Modeling Transport Phenomena of Microparticles
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Lecture – 17
Flow Through Porous Media - Elementary Geometries

Hello! In the last lecture we discussed introduction about flow through porous media. In particular we discussed some basic models as downing equations. The first model that we discussed is a Darcy equation which indicates that the volume flux is proportional to the pressure gradient. And in the second we discussed about Brinkman equation so where you have a viscous terms and then we discussed about various possible interface conditions.

And if you recall the exact set of interface conditions at a porous liquid interface is really not settled yet. So there are popular set of boundary conditions and then these are well accepted in the literature and it is used. So today we are going to see some simple configurations like, in case of a clear flow channel already we discussed flow between two parallel plates and flow inside a tube etc.

So similar models however in this case we are going to see with some porous packing's okay. So let us look at it.

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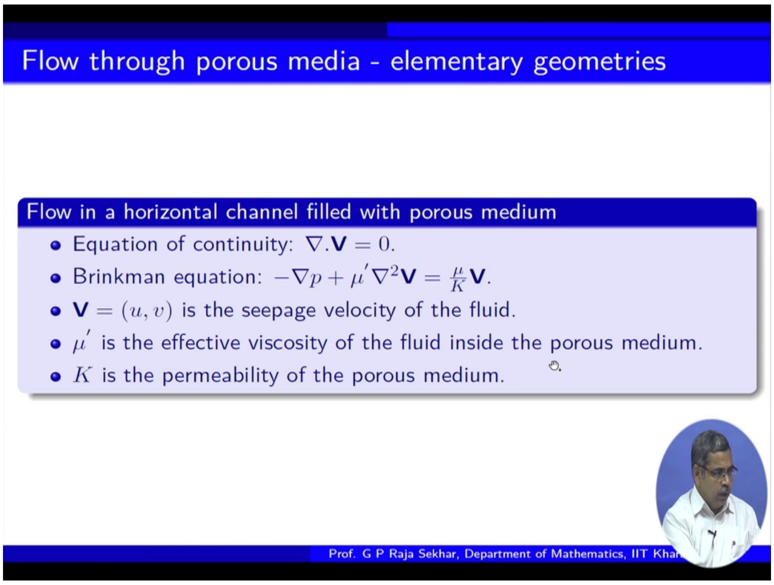

So the first case is a flow in a horizontal channel filled with porous medium. So as usual we are considering the steady viscous incompressible flow. So this is the corresponding equation of continuity and then we are going to discuss why assuming the Brinkman equation for the flow inside the porous media. So this is the pressure force and this is the viscous force and this is a corresponding damping by virtue of the porous obstructions.

So we have discussed a bit about this. This Mu prime is called effective viscosity, which is in general is supposed to be different from the dynamic viscosity Mu. But as I mentioned various studies consider many of the time these two are equal. But there are some correlations we will discuss about that. So K is the permeability okay. So then here for a generic set up in two dimensions so this is the seepage velocity of the fluid okay.

And this already I have indicated and K is the permeability of the porous medium okay.

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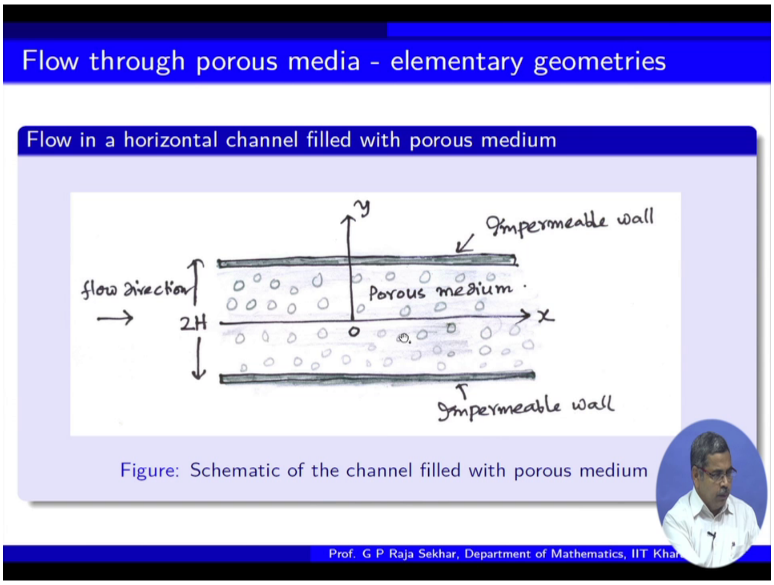

So let us consider the configuration. So the first problem that we are going to discuss is flow bounded by two plates and we have a porous packing and we are discussing flow inside. And as you can see from the configuration both upper and the lower plates are impermeable. So naturally we expect no slip condition there okay. So with this configuration if you recall for clear flow we have considered unidirectional case and then reduce the corresponding downing equation.

