Modelling Transport Phenomena of Microparticles

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Lecture-2
Cauchy's Equation of Motion and Navier-Stokes Equations

Hello in the last lecture we discussed briefly about the stress vector and the stress tensor and the corresponding relation. So now we use the applications of that, they are based on which we derive the corresponding governing equations okay. So let us see how to derive the corresponding relation.

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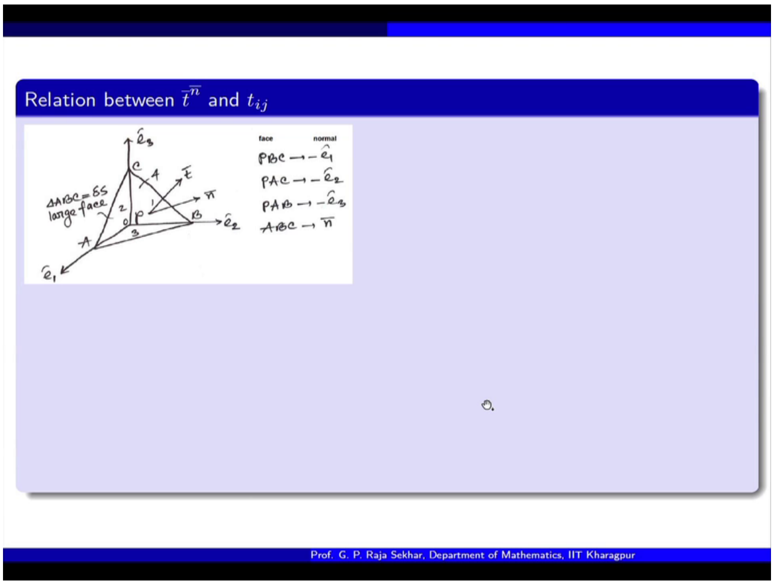

So there are several approaches, the one which I am going to explain is a particular approach, but depending on the methodology and then books different authors use different approaches. So what we are doing is we are considering this kind of a volume element okay, so it is a tetrahedron you can see with a specific property so what is the property if you see, so the face this particular one okay, this point we have we are calling P okay.

So I am referring the face A, B, C so the assumption is this is the largest face okay. A, B, C okay so now if you consider we have the corresponding unit vectors like this e1, e2, and e3 okay. So now let us consider the face P, A, C, this one okay, so if you see P, A, C, the normal is see this is e2, so therefore the normal to P, A, C, is –e2 to right, similarly if you consider P, B, C, this, this one okay.

So the normal is you see e1 this side so therefore normal to the face P, B, C, is naturally –e1 because that is other direction right, so similarly this bottom one that is P, A, B, so normal is –e3, e3 because this is e3, so normal to it is –e3. Now the phase left that is A, B, C so that is it arbitrary direction, so the corresponding normal is n bar okay. Right and the t bar denote the stress which just now we have defined okay.

So now what we are trying to do is we are trying to consider this particular volume element and then try to balance the forces okay. So of general approach as I indicated, so people consider a parallelepiped and tetrahedron. So here we have considered a tetrahedron and trying to balance the forces, so the approach is very standard what is the approach? We try to sum of the forces in a particular direction and then take the limit as the volume element is shrinking to a point okay. So that is what we are doing.

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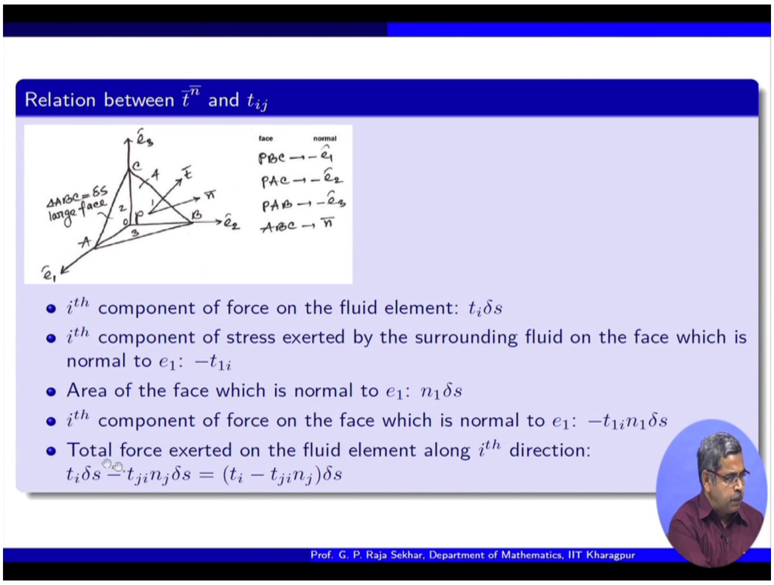

So let us consider ith component of force on the fluid element okay. So since t is a stress ith component is the stress vector multiplied by the corresponding surface okay. Well when we say ith component of the force on the fluid element, what we are this is the larger phase A, B, C, so the total contribution it is an approximate sense the total contribution is coming from only this phase because this is dominating compared to the other 3 okay.

Then ith component of stress exerted by the surrounding fluid on the face which is normal to e1 so what is the phase which is a normal to e1 okay, so that is P, B, C, okay. So what will be the ith component, ith component of the corresponding on this stress okay, this is P, B, C, it is having a normal –e1 that is what we have indicated earlier, so this is the corresponding stress and as you know why this one is one is denoting the surface where you have the normal and i denotes the ith component okay.

Then area of the face which is normal to e1 that means we are talking about area of the face P, B, C, this is just by general geometry normal times the corresponding surface area of this okay. So once we have that then ith component of force on the face which is normal to e1, so this is the stress, this is the area, so this will give you the total force stress along ith direction on a face having normal e1 this is area of that phase and the hence the product will give you the force on that face okay.

So that is what we have obtained now this must be equals to the force acting on that that by virtue of the face A, B, C, using stress vector the ith component of the stress vector multiplied by the area, this is the force acting in terms of the stress vector and in terms of the individual component, we have obtained this. Now we must balance, so the total force exerted we are balancing, so this is the total force okay so before balancing this is the total force we are summing it up, so you must understand this is a stress vector under these are the stress tensor okay.

So this total force when I said we are summing it up okay with the e2, e3, okay. So that is what because on a face having normal e1 and ith direction, now there are two more say is having normal e2 along either direction faced having normal e3 but along ith direction so that will be also t2i, t3i, so these are also included, so therefore total force along either direction is this plus this okay, so that is what we have obtained.

