Modelling Transport Phenomena of Microparticles
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Lecture - 05
Dimensional Analysis - Non dimensionalization of Navier-Stoke's Equations

Hello in the last class we discussed some examples on exact solutions of Navier-Stokes equations. So in particular if you consider flow between two parallel plates where the upper plate is moving, so the velocity profile is just a function of the velocity of the plate and the co-ordinate okay and the distance between the plates okay. So now if you try to plot so what would happen is somebody may fix some particular pressure gradient and then try to plot with H okay.

Somebody may fix another pressure gradient and then try to plot H and then they claim the values are the same or different so that means there is a trade-off in some sense okay. So how to resolve the trade offer okay. So if you particularly pay attention to these velocity profiles they are dimensional okay, if you consider shear stress they are dimensional, so whatever the physical quantity that we have computed so they are dimensional okay.

Now we will ask a question so really do we need dimensions and then now if we have what is a scenario and then if you want to get rid of them what is the advantage okay.

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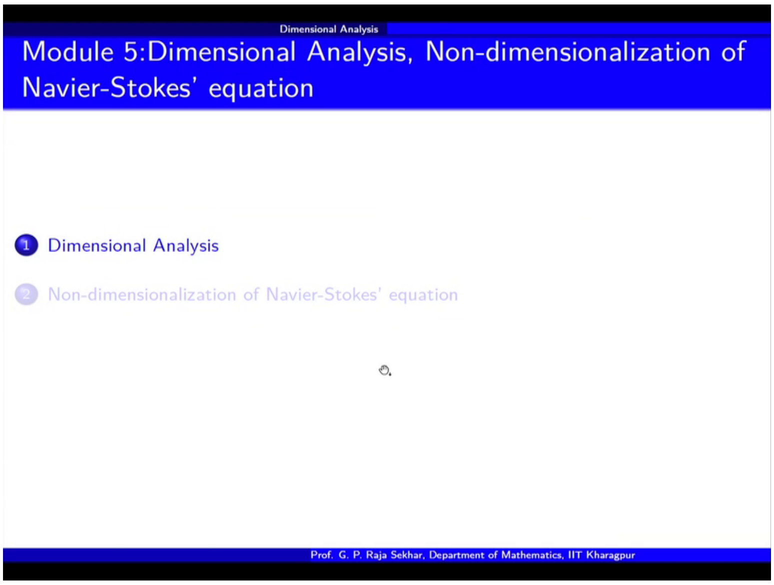

So correspondingly we are discussing dimensional analysis and non dimensional of Navier-Stokes equations. So to start with the dimension analysis.

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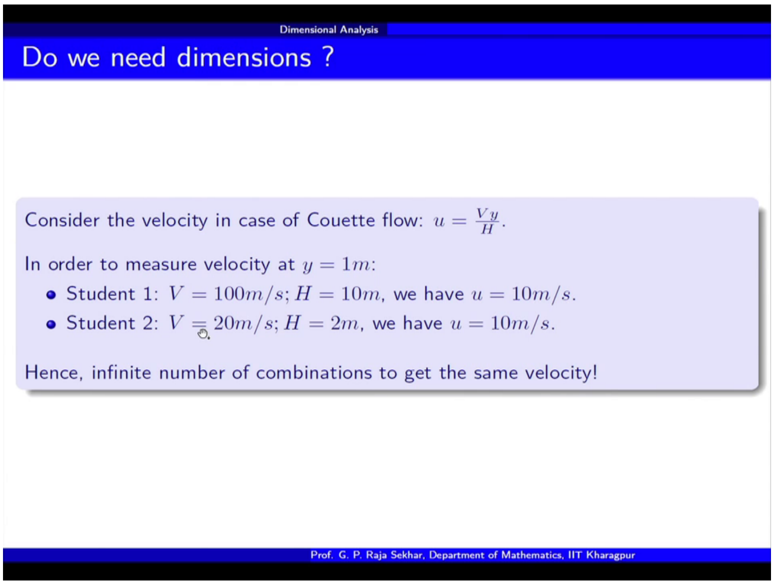

So the first question that we ask is do we need dimensions? Okay, so why we are asking this question so we can address using the following example, so consider the velocity in case of Couette flow so as I indicated velocity is depending on the velocity of the upper plate distance between the plates and the y co-ordinate okay.

Let us say in order to measure velocity at y = 1 meter, student one has considered V is 100 meters per second. H is 10 meters then we have accordingly u is 10 meters per second suppose student two V is 20 meters per second H is 2 meters then also we have u the same okay.

So naturally these two students may get on a quarrel no my solution is correct or no my solution is correct okay. So indeed both the solutions are correct okay, so in some sense if you observe closely you have infinite number of solutions right. So how to resolve this okay, you see carefully since this depends on these quantities we would like to measure the velocity at y = 1.

So you have several combinations in order to get u 10 one can take various combinations of V and H okay. So that is the reason infinite number of combinations to get the same velocity okay.