So the same we are going to do here okay.

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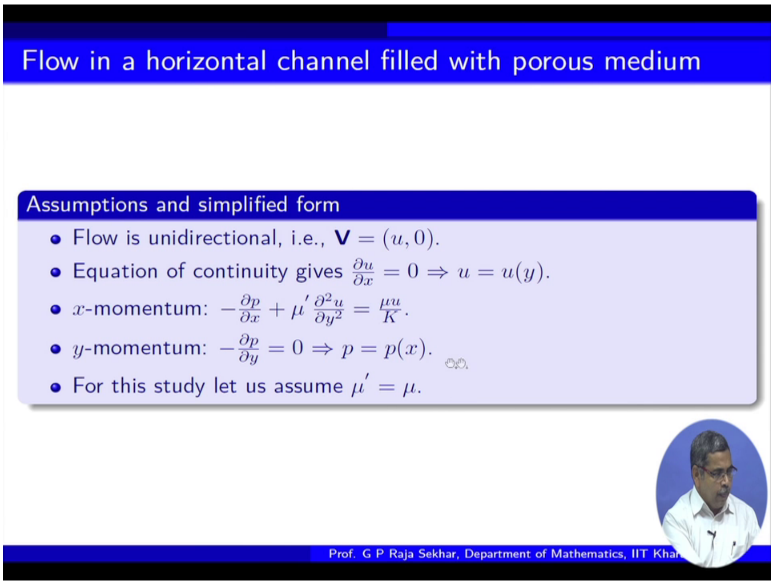

So first assumption is flow is unidirectional so correspondingly we have u comma 0. Then once we have this from equation of continuity we get u is function of y alone okay. So then the corresponding x momentum reduces to this because Dow u Dow x is 0. From the Laplacian term, we get only one of the terms other term vanishes okay. So then the corresponding y momentum since V is 0 we get the net inference is, p is function of x alone okay.

So this is a completely analogous to the flow inside parallel plates in case of clear flow except that we have an additional term which is a due to the resistance offered by the porous packing okay. So naturally if somebody takes K goes to infinity that means it is a large permeability so this vanishes and the equation reduces to clear flow stokes equation okay. So with this we are assuming for a simplicity Mu-dash = Mu that is effective viscosity equals to the viscosity okay.

So now with this assumption let us try for the solution.

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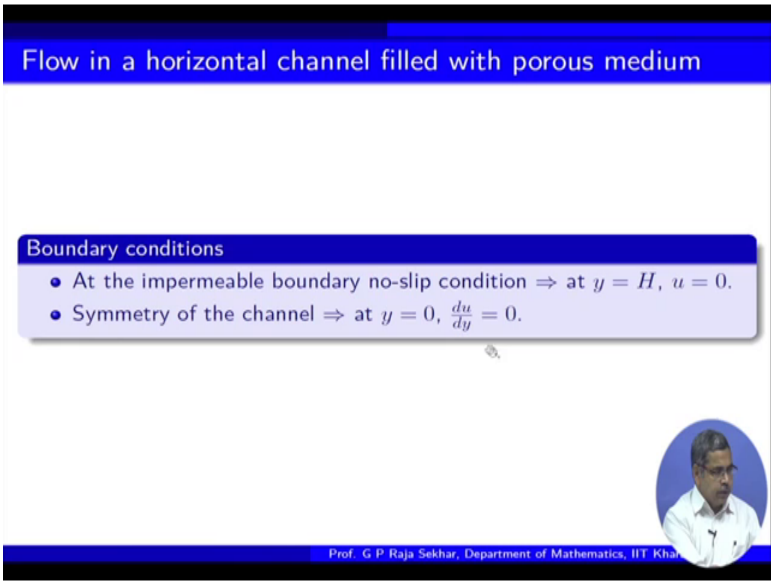

Before we compute the solution we set up the boundary value problem by fixing the boundary conditions. So at the upper plate the no slip condition u is 0. And then so if you see the configuration what we have is say this is -H 2H because this gap is of size 2H. So therefore what we are considering is either we can fix at y = -H no slip and then y = H no slip and then solve the complete domain or consider either upper portion or lower portion and then use the symmetry condition.