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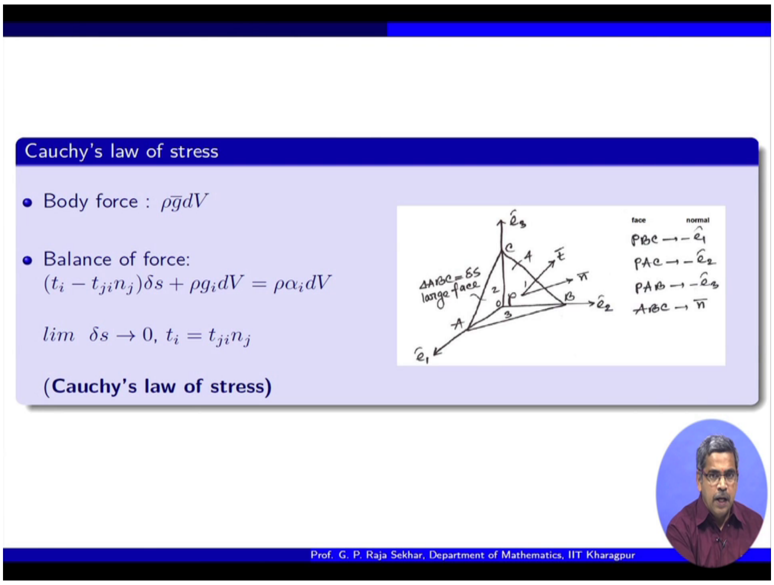

Now what is the, if you consider any volume element, so you have a body force okay. So as you can see the body force is the density and a gravity and the volume element, so total force balance we are doing, so this is the total force on the element along ith direction, then we have added the body force that must be equal to the total external force, if at all there is some acceleration alpha i okay.

So once you have this is the total balance as I indicator what, what is the general technique of deriving we assume that the volume is a shrinking to a point, so therefore we take the limit delta s approaching zero okay. So you might be surprised on this limit how we are getting this okay. What is this delta s, delta s is the surface element so with respect to length scales this is of order say L2, L square and this is volume element this is volume element okay.

Why because the body force is acting on a volume okay, and also this is a actual force due to the fluids momentum okay. This is again volume but stresses add from the surface therefore this surface element, so with respect to length scales this term behaves like L cube this term behaves like L cube, whereas this term behaves like L square okay.

So if this is approaching zero, so these two approach faster to zero therefore we get this equals to this and this is very important law that is a Cauchy's Law of stress this relate this stress vector with the stress tensor okay. So this is very useful relation which will be used time and again for deriving various balance loss okay. So I hope you get the corresponding relation so one can expand for example if you would like to write t1 it is simply j1 and j so now summation on n j.

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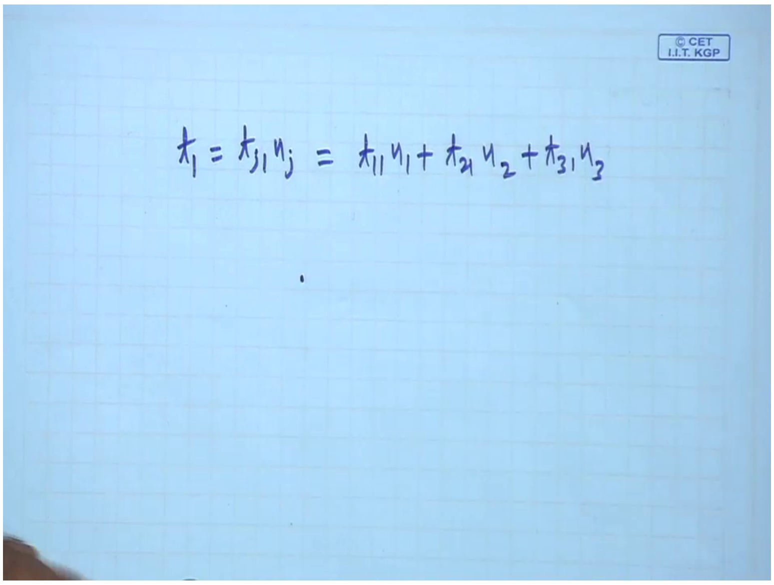

So you can expand like this suppose t1 is nothing but tj1 and j. So now summation on j is going on so therefore t11 n1, t1 n2, t1 n3 okay. So similarly other you can vary the corresponding index and obtain okay.

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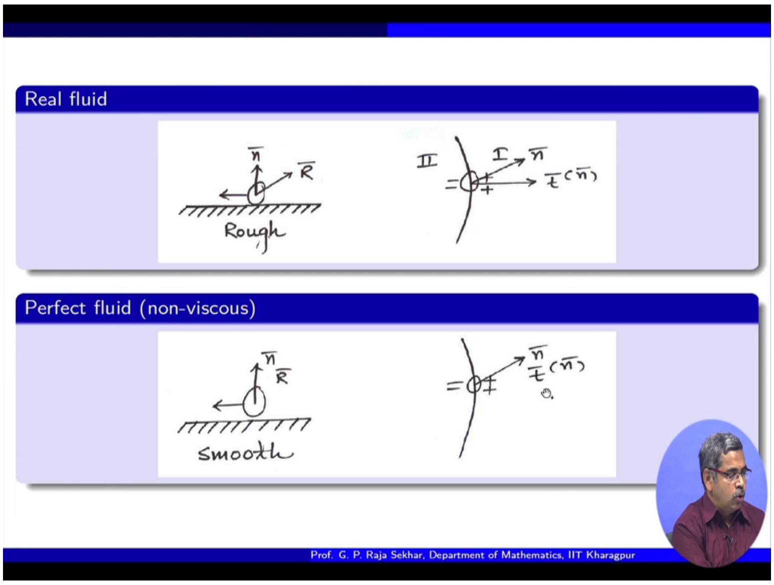

Now we classify depending on the, on the roughness so what is the reaction and then classify the fluids accordingly, so if you take a surface element where it is a rough surface then if the normal is in this direction and let us say you have a fluid on top of it. So corresponding reaction will be in an arbitrary direction that is accordingly it is shown here if this is n bar the corresponding stress factor is in some arbitrary direction okay.