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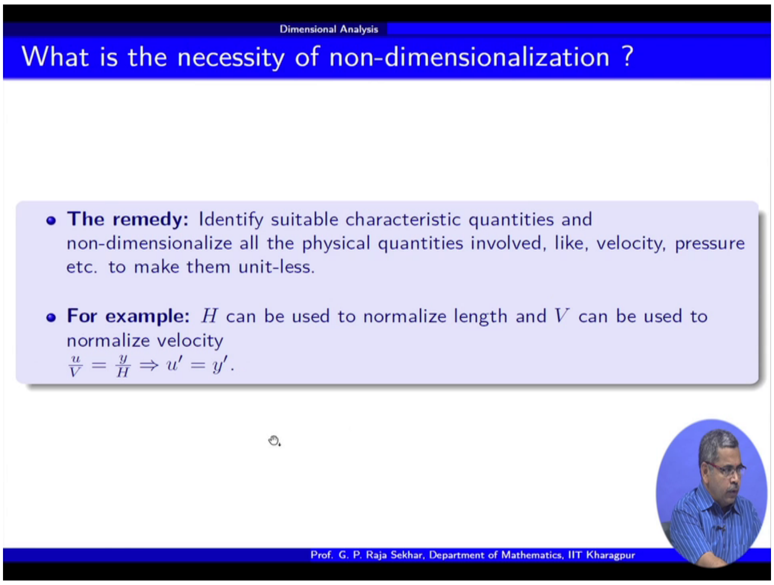

So what is the necessity of dimension okay and what is the necessity of non-dimensionalization okay. So what global aim is let us say somebody has done some experiment on a smaller, let us say small environment with some dimensions so in a hand, Let us say and then they want to scale it up okay. So how do they do? So in these cases of the corresponding dimensional analysis play a role, so let us see how this can be achieved?

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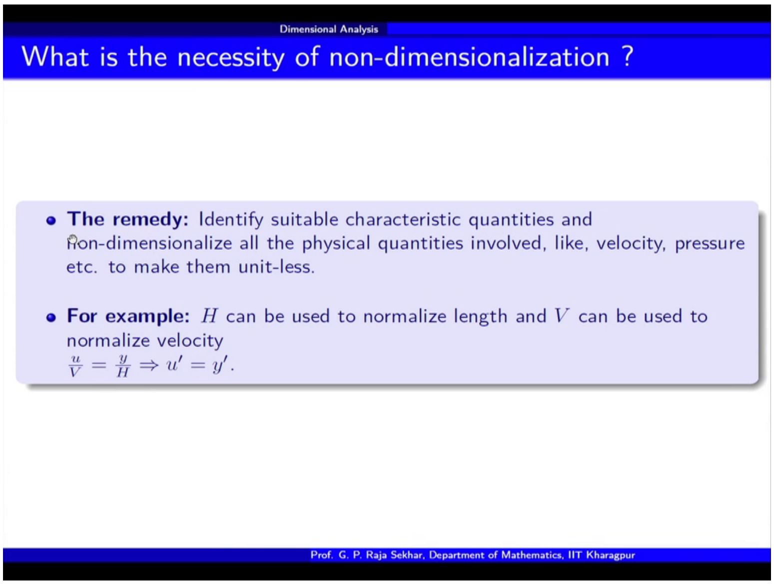

So what is the remedy? identify suitable characteristic quantities and non-dimensionalize all of the physical quantities involved like velocity, pressure, etc., make them unit-less. For example you take H. H is nothing but the distance between the plates. So one can normalize using some length V is velocity so one can normalize okay. So H can be used to normalize any length and V can be used to normalize velocity, so you define u by V, so this will be non dimensionalized.

We are calling it u prime, y by H, which will be a non dimensionalized length scale which we are calling u by prime. So now if you consider u prime = y prime, so this is non-dimensional quantity, so any variations if you would like to discuss so simply u varying with y okay. So this is, this in some sense incorporating this structure the entire trade-off that we have, they are hidden in this and the simply use function of y okay, so this is the advantage.

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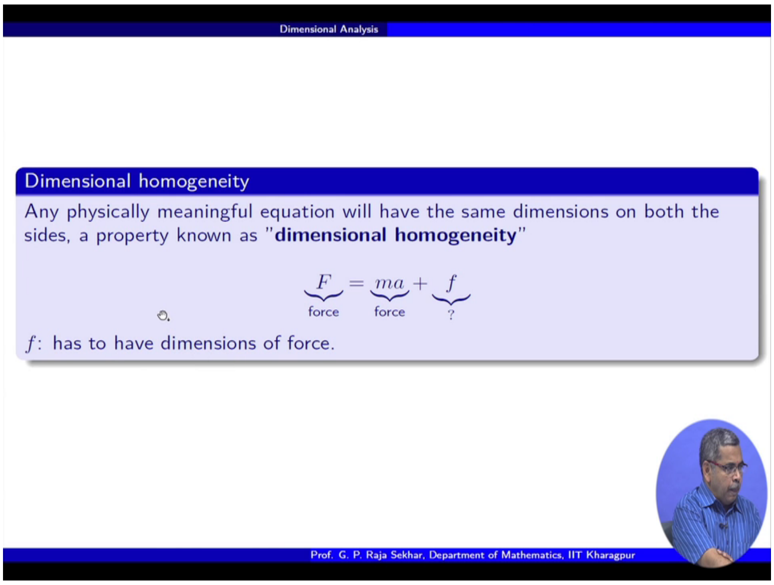

So therefore we would like to discuss more about the dimensional analysis and look into the corresponding the algebra. So dimensional homogeneity any physically meaningful equation we have the same dimensions on both the sides the property known as the dimensional homogeneity. What does it mean? So let us say you have a standard equation which is force balance so F equals to ma + some quantity okay.