So in this case so we are using the symmetry condition with respect to y = 0 and then solve the problem in the upper portion. So correspondingly, you can see the boundary conditions. So this is the no slip on the upper plate and this is the symmetry condition at y = 0. So du/dy is 0 okay. So this sets the boundary value problem.

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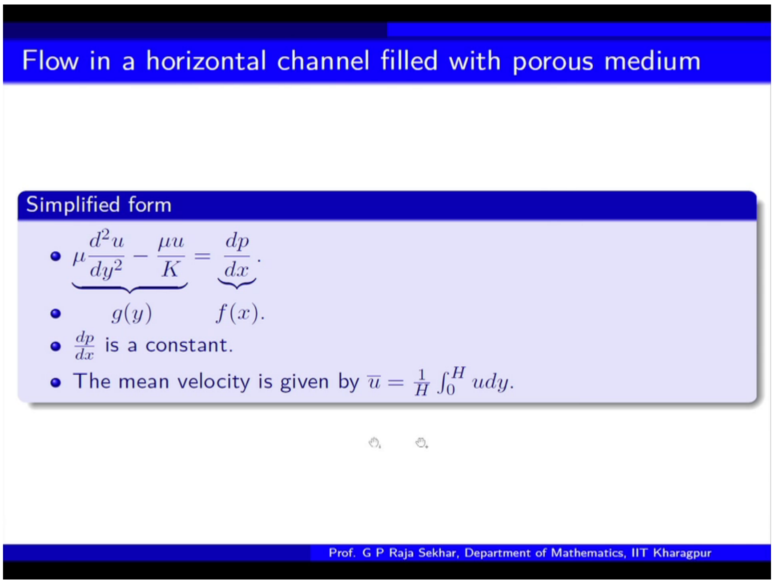

Now if you consider since ux is 0, so this is the total derivative and since u is function of y. So the left hand side is completely function of y and then since we have py is 0, that is the Dow p Dow y is 0, p is function of x that is the inference we have. So, therefore, this is function of x. So each must be constant. So hence, dp/dx is constant. So this is again exactly analogous to the clear flow scenario.

Repeatedly I am telling, so you have the damping force due to the porous packing and then naturally when we non-dimensionalize we expect some parameter controlling the permeability structure and then we expect some limiting scenarios okay. So the mean velocity in this case is given by so we are integrating the velocity u across the domain and normalizing by the corresponding scaling okay. So this is the mean velocity.

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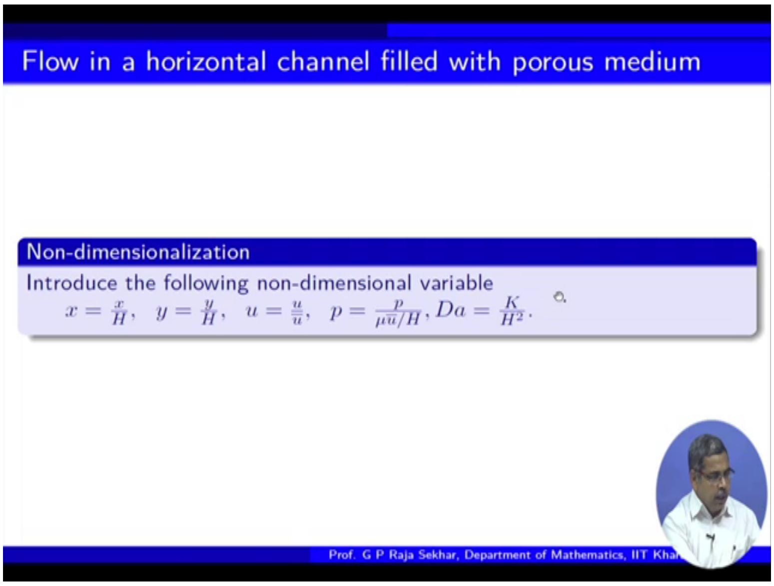

Now let us introduce non-dimensionalization. So since we have only one length scale involved so that is the spacing between the plates. So this is x and y both are normalized by H and then this is some characteristic velocity. So u bar in this case is mean velocity and pressure. So the corresponding non-dimensionalization is already been seen in case of clear flow. So one parameter that is additional here is not seen yet.