So why is this because the fluid is sorry the surface is rough so correspondingly you expect this is a typically real fluid that is because, so you have a surface and then fluid on top of it so, if it is real fluid so due to the viscous effects so there is a transfer of the viscous effects from the boundary to the top, so due to the surface roughness and then the reaction of the molecules with the surface roughness, you expect it is an arbitrary direction.

But on the other hand suppose it is a perfect fluid that means very smooth, we expect the direction of the normal and the stress tensor both coincide okay. So this is a typically on a hypothetical situation perfect fluid a non-viscous so in case okay.

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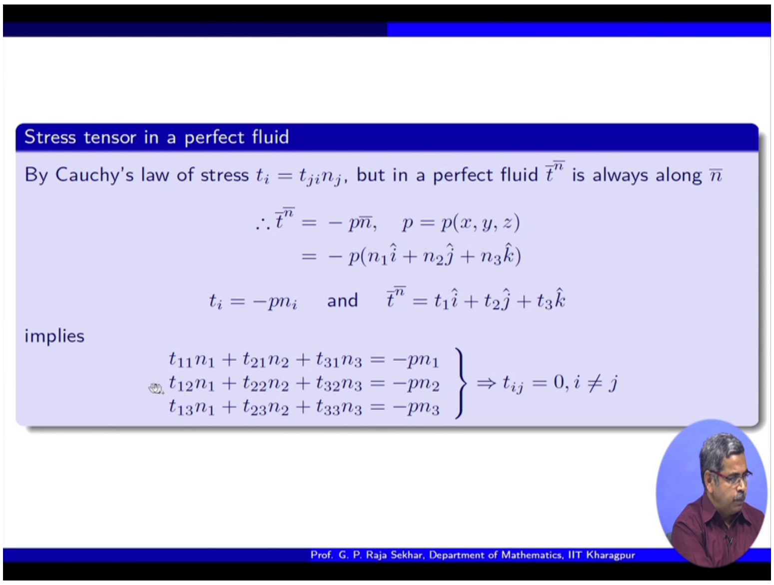

Okay so now let us discuss more about the stress tensor so this is our Cauchy's Law but in perfect fluid the stress vector is always along the direction of normal, just now we have seen perfect fluid okay. When the surface is smooth in a perfect fluid that what is happening, then since stress factor is always along the direction of normal you take the proportionality okay.

As -p where p is function of x, y, z, consider until dimensions now n bar can be resolved like this okay. But t is a stress vector so that can also be resolved like this, so the same thing in shorthand notation okay. Because t is nothing but t1, t2, t3, so therefore from this we can introduce the shorthand notation ti is t_1 is –pn1, t2 is –pn2, t3 is –pn3 and okay.

So from this if you expand t1 in terms of stress tensor this one, but t1 component in terms of p is this, similarly t2 we can expand and the corresponding component is this, t3 can be expanded and the corresponding component is this okay. So this must be satisfied identically but as you can see right hand side you have only n1, n2, n3. So therefore in order to have this identically satisfied what we have is the corresponding off-diagonal entries will be 0 okay.

So when i not equal to j, so tij is 0, therefore what we get is this okay.

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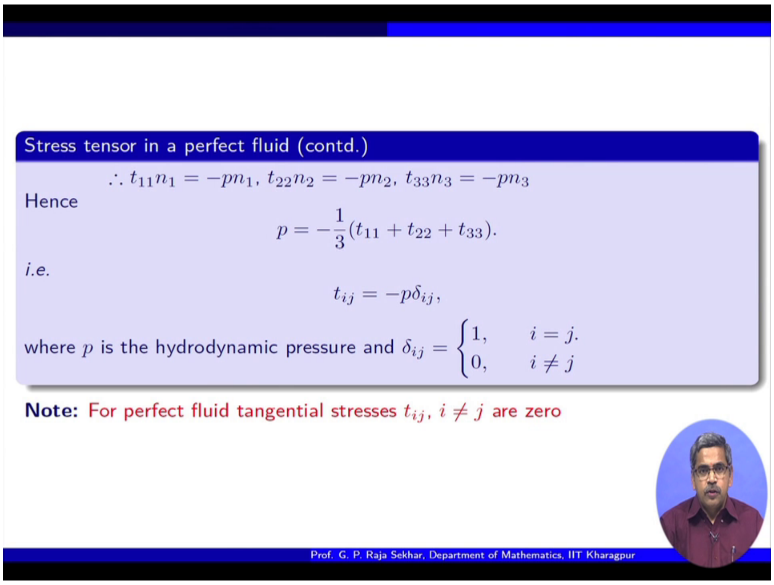

So only the normal corresponding diagonal entries are equal and the one can define p from this so p is nothing but the average of the diagonal entries okay, so that is what we got okay, so therefore for a perfect fluid the corresponding stress tensor is tij is –p delta ij, okay and the p is defined as the hydro dynamic pressure okay.

So in perfect fluid corresponding stress tensor if you consider the off-diagonal entries will be 0 that means there is no shearing effect okay, so that is an important remark.

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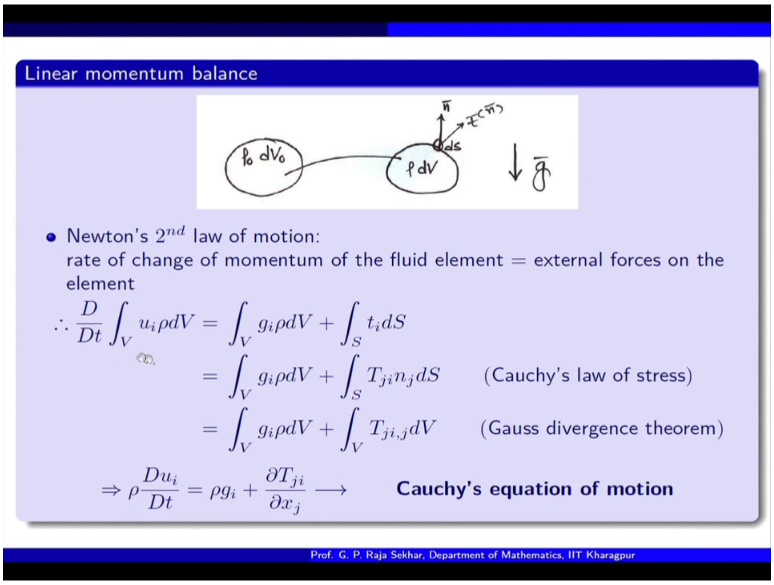

Now let us move on to the Linear Momentum balance which is a most important governing law, so what we do is we consider a fluid element and we are representing it by density Rho 0 dV0, so then now this is a after convecting and this is the corresponding external gravity forces okay.