So now for a dimensional homogeneity left hand side denotes force, right hand side first part denotes force, therefore F has to have dimensions of force, so that is the dimensional homogeneity okay.

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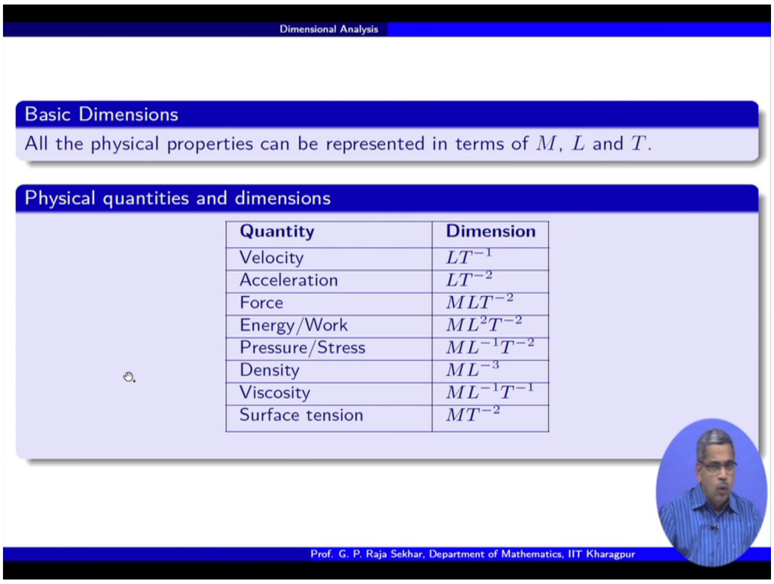

Now what are the fundamental quantities that that are used indicating the dimensions so these are the three which are mass is M, length L and time is T, so correspondingly all the physical quantities will be represented in terms of MLT. So let us look at quickly some quantities, so for example velocity has dimensions LT power -1, because distance by time right. Similarly acceleration distance by Time square, so therefore LT power – 2, force mass into acceleration.

So energy, so correspondingly pressure, density, viscosity, surface tension okay, so these are represented, the dimensions are represented in terms of MLT okay.

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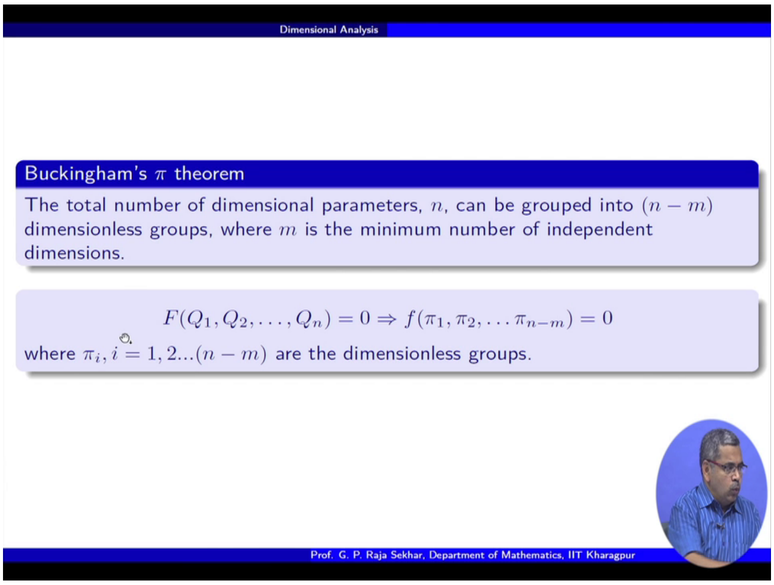

Now for a particular physical problem how do we figure out? How do we group this? This non-dimensional group's okay. So that means our global aim is to identify some non-dimensional groups, so that you do not talk about a particular pressure gradient particular distance etc. So you identify a non-dimensional group and then you fix that for this value this is what was happening, so we can indicate okay.

All the trade-offs are completely hidden into that non dimensional group okay, so our aim is to identify such non-dimensional groups. So for this powerful tool is called Buckingham's Pi theorem, so what does it state the total number of dimensional parameters can be grouped into n – m dimensionless groups where m is the minimum number of independent dimensions okay.

Which means suppose you have n dimensional parameters then m minimum number of independent dimensions then n – m dimensional groups can be identified okay. So in a little formal sense suppose Q1, Q2, Q3 are the dimensional parameters okay.

In a balanced equation or whatever then what we are trying to say is, this can be put it in a structure like m are the minimum number of independent dimensions, therefore this can be grouped in such a way, that Pi1, Pi2, to Pi n – m where each of these Pi is are dimensionless groups okay. So this is appears very abstract but let us look at the more categorically.

