Modeling Transport Phenomena of Microparticles
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Lecture - 34
Numerical Methods for Coupled Set of BVP

Okay, the derivative boundary conditions we introduce the technique is that you would choose to fictitious points symmetrically beyond the domain of the grid and extend the grid from 0 to n and replace the boundary conditions from the body condition they are fictitious find values to the given equation and still you get a tri-diagonal system. So one can try this to solve this problem.

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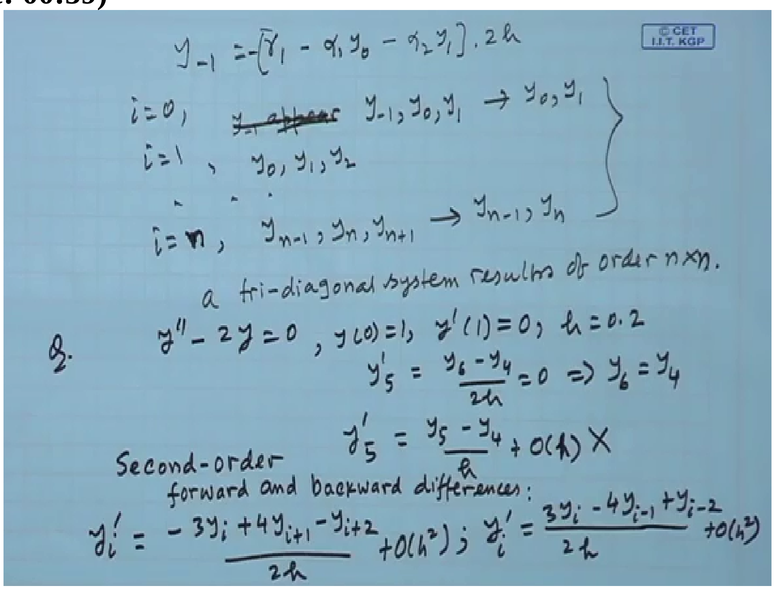

So y double dash -2y = 0 with boundary condition has y (0) = 1, y dash (1) = 0 with the two boundary conditions and let us take h = 0.2 so you can derive the tri-diagonal system by replacing this backward directions 1 by a fictitious there so that is y dash this is 5 will be so this is y5 we write y6 – y4 by 2h = 0 so this y6 = y4. So whenever the y6 is appearing in the disc dye equation they are you substitute y4.

And you get it hetro-diagonal system as mentioned in the previous one that if it is a nonlinear one then it is not advisable to go beyond the domain of the competition so in that case we would like to be within the domain. So for that one can say that why not first order logic first order say this is a backward difference is suppose if I take y dash 5 = y5 – y4 by h.

So this is one of the remedy but the problem is that it is a order h, so this lowers our order of accuracy which is a we have taken a backward difference formula similarly if it is a given at the first point so I can take the same way forward difference formula, so that we do not have to go beyond the domain of the competition but here we are compromising with the order of accuracy so this is not advisable but we can derive second order forward difference.

And backward difference formula and this second order forward and backward difference formula for the first order derivative as this yi dash = -3yi + 4yi + 1 – yi + 2 by 2h which his order h square similarly one can obtain y i dash this is forward difference, similarly one can obtain the other formulas yi dash = 3yi – 4y - 1 + yi - 2 by 2h order h square this is the backward difference if I use this forward and backward difference formula for the derivative conditions.

So when you have first grid points with the derivative yi dash is described so I can nicely use the forward difference condition and we did not have to go beyond the domain of the competition and if I use the second order backward difference formula at the end points again will be confined within the domain so this is the one of the remedy and it works nicely because it is a get order h square situation so this one can try and get the solution now with that again will have a tri-diagonal system.

So because once in the technique what do you do if we place this y0 and yn, now if I instead of y0 if I have this condition y0 dash what I find is we have a relation of a y5 I = 0, y0, y1, y2 so the relation between y0, y1, y2 from there y0 can be expressed in terms of y1, y2 which can be substituted in the first equation of the algebraic system similarly yn - 1 by yn if I put to ensure I am can be expressed in terms of yn -1 and yn-2 you substitute of the last equation.

So that involved n - 1 and n - 2 so again will get a n - 1 order tri-diagonal system so those one can work out very easily this is a very simple way to derive those formulas I am not going to details know. What next topic is higher order possible so it is not necessary that will be always having a second order so if you have the higher order BVP.