And this is a small surface element normal and the corresponding stress, so now what is the first basic laws that we have derived is a conservation of mass, so the next important principle is nothing but the balancing linear momentum which is nothing but Newton's second law, so that is what we are doing by virtue of this element.

So what is it says rate of change of momentum of the fluid okay, element is equals to the external force on the element, linear momentum. So with the index notation of momentum is nothing but mass into velocity so Rho dV that is a mass over that element and then this is velocity.

So we would like to consider rate of change therefore we have taken the material derivative that is equals to the external forces acting which are due to gravity, let us say or it could be anything g bar need not the gravity you can have some electric field or etc., magnetic field and surface forces so when you are balancing, what are the total forces? You have body forces and then now you have corresponding external forces okay.

And then now surface forces so typically what contributes to the surface force, the stresses contribute to the surface force, so that is what so this is a the external force due to g and these are the surface forces, but we have Cauchy's law of stress so we are using that ti is tij. So it is a I have already introduced a small tij and a capital Tij, generally we misuse this time and again we can misuse okay and people use it okay you can consider both are the same, so using Cauchy's law we get this then we use a Divergence theorem to transfer the surface integral to volume integral.

So the notation here this is a stress tensor gamma j means this is exactly nothing but the derivative.

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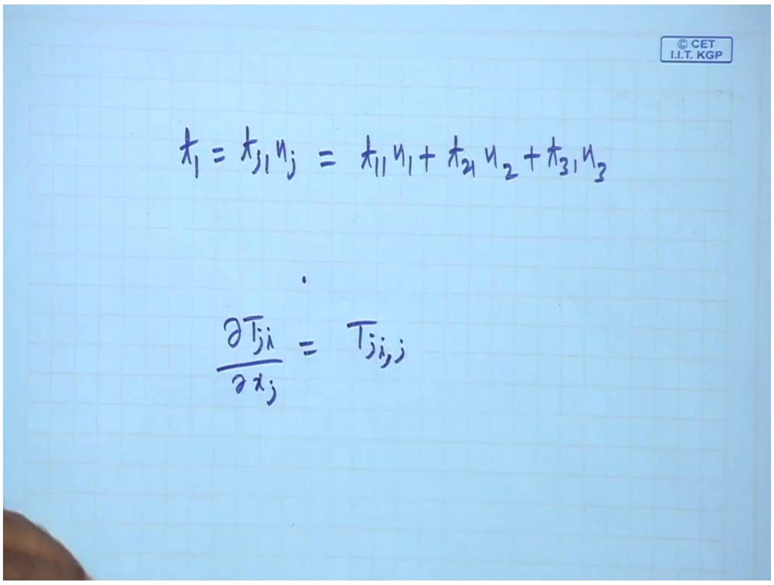

So Tji by xj so this is a shorthand notation, we use okay so this is a shorthand notation we are using fine, so now we brought all the quantities under the same volume element therefore what we get is, so since we are in the material frame, we can take it inside so this happens for any arbitrary volume element, so you make it equal to 0 so Rho Dui by Dt okay, so for this you have to use the identity Rho F bar identity that is very helpful here okay.

If you see the corollary where we have defined as a, as a corollary of Reynolds transport theorem we have considered F is equal to Rho F okay, so you consider the corresponding quantity here Rho F, so that corollary we have used and hence we get this okay, and this is this term and this is this term, so this is nothing but are the corresponding linear momentum balance okay, so this is very much important for us because of any fluid flow of first thing is we try to go for the corresponding linear momentum balance and then try to proceed further okay.

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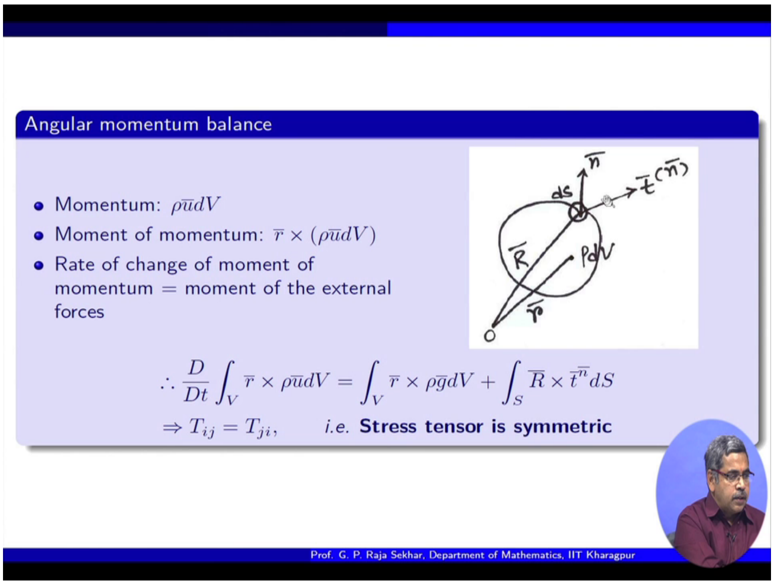

Fine so next is naturally angular momentum balance okay, this we would not discuss the complete derivation, I give an idea okay so let us consider a fluid element and this is a surface element so any interior point the position vector is denoted by our r bar any point on the surface the corresponding position vector is capital R bar okay, and again this is a normal and this is the corresponding stress vector so now what is the angular momentum? Angular momentum is you have to take r bar cross MV okay.

So that is the angular momentum, so now this is momentum then moment of momentum that is r bar cross the momentum, then what is our conservation principle tells rate of change of moment of momentum equals to moment of the external forces okay. Please do not be confused, so let me explain rate of change of momentum okay, moment of momentum equals to moment of the external forces okay.

So you consider the momentum then you consider the rate of change of this, this must be equals to moment of the external forces that means you take let us say F bar is a external force so you take the corresponding okay that is r bar cross, so that is what we are doing. So rate of change of the momentum and since we are having external forces that is g bar so we are taking and why we are taking r bar which is on the interior because this is the body force which acts on the entire fluid element, therefore we are considering this.

Whereas here we are taking capital R bar which is nothing but the position vector corresponding to the surface the reason is straightforward. Because the stress vector this is acts on the surface, so this is a surface force therefore the corresponding moment should be taken with respect to the surface position vector, whereas this moment should be taken with respect to the any fluid in interior okay.