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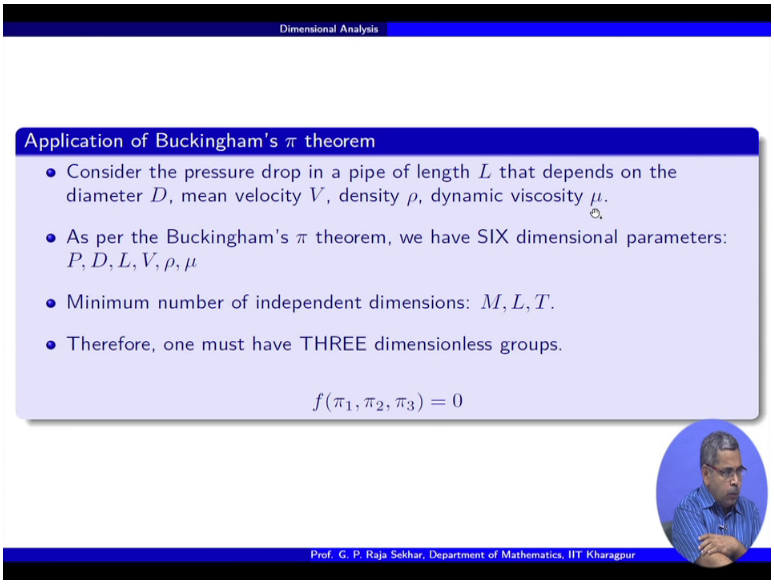

So for this we would like to take an example and then display, so consider the pressure drop in a pipe of length L that depends on the diameter D, mean velocity V, density Rho, dynamic viscosity Mu okay. So please pay attention we have various physical quantities length, pressure drop, diameter, mean velocity, density, dynamic viscosity.

So as per the Buckingham's Pi theorem we have six dimensional parameters what are they? Pressure drop also included diameter, length, velocity, density, and viscosity and minimum number of independent dimensions M, L, T. So therefore as per Buckingham's Pi theorem n - m that is 6 - 3, so we must have three dimensionless groups. So they are given by Pi1, Pi2, Pi3, okay.

So let us try to get some lights, how do we identify these dimensionless groups and that is the task at hand okay. So if you see we are saying one can choose these options okay.

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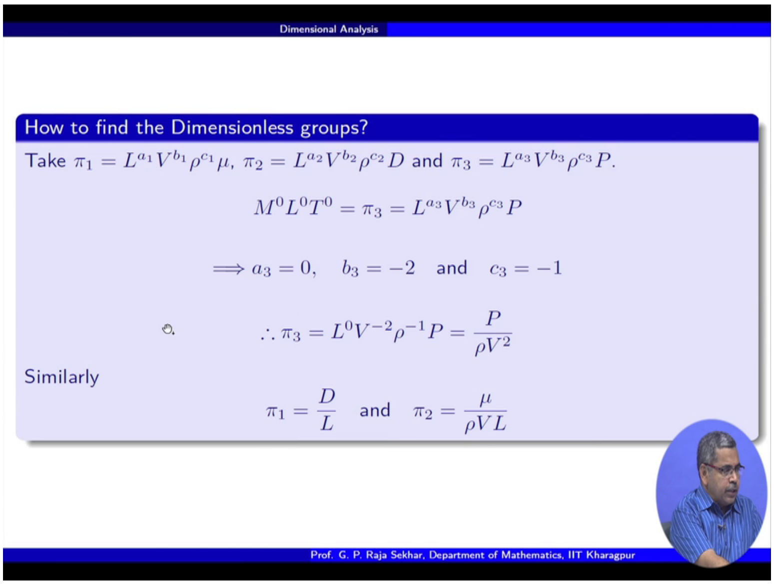

But if you see for what variables we can have such powers and what are the free variables, so there are there is a trade-off one okay, so how do we fix it, if you see closely length this is attached to the geometry, velocity it is the flow property, whereas the density is the fluid property okay

So this is a flow property depending on the context, similarly length and the velocity flow property and this is a fluid property okay. So while choosing these dimensionless groups we must have these three with some power, we do not know what is the power we have to determine so that Pi1 is non-dimensional and once these three are fixed other ones we can vary so that we have these are non-dimensional groups involving viscosity, involving diameter, involving pressure okay, so we will explain with one of them

So let us consider the Pi3 what we are trying to do is we are trying to get a non-dimensional group that involves pressure, length, velocity, and density. So that means in some sense we are trying to identify some quantity that will have dimensions of pressure, so that we can normalize and that Pi3 will be non dimensional so once we expect Pi3 to be non dimensional naturally the powers are going to be 0 and right-hand side we have this powers okay.

So correspondingly we get these values right? So how do we get these values.

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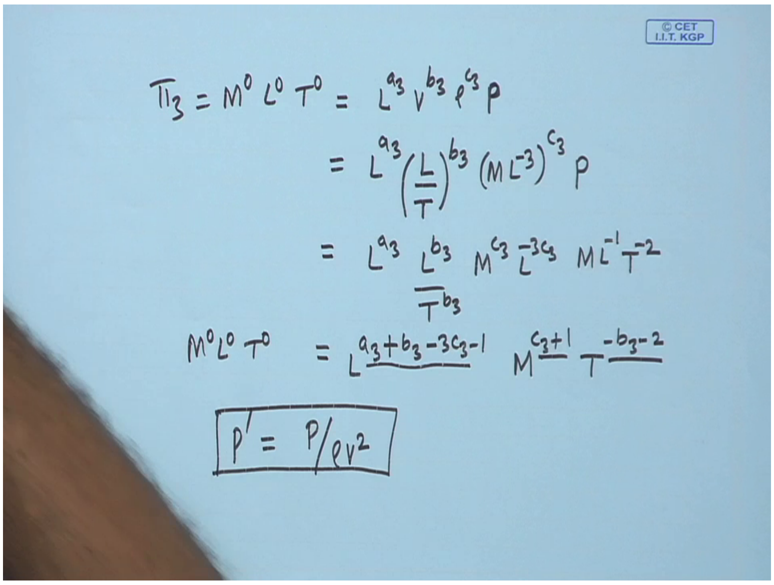

So let us have a look at this so what we have is Pi3 okay, so this we have written as a combination of length, velocity and pressure, this pressure. So now we expand, so velocity as a dimensions L by T, then density, so density, then P okay, we expand then pressure, we have so now we would like to get the powers of M. L, and T, so correspondingly if we consider what we get okay.