So if you see in the last lecture we indicated that the permeability K has dimensions of length square and then if you see here we are normalizing by length square hence, this particular parameter which is denoted by Da is supposed to be non dimensional parameter and it is called a Darcy number. So let us see the non-dimensionalization and how we can get this Darcy number in our governing equation.

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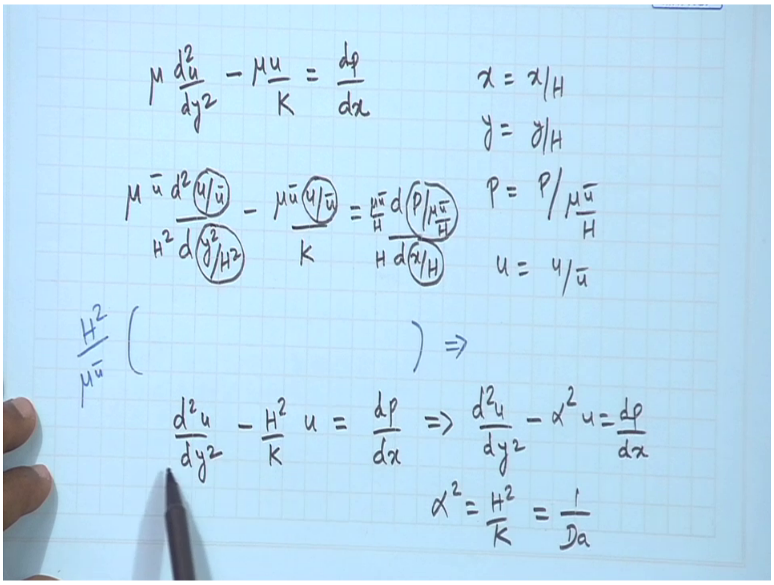

So let us consider the governing equation so which is Mu d square u by dy square equals dpdx. And we have the non-dimetionalization given by x is x by H y is y by H and the P is P by Mu u by H okay. Of course u is u by u okay. So I mean I am not using primes so just it is indicative. So these are the non-dimensional. So now let us non-dimensionalize so we have μ so now we supply u by so therefore there will be u bar.

Then so we get H square there okay. Then k we keeping as it is, then p by H. So here we get then okay. So now we try to simplify okay. So this is non dimensional velocity, so non dimensional quantities okay. So with this if you simplify what the common factor is let us say you normalize by H square by Mu u. So that means we are multiplying by H square by Mu u throughout, so this equation if you multiply throughout what we get is the following.

So here this will be one so therefore we get. So this is normalized so I am not so this is normalized quantity therefore I am just. We could have written primes but I am just as a notation just we have dropped the prime. So now look at this quantity so you have Mu u bar by K and we are multiplying by H square by μ u bar. So Mu u bar get cancelled and then we get H square by K u equals. And here also we have exactly Mu u bar by H square.

So you get so this can be written as, you give a notation some alpha square u okay. So where alpha square is H square by K. Still you might be wondering where is the corresponding the Darcy number that we have defined. This is nothing but one over Da okay. So the non-dimentionalization of the corresponding governing equation introduces a non-dimensional parameter okay which we are defining it as a Darcy number or in this case one over Darcy number okay.

So if you see K goes to infinity so that will be producing stokes equation okay, large permeability case right. So this is limiting. So now let us see that is what we have defined here.

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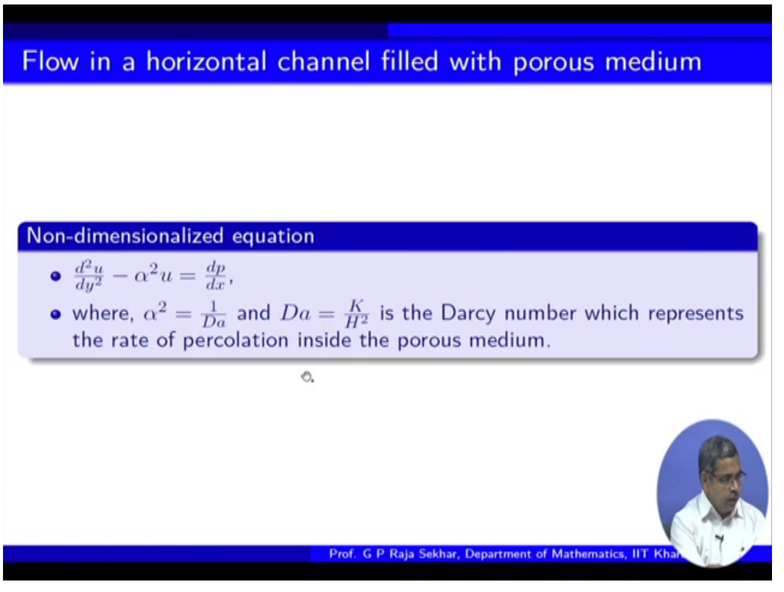