Now similar arguments only thing is a cross involved, so therefore we use Divergence theorem here but earlier we have used on this straight forward, but now we have a cross so one can use Divergence theorem. Let us say you put these two into summation notation then use Divergence theorem okay, so then this also put it in summation notation and then use the corollary that we have then if you bring it everything to volume element.

And balance what you get is the stress tensor is symmetric, so that is that net result will be obtaining okay. So this final calculations can be done so I am are leaving it as an exercise, again this is not very difficult as I indicator just put it everything in summation notation and do as before whatever we have done you will be able to get it okay fine.

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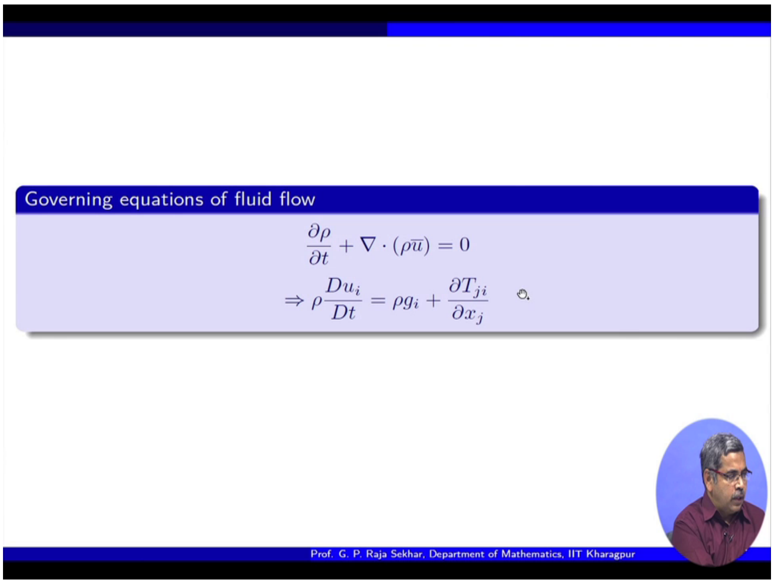

So now we are ready to define the two fundamental equations of fluid flow, first one is conservation of mass and the second one is linear momentum balance okay. So the first one is as we indicated, so this is a conservation of mass under this is a linear momentum balance, where u denotes velocity Rho denotes the density and a tji denotes the stress tensor okay.

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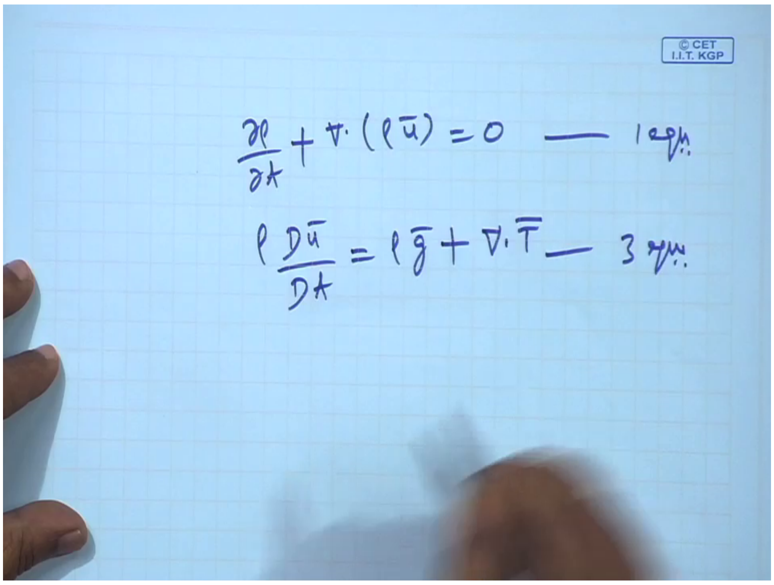

Okay now at this stage if you see, if you balance the number of equations and unknowns, so what we have is so I am just repeatedly writing this, so this is one equation and then in vector form if you write, let us say this is Rho g bar + divergence of T okay. So this is one equation and these are 3 equations each has component form okay. But then how many unknowns we have, we have u 3 unknowns and then stress tensor okay.

So we have nine of them, but if it is the symmetry if you use symmetry six of them, so we have only four equations. That means there are more unknowns okay, sorry there are more unknowns yes than the equations, so that means still there is a lack of information okay. So still we have to reduce, right so then who will do this job? So this is very hypothetical.

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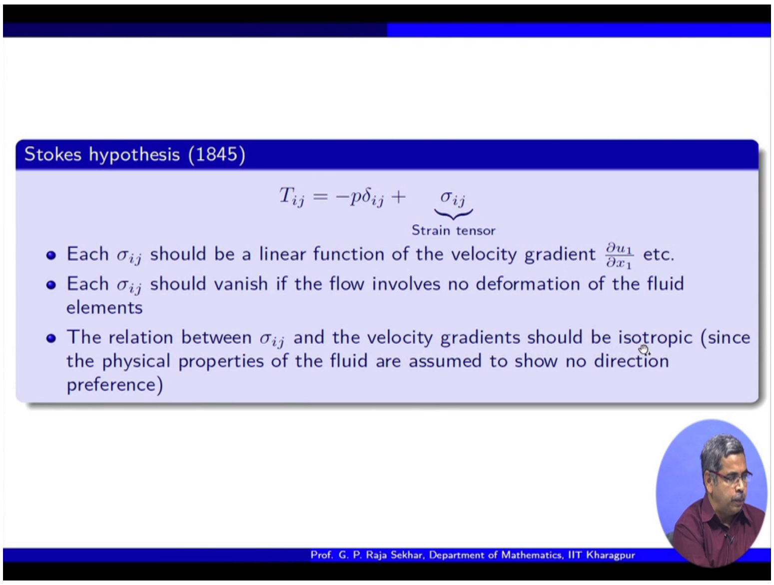

This is Stokes hypothesis, so Stokes has postulated that the stress tensor tij can be decomposed as this is the along the diagonal entries, and then something called strain tensor, then what is the structure of this strain tensor? So that is a question okay, so to start with the structure is it should depend on the, see stress tensor is resolved into principle that is along the diagonal so then the off diagonal, so these are nothing but they indicate the deformation okay.