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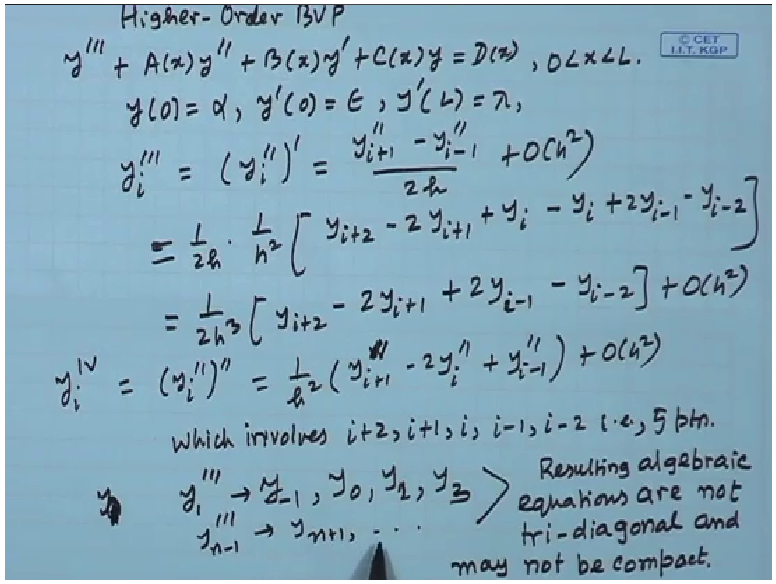

So in that case higher order boundary value problem so in that case what should be our strategy so let is take a problem as y triple dash + A(x) y double dash + B(x) y dash + c(x) y = d(x) with a y(0) = Alpha, y dash (0) = Epsilon, y dash (l) = Lambda, so here we are taken 0 less the x l at the starting point. We can always take this 0 because I can change the co-ordinate from some value x = 8 , x = 0.

So without loss of generality know any higher order derivative can be obtained in terms of the whatever we have derived for first order and second order for example, yi triple dash one can write in this way so I am treating yi triple dash as the derivative of yi double dash function. So this can be written is yi triple dash – 1 yi double dash by 2h now each of these yi double dash can be expressed in a form 1 by 2h into 1 by h square [yi + 2 – 2yi +1 + yi – yi + 2yi – 1 – yi - 2].

So this is becoming 1 by 2h cube [yi + 2 – 2yi + 1 + 2yi – 1 – yi – 2] + order of h square, same order one can if required we can also approximate yi4 in the same manner that means 1 by h square (yi + 1 double dash – 2yi double dash + yi – 1 double dash) + order of h square short, now here this is not a representation which is involving three points, which involves i + 2, i + 1, i, i – 1, and i – 2, so that means that is five points.

So obviously tri-diagonally system is ruled out, another thing is that the compact, we may not have a compact system so that means compact system is number of equations and number of unknowns are may not be tally, because say this i = 1, which is the first grid, sorry second grid, for first grid i = 0, i = 1, if I write the equation say y1 triple dash, involves 1 -1.

Because of i – 2, -1 is there and y0 is there, y1 there is no problem this two does not have any problem, so here it is yi1 y2, y3, but we have introduced a y – 1, similarly y – 1 can be replaced by that way and similarly when we talk about yn – 1 triple dash so that gives you, yn + 1 ans so on. And so the, what are the if I doing this way direct method so we can write that resulting system algebraic equation equations or not tri-diagonal and may not be compact c.

In this case this is a very easily find out because she when I describe this third order equation yn - 1 double dash so it involves yn + 1 yn, so yn it is not prescribed yn – 1, I can write because y dash n it is described so I guess it can be n dash is given. So through that one can write yn + 1 in terms of yn – 1, but yn is not given so we have a may not be compact system will appear so that will see number of equations and number of variables may not be equal.

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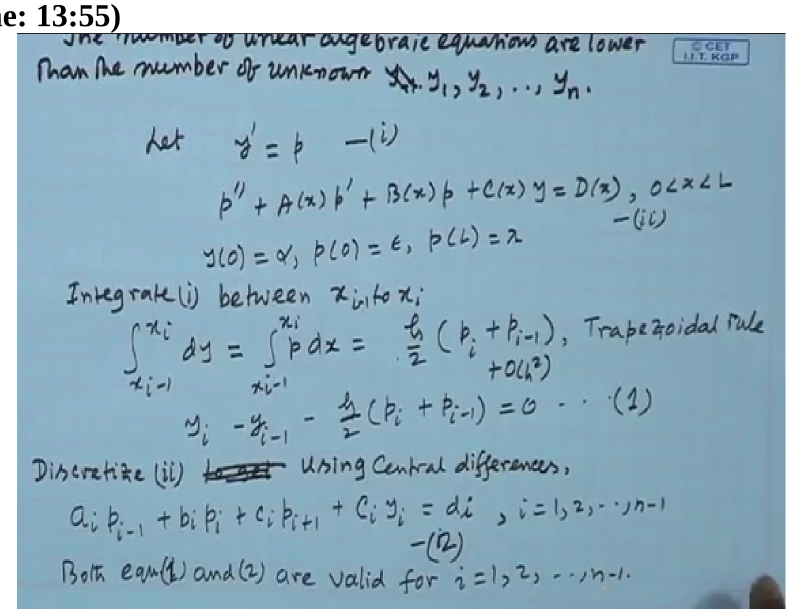