So the non dimensional equation just now we have derived and alpha square is one over Da where Da is K by H square. This is the Darcy number and this represents the percolation inside the porous medium because this is a proportional to the permeability. So large Darcy number means large permeability small Darcy number means small permeability okay. So we have a non dimensional equation so now one can get the general solution and employ the corresponding interface conditions okay.

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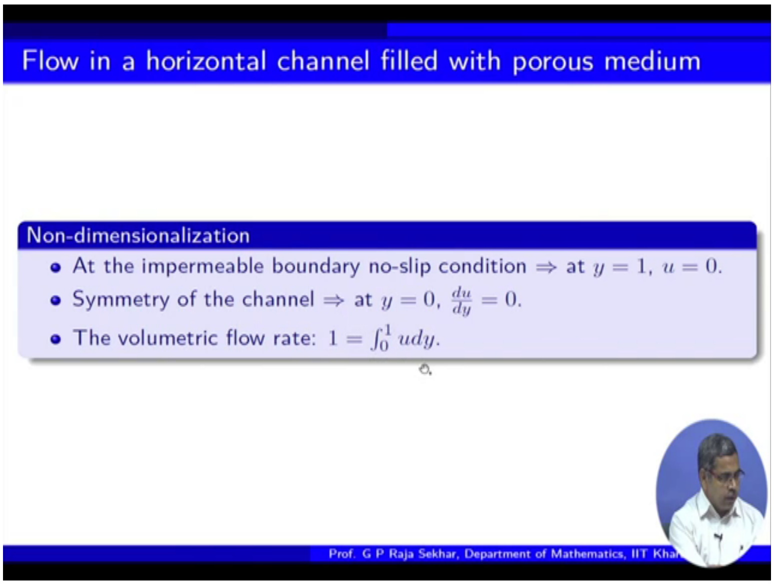

So this is a non dimensionalized boundary condition. So here there is no interface you have a boundary and then this is the symmetry condition and the normalized volumetric flow rate is given by this okay. So you will see how to use this volumetric flow rate and then even while interpreting a physical some physical insights we come across the volumetric flow rate balance okay.

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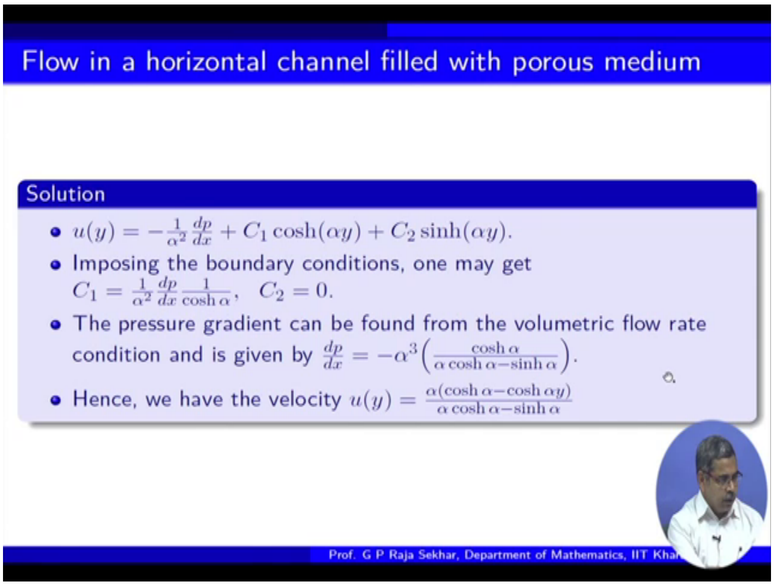

So it is very straight forward to write down the general solution because we have dp dx is constant. So one can get the corresponding solution in this form where C1 C2 are constants to be determined using the boundary condition. Now imposing the boundary condition we get these constants and once we determine u with the help of these constants this can be utilized to estimate the pressure gradient which is given by this okay.