So correspondingly some structure is put in okay, so what is that each sigma ij should be a linear function of the velocity gradient okay, so that is the shearing linear function of the shearing and each sigma ij should vanish, if the flow involves no deformation okay. If there is no deformation then this 0 and the relation between sigma ij and velocity gradient should be isotropic, that means we are not attaching any directional dependency okay.

So the isotropy is so this is our initial hypothesis by Stokes, so then what happens it appeared that the number of unknowns are reduced, but still these are still unknowns and these are off diagonal entries okay. So in place of 3 diagonal entries we got one unknown, that means a reduction of two unknowns, but still we have off diagonal entries right, so still if you balance the number of unknowns will be more okay, so now what is the corresponding remedy okay.

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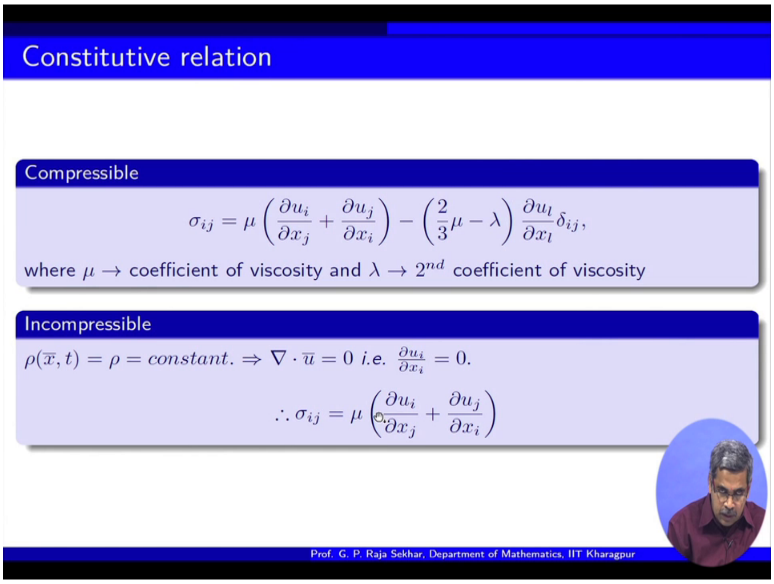

So again it is proposed to a constitutive relation this is popularly known as constitutive relation, what is that the corresponding strain tensor sigma ij, is related to the velocities as follows okay. So this is a first coefficient of viscosity typically we call it a dynamic coefficient of viscosity and this is a second coefficient of viscosity okay.

So this is the hypothesis okay, and this is for compressible and if it is incompressible you can see this is nothing but dow ui by dow xi kind of, because it is a dow ul by dow xl is a dummy index this is nothing but divergence of u0. If it is incompressible we have divergence of u0, so this goes to 0, so if it is incompressible we have this correspondingly as I already mentioned this goes to 0, so therefore sigma ij reduces to this okay.

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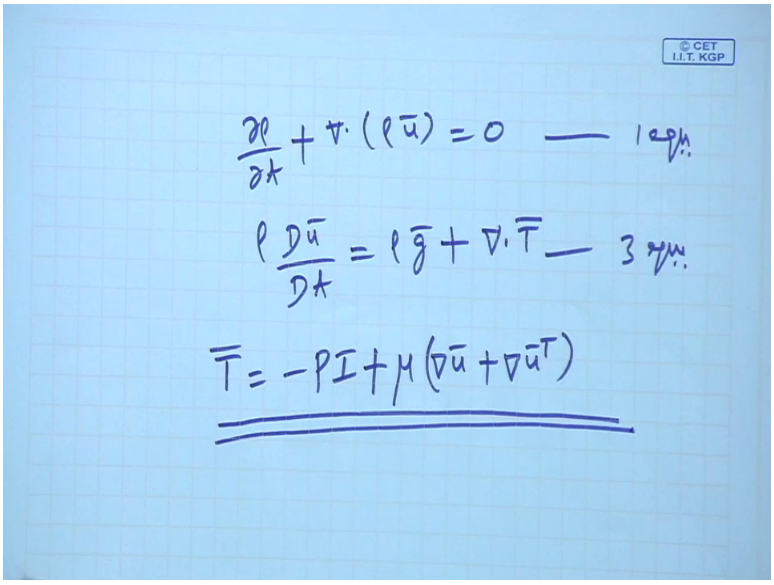

So once we have this what will be the corresponding stress T bar is -P I + Mu, so I am just writing some notation, which is not there on the slides okay. So this is nothing dow ui by dow xi this dow uxj by dow xi okay. So this is exactly we get it done, that is what we have this is a corresponding strain part and you have a normal part p so that is what exactly I have written here okay.

Right so now with this if you balance the equations, you will see the number of equations and the number of unknowns, will be matching how you have velocity vector 3 components, so therefore 3 velocity equations okay, and then you have one conservation of mass that is one equation four equations and how many unknowns? Pressure and 3 components of velocity your four unknowns so the system is closed okay.

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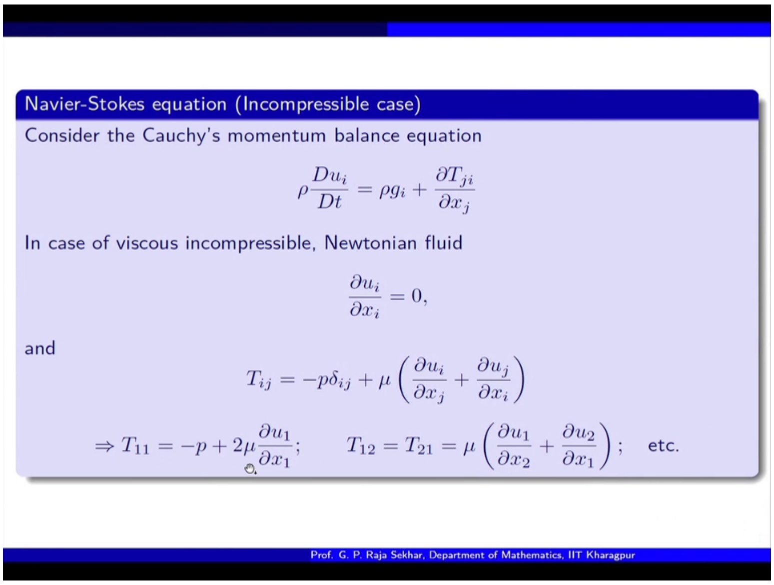

Now there are some limiting cases as I already indicated if it is incompressible okay. So we have a general case and then if it is incompressible limiting this is the what already I have explained summation notation the corresponding vector form just now I have written okay. So if somebody is interested in writing explicitly you can see p contributes to only the diagonal entries so as you can see t11 contains p whereas t12 and t21 do not contain p, similarly if somebody writes t22 you will get definitely p and then the corresponding strain okay.