So one can simplify this, so that we get this okay, because we have left hand side M power 0 and L power 0, so we equate this is 0, this is 0, this is 0, and solve the system so that we get. So once we get this what is our Pi3, L power 0, V power -2, Rho power -1, which is nothing but P by Rho V square, which means this is a non-dimensional group that means if we have to non dimensional pressure we can use Rho V power 2.

So that Pi3 is a non-dimensional group, so now what does it mean the trade-off between density and velocity is hidden and one can non-dimensionalize and then get non-dimensional pressure as you declare let us say P prime which is non-dimensional like this, so this is a non dimensional pressure and hence we can deal with the non-dimensional quantity okay.

So similarly if we work similar algebra we get another two non-dimensional groups one is D by L and another is Mu by Rho V L okay. So with this non-dimensionalization, so we are ready to go for our non-dimensionalize or Navier-Stoke's equations, which is very much essential to understand the additional insights on the Navier-Sroke's okay.

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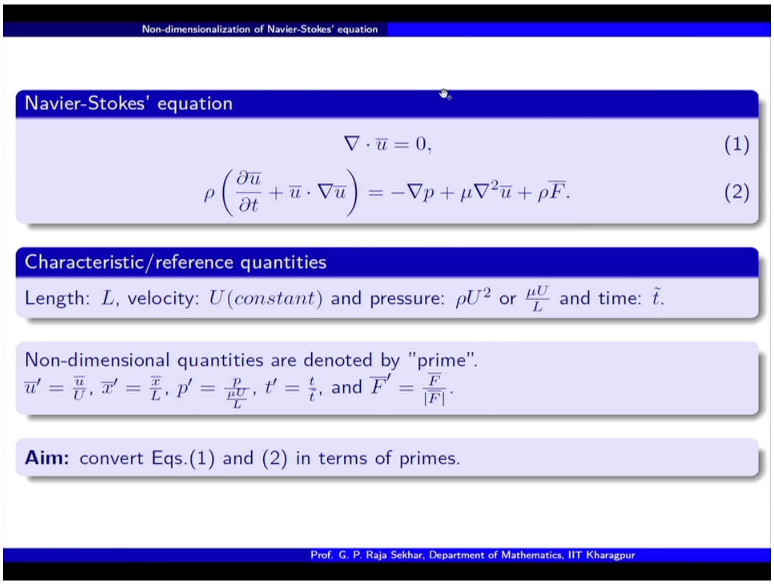

So that is the next section non dimensional Navier-Stoke's equation, so let us consider a Navier-Stoke's equation, given by equation of continuity and we are considering unsteady viscous incompressible case and we have also external forces. Now the question is what are these are non dimensional variables that we are using, so that depends on the geometry. So in the case of say flow between two parallel plates, so one can use for velocity, the velocity of the upper plate.

And for length one can use the distance between the plates okay. So since they depend on the geometry of the problem we call them as characteristic quantities, which vary depending on the geometry okay. So correspondingly we are introducing some characteristic or reference quantities for example for length L velocity U which is constant and the pressure, just now we have seen in addition for pressure this is also another non-dimensional group and the time.

So which means we are defining non-dimensional physical variables, which we are denoting with primes so here U prime is Mu by U and any length is a corresponding length by L and the pressure P by this or one can take P by Rho V square and the time and similarly the only external force F bar, so we are assuming some force dimensions of the magnitude and then now we are normalizing okay.

So once we define such non dimensional quantities what is our aim? Our aim is to convert these two equations in terms of these prime variables which are non-dimensional okay. So how do we do it so we have to play with the equation?

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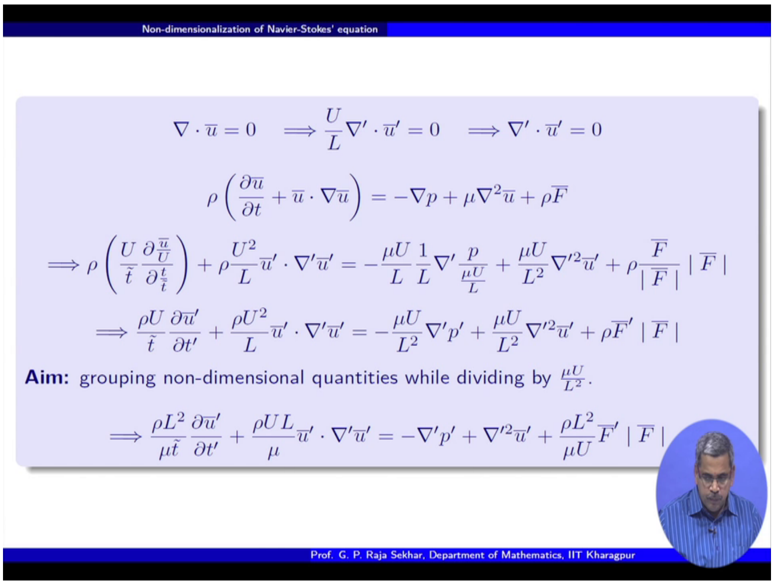

So please pay attention those are comfortable with the vector notation one can follow, but I will explain also in component form. So consider the equation of continuity, so in order to normalize the gradient operator, we have to introduce L in the denominator because we have Dow u by Dow x, so xi we have to non-dimensionalize that means xi by L into L okay and similarly u so you divided by velocity and multiply by velocity.