So total volumetric flow rate balance is used to estimate the pressure gradient okay. So now we have the solution so we would like to do some analysis okay.

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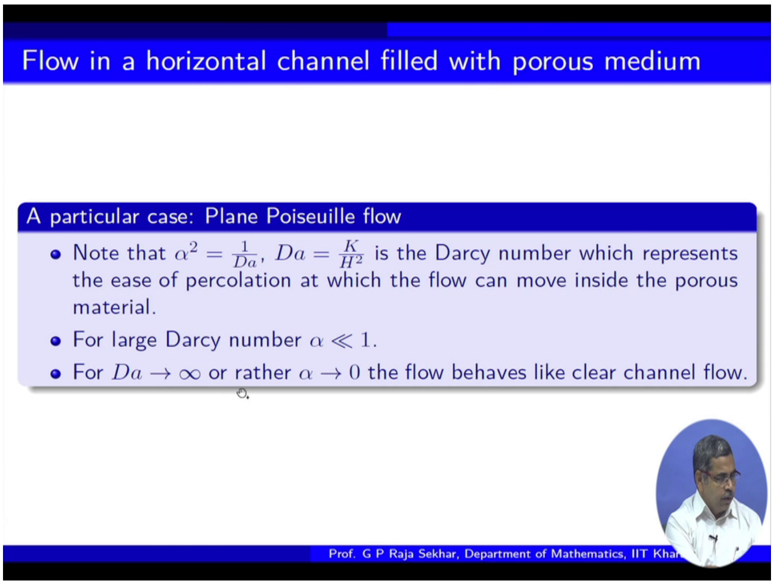

So the compact form of the velocity can be represented like this right. So already I have indicated you have a porous packing. So if permeability is large we expect a clear flow scenario. So if a permeability is small so then now no flow right. So permeability large case we would like to compare our results with a plain Poiseuille flow okay. So let us see so this is the Darcy number. So for large Darcy number alpha is much less than 1.

So these two are equivalent like the asymptotic case of Da goes to infinity or alpha goes to zero. So the flow behaves like a clear flow channel. So this is a one can verify.

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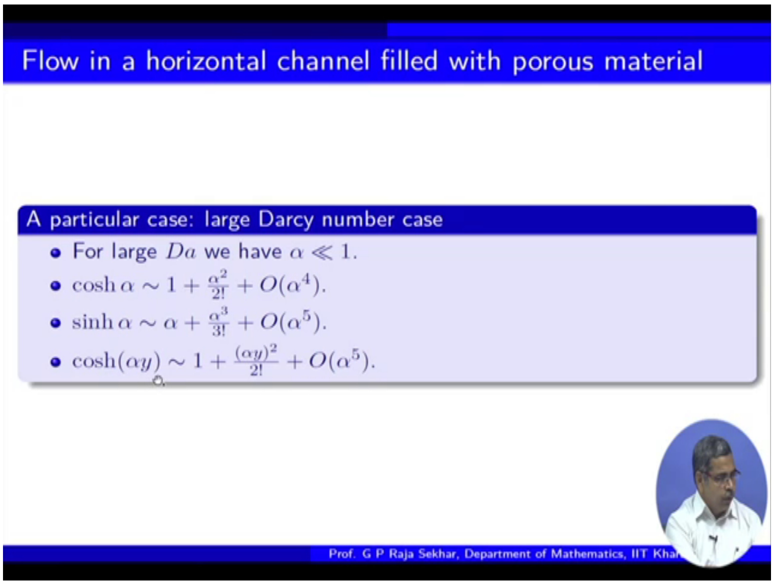

So let us see so the solution contains some hyperbolic functions. So for large Da alpha is much less than 1. So we can do some asymptotic analysis and consider this approximation. So these are the hyperbolic functions involved in the velocity field okay.