So this indicates the strain so for example dow u1 by dow x2 indicates the baseline deformation okay. So that is a limiting case okay.

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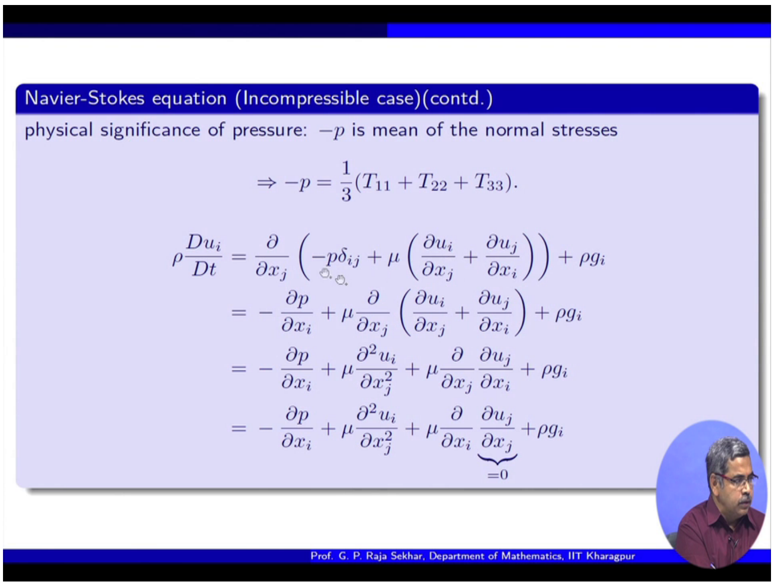

So this is the mean pressure then in the corresponding equation we take the divergence and then now substitute the incompressible stress tensor okay. And this is the body force and little bit of algebra one can very easily verify it is very straightforward, we have not done any magic simple operation of this into each of them.

Whenever xj is equal to xj, we get a double derivative with the repeated index so that is the xj square whereas here there is a difference of index so that is this term okay, but this is a commutative so you bring it inside and the dow xuj by dow xj is nothing but the incompressibility condition this is divergence of u0.

So therefore this vanishes and what we get is the corresponding Navier Stokes equation for incompressible. Typically this is used so the same in vector form.

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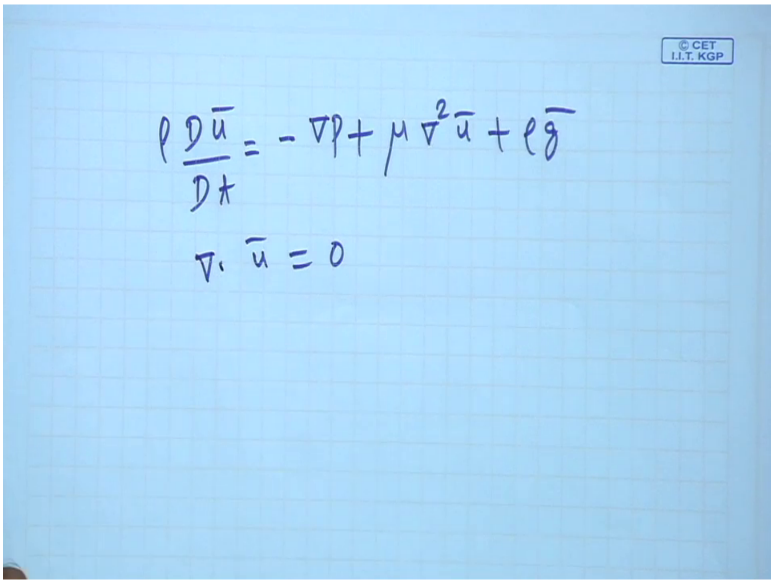

This is equals - Grad p + Mu Del square u + Rho g and this conservation of mass okay. So this is a typical incompressible Navier-stokes equation, so probably we have given okay.

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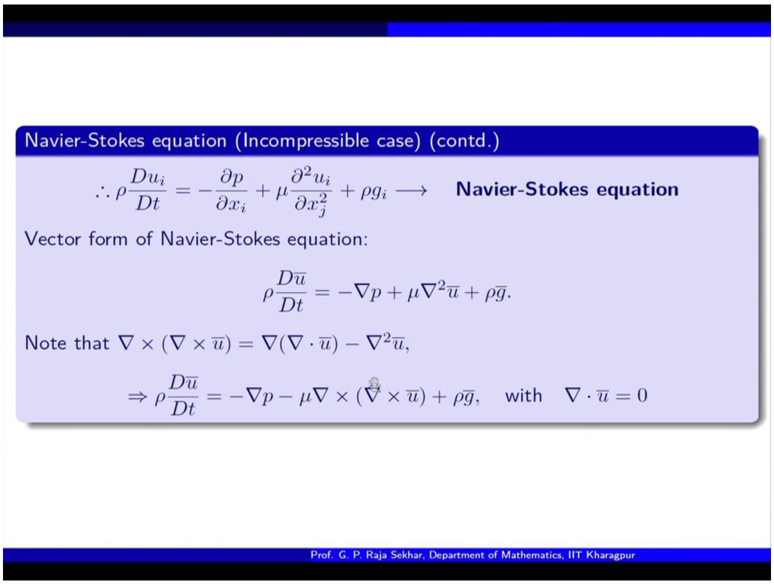

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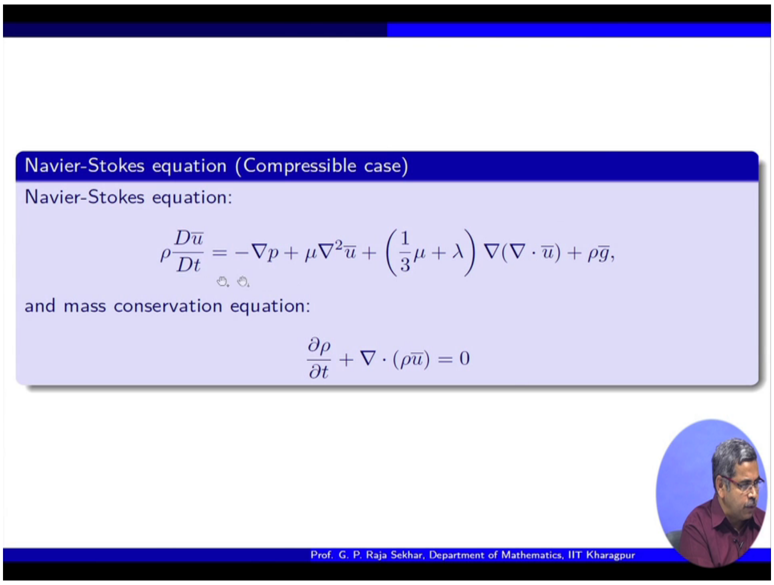

Let me compressible case this one so incompressible this is 0 so that what just now I have written okay, so these are just to give you some idea about the various limiting cases okay. So that is not a big deal once you get the complete equations you can always get the limiting cases okay.