So that is how this is coming so I am sure some of you will have difficulty so let me explain.

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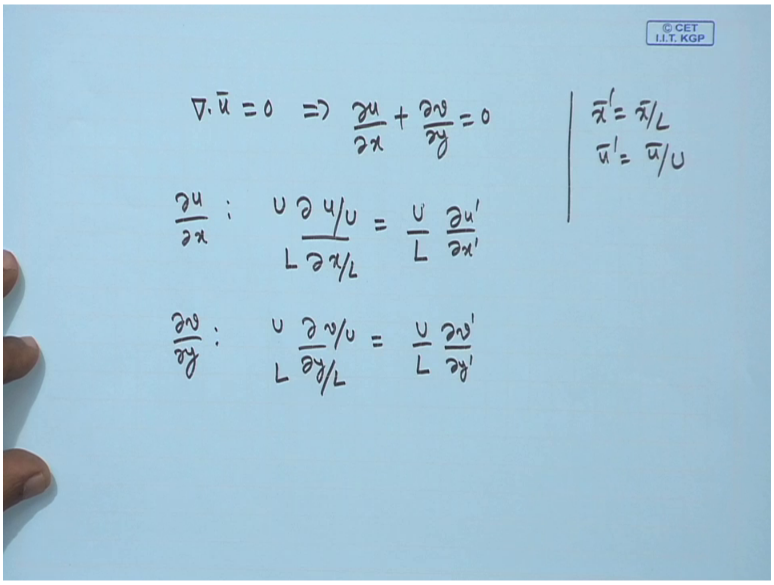

So this is what we have which means say Dow u by Dow x + Dow v by Dow y goes to 0, so now our non dimensional quantities are x prime is and u is, so what we do is consider so for U divide and multiply, for x divide and multiply, this will be U by L and this is nothing but according to this U prime, this is according to this x prime, therefore this is nothing but similarly so V, so U, this will be V by L, so that is what we have written you can see U by L, these are in terms of primes okay so therefore it is invariant okay.

So now let us look at the momentum balance equation, so what we are trying to do is the same process since we have U we divide and multiply by U, then T divide and multiply by T, similarly U dot Grad and the corresponding pressure etc. So I am sure it will be slightly difficult so let us do it so that we understand the in a better way.

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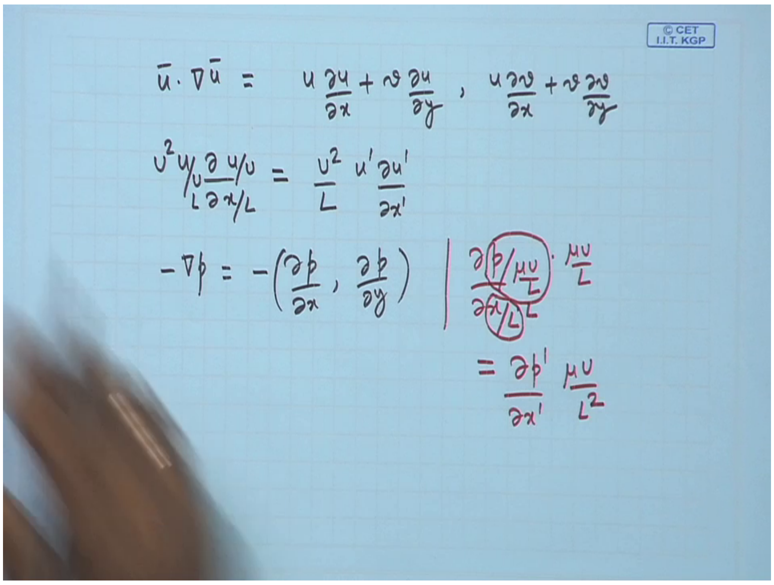

So I will explain U dot Grad U, so this is in component form, so this is X component I have written okay, so this is a let us say X component, so we can write the Y component as well okay the Y component will be in 2D. So if we do it for one of them this one is understood so let us consider this one so we have Dow u by Dow x, so we have a u there multiplied and divided.

So therefore this becomes u square L, so here x is in the denominator, so therefore L has to be given, so correspondingly what we have got this becomes u prime, this becomes u prime, this becomes x prime, so what we are getting is U square by L, u prime. Since we are using the same length scale for y, same velocity scales for V, so even if we non-dimensionalize this we are going to get the same pre factor okay.

So you can see so that is what we are getting U square by L, however since we have a density so correspondingly Rho is multiplied and the these are now similarly pressure, so let us see this is minus okay, so we can take one of them say you consider, so what we have decided is for x, L and P, Mu u by L, so that is what we have discussed. So P u Mu by L into Mu u by L, into L, so we get this quantity is P prime, this quantity is x prime.

So therefore we get and what we get this is the pre factor, so that is what we are getting Mu u by L square and correspondingly these are non-dimensionalized so this is our P prime. So similarly Laplacian.