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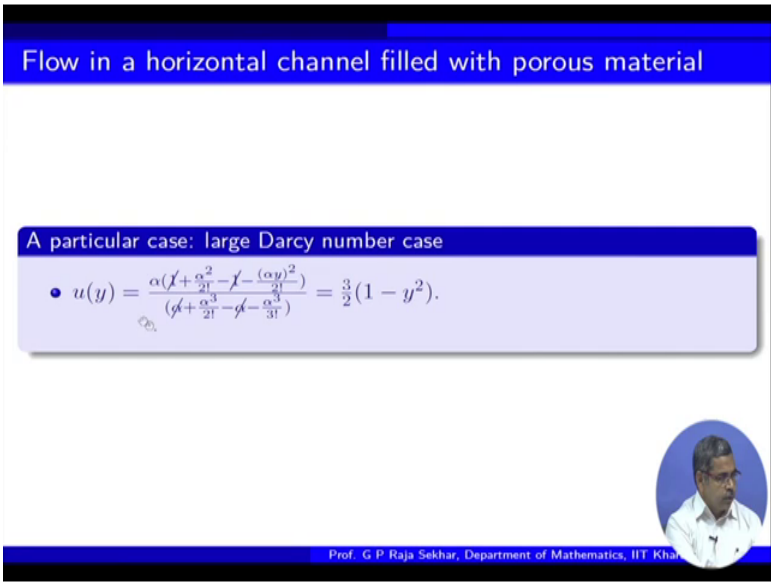

So once we have these approximations you substitute and consider the corresponding limiting case what you get is alpha is much less than 1 right. So if you take corresponding limiting case of a alpha going to 0, So what we get is exactly the plane Poiseuille solution okay. So large Darcy number the solution gives this okay. So that is limiting case one can verify very easily. So this is as I indicated this is the plane Poiseuille flow okay.

So now once we have this asymptotic agreement so we would like to see some results.

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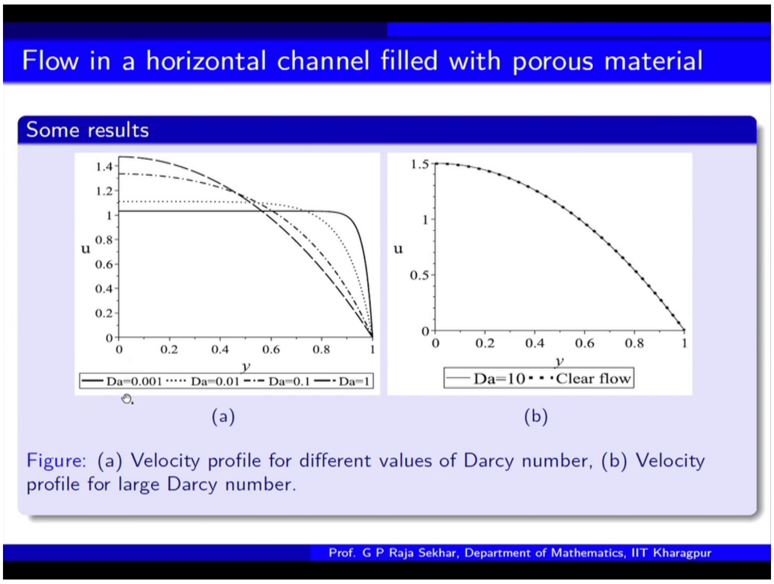

So the solution depends on the because it is non-dimensionalized so the only parameter that controls is the Darcy number okay. So you will see the velocity with the y coordinate. So here you see these legends indicate the Darcy number so this is a small that means this represents a low permeability and this represents a high permeability. So if the permeability is less than what you would see is the resistance is more so the velocity will be less.

So you can see the velocities less and moreover you have the corresponding plate is here because y = 1. So here the interface viscous effects are more so the velocity is low here compared to the centre of the channel okay. So this is only one portion we are showing okay right. So if you increase the Darcy number so then you expect more flow so you see the velocity is increasing then towards the boundary it is meeting the no slip.

Further you increase so then the velocity at the centre it is increasing okay. And further if you increase so this is almost you are reaching the fully developed clear flow channel okay. But you can analyze a little bit more you see here for a particular Darcy number suppose you consider a particular Darcy number so the flow behaviour is the flow velocity is more towards the centre and less towards the boundary.

But so compared to particular Darcy numbers let us say these two so this and this. So towards the centre if the Darcy number is less towards the centre velocities is less and if the Darcy number is more towards the centre velocity is more. But towards the boundary there is a crossover and for low Darcy number it is more okay. So this is because total volume flux is balanced. So whatever amount is low here that is being adjusted here.