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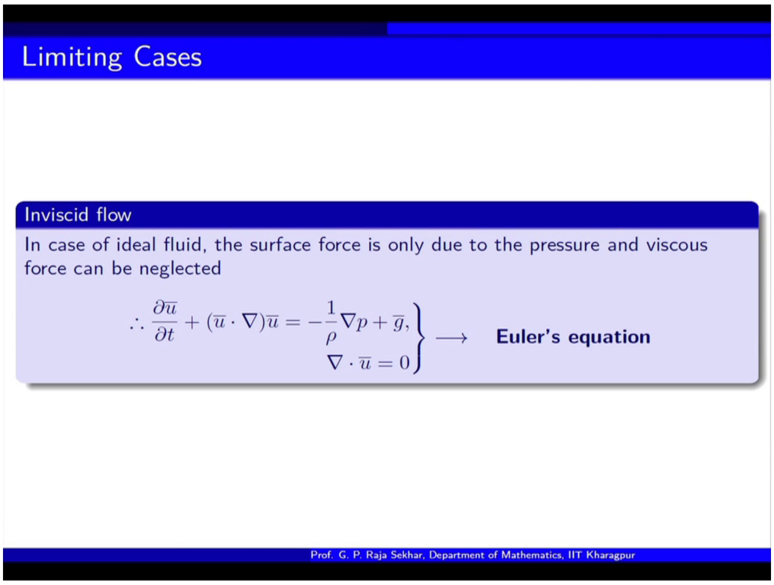

So then what is the other limiting case there will be competition across the forces so what are the forces available? So you have left hand side you have their corresponding the convective derivative okay. And then right hand side you have the pressure forces viscous forces and body forces okay.

If you see the convective derivative so some of the forces are, if you expand this.

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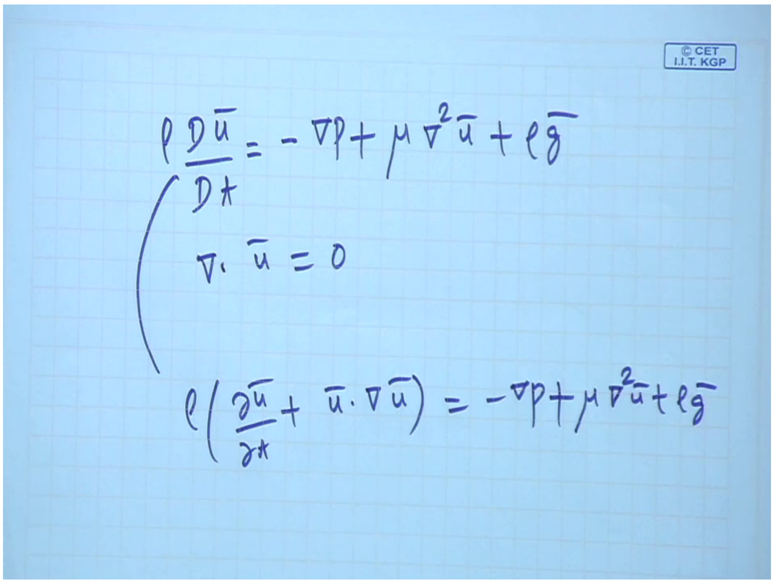

What you will have is the local derivative plus the convert to derivative okay, right now if you assume that the viscous force are neglected then you get this, if the viscous force are neglected this, viscous force are neglected and these equations are called Euler's equation you can see there is no viscous forces okay.

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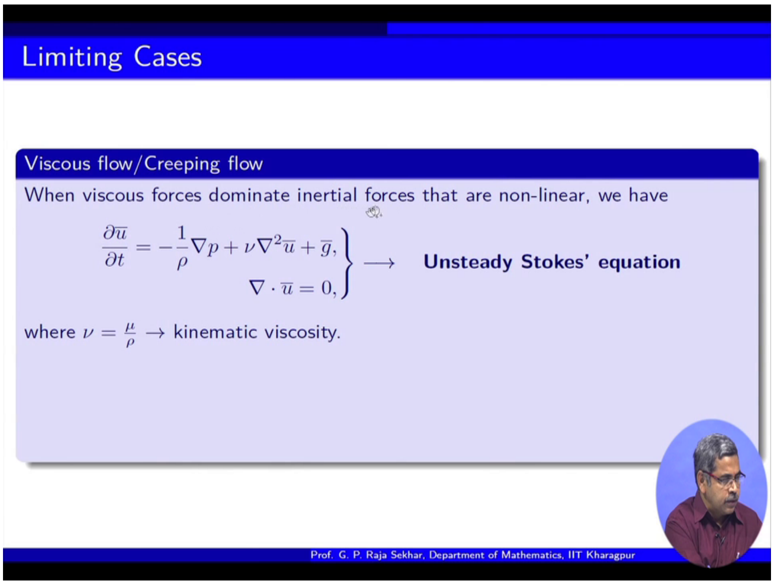

In this similarly if you neglect inertial forces under the assumption that viscous forces dominate inertial forces then you get a linear equation this is unsteady stokes equation okay. So these are some limiting cases ofcourse we will pay much closer attention to the competition between viscous and inertial forces, when we discuss about the non-dimensionalization okay.

So otherwise once you have the Navier-stokes equation one can obtain various limiting cases, suppose the first case we have considered is we have considered that viscous force of neglected, second case we have considered that inertial force of neglected and get the corresponding limiting cases, so with this I hope you get an idea about the role of stress tensor and then how that is useful in deriving the corresponding linear momentum balance thank you