So the one can check that the number of linear algebraic equation are lower than the number of unknowns means y - 1 may not be unknown or not because y - 1 can be replaced by y0 is given so let us start with y1, y2, yn, n is not given, so n number of unknowns and you will have n - 1 number of equation so this is one problem another problem that if a n is large so since equations are not tri-diagonal so it will be difficult to solve by a iterative process example.

If I solve Gauss elimination or any other method like sidle iterative methods and all they are quite slow so it is not advisable if n is large, so the remedy is to reduce this to a system of equation so what are the way we can do let y dash = p a new variable I introduce so what will have p double dash + A(x) p dash + B(x) p + C(x) y = D(x) and the given condition are y(0) = Alpha, p(0) = Epsilon and p(l) = Lambda.

So 0 is less than l, now instead of a single equation we have two set of coupled setup boundary value problem. Because it solution of these all depends on p, p also depend on y say the same way we have to describe this equation if we have the second order problem this one should difficulty in discretizing this but how to describe this y dash = P so what I do integrate so let us call this is equation one and this is equation two, integrate 1 between xi - 1 to xi.

Sso that means what I do is xi – 1 dy = integral p dx xi – 1 to xi to xi, now xi is independent variable. So this equation we this integration with replaced by the trapezoidal formula pi + pi - 1 trapezoidal formula and this one is nothing but yi – yi - 1 if we take it out h by 2, pi + pi - 1 = 0, so let call this is equation one so this is this discretize equation of the given equation one.

Now this is order h square because the trapezoidal formula is second order accurate basically it is hq by n number of grid points so one can write it as a order h square. So we get we are not doing any compromising the order of accuracy and here this one is an exact form because dy we integrating with xi – 1 to xi is nothing but yi yi - 1 I so in this stage the approximation only approximation is here where we are replaced by trapezoidal rule.

So this is trapezoidal order square the other one if we discretize this equation so one can get discretize to get so we get a situation so using ways to using Central difference formula, We get a set of equation like the system of equation ai pi - 1 + bi pi + ci pi + 1 + di here it is of course so capital ci yi = di because there will be some known from this site so this is i = 1,2, n – 1, so this is let us called equation two.

So both equation 1 and 2 valid for i = 1 to N -1 shit together modified is this together which forms.

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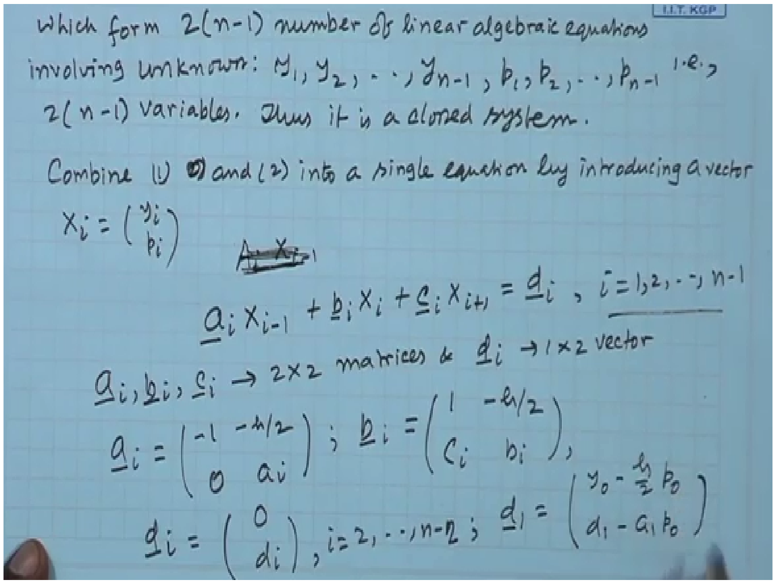

2 into n - 1 number of linear algebraic equation involving how many unknowns. We have involving as y1 because y0 is given see if I put first one is i = 1 so I get y1, y2, up to yn – 1, p1, p2 up to pn – 1. So there is 2 into n - 1 variables, Thus the system is a closed system I can say a compact means number of equations and number of unknown are same. Now what is the advantage? Advantage is that what we find that we have i, i – 1, i + 1, now if I introduce so let us combined 1 and 2 into a single equation.