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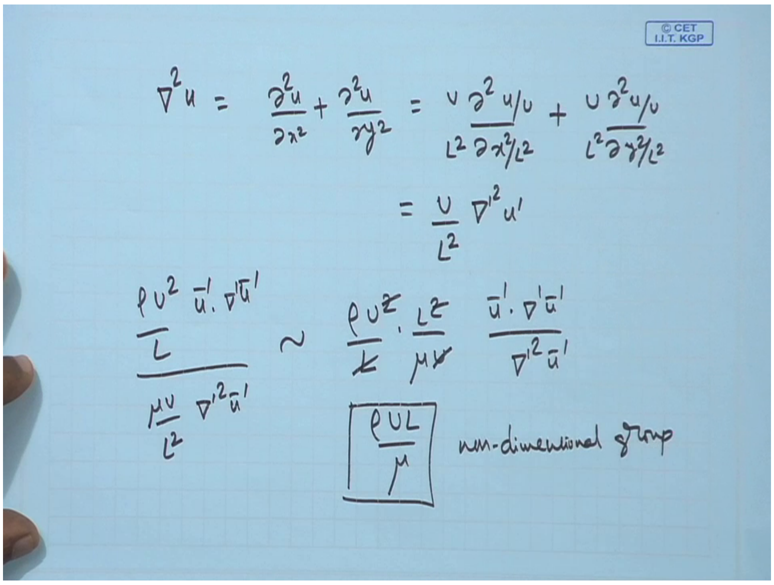

Laplacian is again slightly involved so we have let us consider Del square u, so we introduce u by u, L square, similarly therefore what we get is u by L square, u prime okay. So that is how we are getting u by L square and this Mu is multiplied, so Mu is multiplied here and this force is normalized by corresponding magnitude okay, which has forced dimensions, so we got the simplified equation okay.

So now what is our aim? We would like to get the non dimensionalized grouping, so in order to do this; we divide throughout by Mu by L square okay, this is common here so you divide so we take it to the other side, divide by this so each of them, for example Rho u square L, I am considering the convective term so this we are dividing by Mu U by L square and what is the corresponding term from where this is coming, so let us consider this is coming from this viscous term.

So that means I am taking the ratio of this and ratio of this okay. So this is like this Rho u square by L okay and there is a corresponding okay, so we have, so we got the non-dimensionalized group here which is nothing but Rho U L by Mu, so this is a non-dimensional group which is coming by virtue of the ratio of inertial forces to viscous forces this is a non-dimensional group okay.

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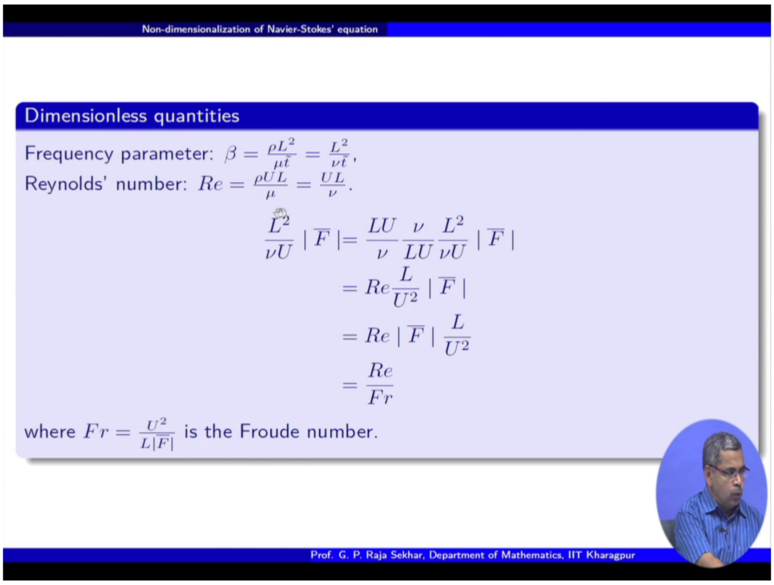

So similarly the first unsteady parameter, if we take the ratio of this coefficient with dividing by this, we got the corresponding parameter here and the one just now what we have got by virtue of ratio of inertial forces to viscous forces, so for example Mu by Rho is Mu, so if you use it, what we got is UL by Mu which is called Reynolds number. So Reynolds number is the one that characterizes the magnitude of the inertial terms okay.

So if the Reynolds number is high, so it is a naturally the velocities are high because the non-linear term is sitting in the inertial term if the Reynolds number is low, so then now definitely the non-linear terms the contribution is less okay. So we discussed so definitely little bit about this in a later lecture, so similarly the body force term so we have the body force term and the corresponding term we can now adjust, so this is the nothing but multiplying and dividing by this.

Because we have body force term, we are left with this so we are playing with this, so multiplying dividing by this then this combination is Reynolds number and we are left with this and one can do some adjustments and define the entire ratio as Re by Fr where Fr is called Froude number, so which is a indicating the force ratio okay. So this is indicating the force ratio, so which is a non-dimensional group.

