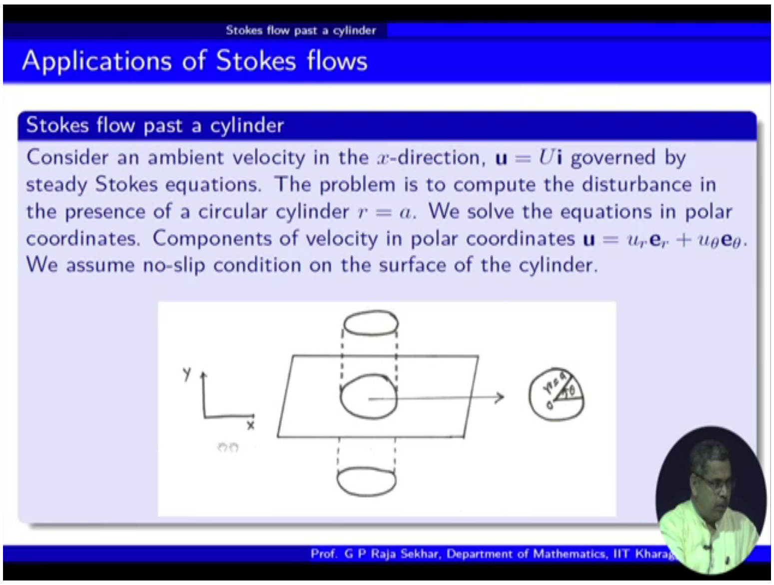

Now let us introduce the problem so as I indicated already we are considering uniform velocity at far field okay and we are placing a cylinder and we are discussing flow past the cylinder. So as I already indicated the problem is 2-dimentions, so we are seeing from top. So we are seeing a circle. So it is essentially flow past a circle okay and the corresponding velocity can be decomposed into radial and then tangential.

And we assume that so there is no flow across the cylinder. So therefore no-slip, right? So we introduce r and Theta coordinates. So now what is the tool the tool? The tool is the stream function.

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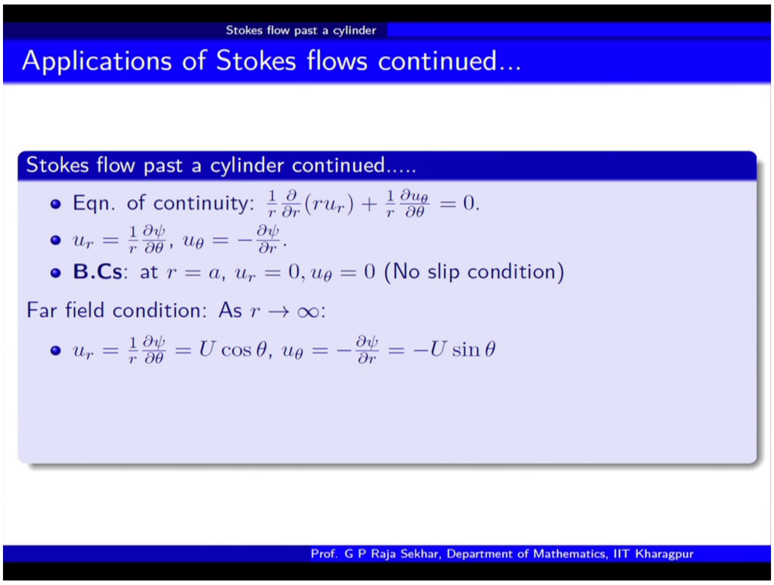

So this is the equation of continuity in plane polar coordinates. Then we have introduced the stream function then the corresponding boundary conditions that we have are the no-slip condition that is the normal velocity 0 and a tangential velocity 0. And far field we have uniform velocity. So this requires a little explanation.

So we are considering a cylinder given by r = a, and here we are considering a far-field velocity.

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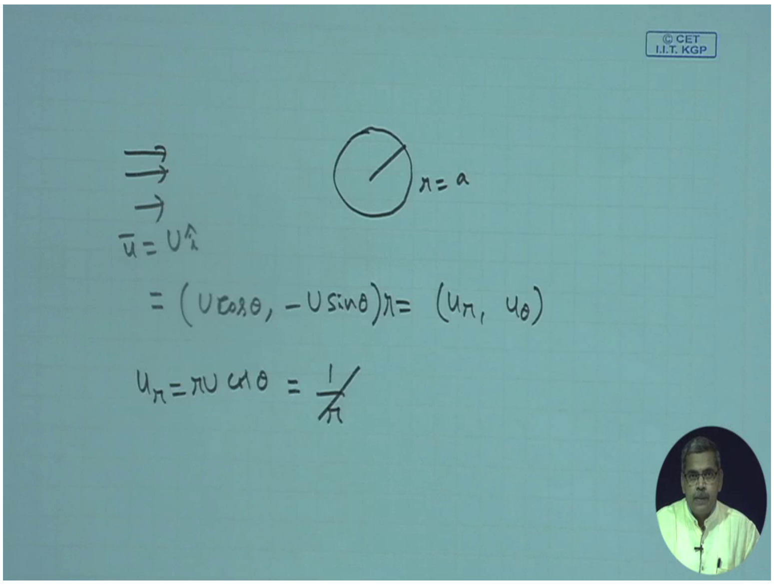

But we are not solving the problem in terms of velocity. We would like to solve the problem in terms of stream function. So corresponding to this we have to get the stream function okay. So how do we do this? This you can decompose uCos Theta – uSin Theta. So how we are getting i has been decomposed in terms of r and Theta okay. So correspondingly we get this.

Now from from this we have the stream function velocity okay relation so we use so this is ur comma uTheta okay. So from here ur is uCos Theta. So this decomposition we do and there should be an r there okay. So this is equals to okay.