By introducing a vector so let us introduce a vector as xi = yi pi so if I now combined 1 and 2 so and put in this way Ai xi – 1, Ai already we are defined so I should not try, so let us call ai bar xi matrix - 1 + bi bar xi + ci bar xi + 1 = di bar, where i = 1, 2, …..n – 1, so what is ai bar, so all this ai bar , bi bar underline where i are 2 by 2 matrices, and this d bar i is 1 cross 2 vector.

Some example one can write a bar i, see this is the way we are writing combined in one and two equations to the first row will be the coefficient of yi and pi in the first equation so first equation yi coefficient yi - 1 and pi -1 - h by 2, and -2 and this second row will be the coefficient for y - 1 in equation two which is 0, pi – 1, ai.

Similarly bi bar can be obtained this way, what will be here it is 1 - h by 2 and here it is ci and bi and so on, so this other 2 by 2 matrix and we get a system of 2 by 2 matrix as this form now and di can be expression this way the same way one can write it d1 are equal to so that comes to the boundary conditions if I put a di = i not equal to what so this is 0.

And this is small di this is for i = 2, to n - 2 and d1 = can if i = 0, what is the first equation so p0 is appearing so y0 – h by 2 p0, and the second equation you have the d1 bar, so d1 n1 is – if i = a1 p0, same way one can find out this, now this system of matrix equations which we can put in a bigger matrix of which is referred as a block diagram form a block Matrix can we define so that way.

If I know write so let us as equation number 3 so this algebraic equation three.

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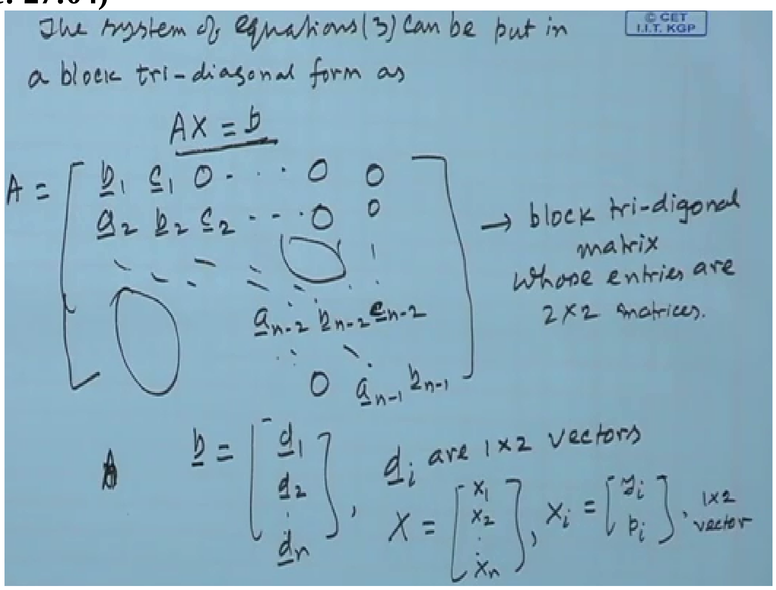

The system of equation, equations 3 can be put array block tri-diagonal form A (x) = say b, where A is with coefficients, were A = b1 bar 1, c bar 1, 0, 0, 0 and a bar 2, b bar 2, c bar 2, 0, 0, 0 and last but one will be a bar n – 2, b bar n – 2, c bar n -2. So this is also I 0, 0, 0, so this one adding and here it is 0, so these all are 0, so this one will be 0 and a bar n – 1, b bar n - 1 so this is the block tri-diagonal matrix this kind of matrix is referred as a block tri-diagonal matrix.

So this a block tri-diagonal matrix, whose entries are 2 by 2 matrices, now this block tri-diagonal matrix is also can be used can be solved through block elimination method quite easily, so some know if I have and b is the vector of vectors. So this is b bar let us call that is the components of b bar are d bar 1, d bar 2, d bar n, so each of this d bar i are 1 cross 2 vectors and x director of unknown x1, x2, xn so each xi are the 1 cross 2 vector.

Yi pi, already we have define so if we can solve this is AX = b, we get X and that gives you the solution for the whole system so that is y1, yi, pi and so on, so there is a procedure through there are several methods for the block elimination procedure, so that will discuss with the subsequent lecture thank you.