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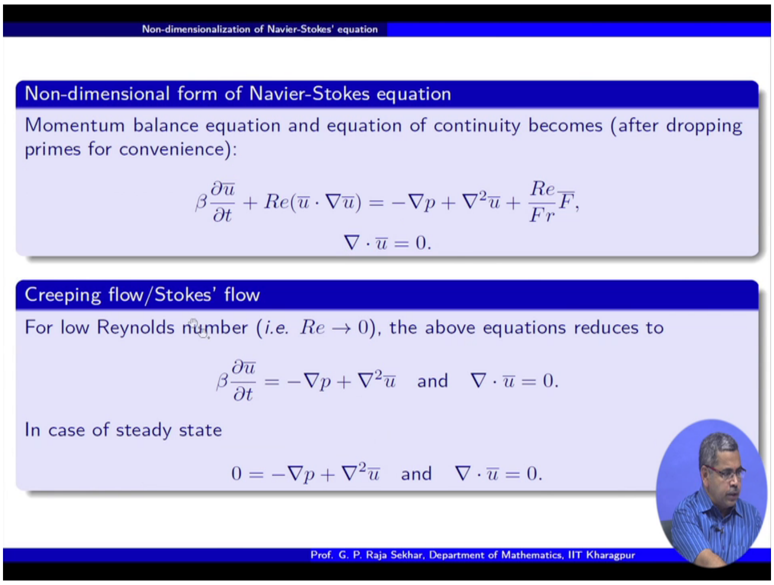

So once we have a non-dimensionalizeation of the Navier-stokes equation, we have three non-dimensional parameters, one is the corresponding frequency parameter, another is the Reynolds number, another one is the non-dimensional force parameter that is called Forde number okay. So we would like to work with the non dimensional system, so that the entire trade-offs across velocity, viscosity, density etcetera.

They are taken care in non-dimensional groups, so that means there is no debate what should be the viscosity, what should be the density, so that I get this velocity, such questions can be addressed that means we are really scaling up okay. So in a non-dimensional sense we are talking all the physical quantities and if you want to scale it up only you talk in terms of these, these non-dimensional groups so that is the global aim okay.

So now creeping flow, so if the Reynolds number is a very low tending to zero, that means we are neglecting the inertial terms okay so we get this right and if it is a further steady case then we get these are called a stokes equations.

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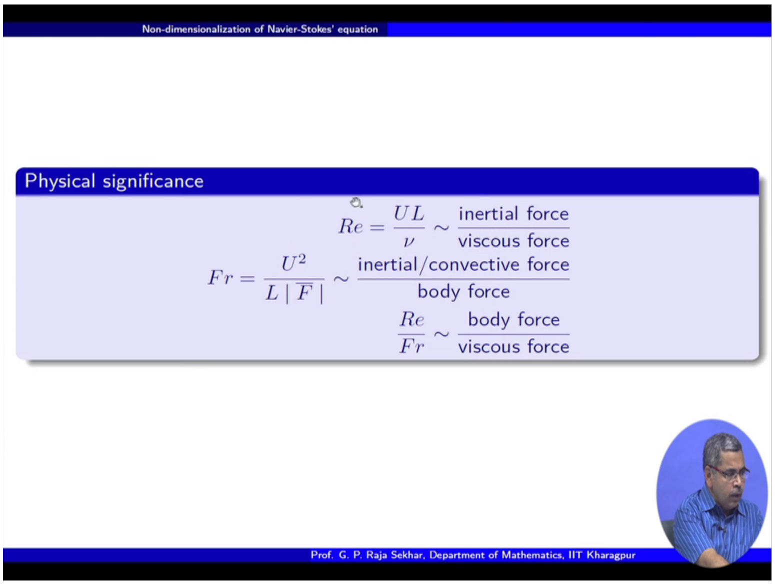

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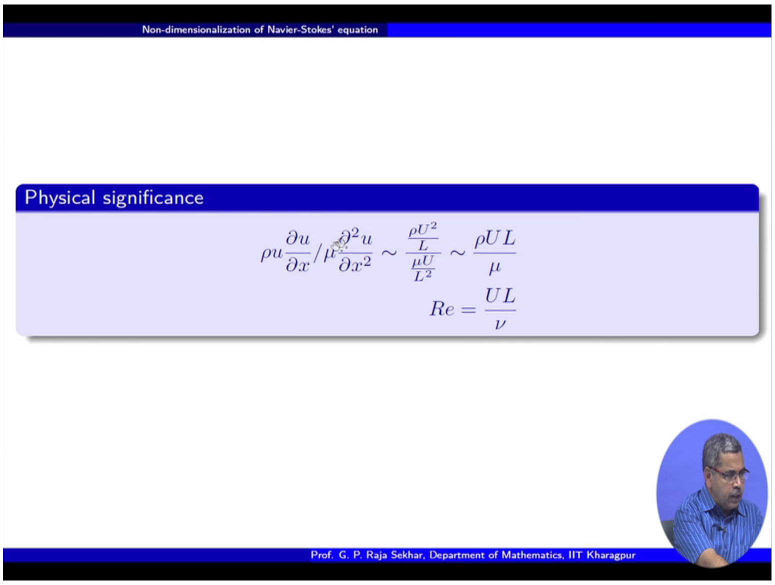

So what is the physical significance, physical significance is as I indicator you take inertial term X component I have taken, then this is a viscous term, this ratio already I have explained we get this which is nothing but the Reynolds number so in some sense, what we are getting after doing the non-dimensionalization is the competition across various forces, so Reynolds number is the competition between inertial force to viscous force.

And similarly Forde number is competition between inertia and convective force to body force and then this particular is body force to viscous force okay. So when we say viscous force is dominating so then the denominator is too large so Reynolds number is small okay. So when we say inertial force a large, so then Reynolds number is large.

So one can control order of the Reynolds number and correspondingly assess the results so these are very useful for scaling up purposes okay. So with this you got some idea about non dimensionalization and how to identify the non-dimensional group's etc., thank you.