So this portion plus this portion is conserved okay. So that can be seen very clearly and you can see one itself is onset of agreement with the fully developed case. Now further if you take large Darcy number and then you superimpose the clear flow solution so they have complete agreement okay. So Da 1 onwards it is almost converging to the clear flow solution and the once it is a Da 10 you have complete clear flow channel flow solution okay.

So this is a so this case is a, this is a low Darcy number that is a hyper, means you can now so hypo-hyper either way you can hyper porous and hypo porous you can define okay. So that is like a highly porous and then low porous you can define okay. So this is the agreement with the clear flow channel for the parallel plate.

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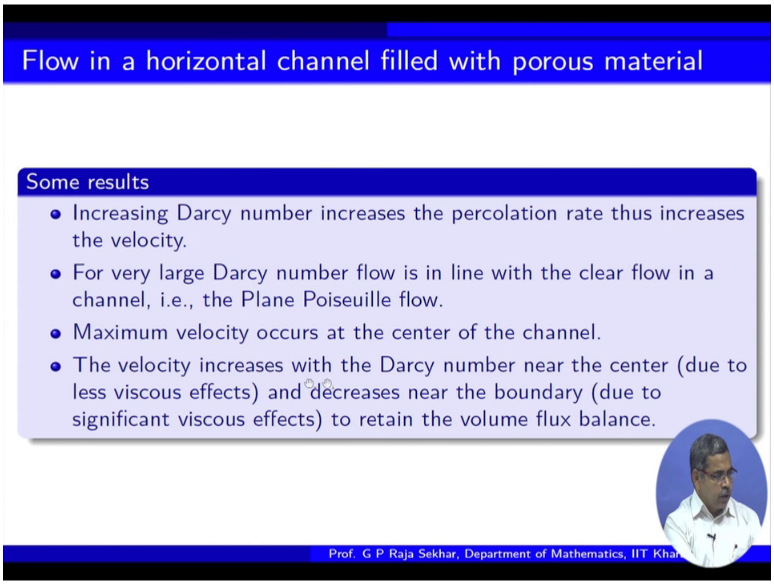

So quick inferences already we discussed increasing Darcy number increase the percolation rate thus increases the velocity which we have already seen. And for very large Darcy number flow is in line with the clear flow. So this is already we have seen and maximum velocity occurs at the centre of the channel. So this we have seen because you see maximum velocity for any given Darcy number maxima occurs towards the centre because the boundary viscous effects are less towards the centre okay.

And the most important observation is the velocity increase with the Darcy number near the centre due to less viscous effects and it decreases with near the boundary due to the significant viscous effects. And to retain the volume flux this adjustment is taking place this adjustment that I have explained okay.

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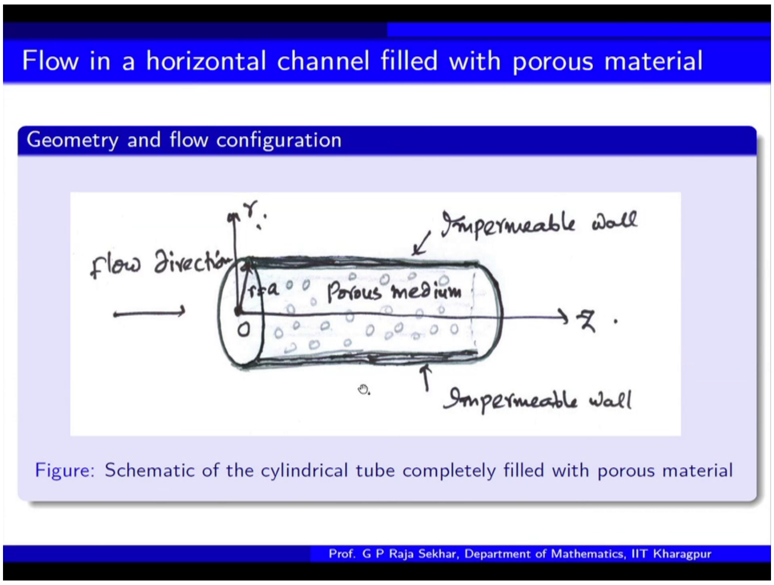

So now we move on to the next problem. This is another simple elementary computation that we discussed for the clear flow case. So again similar to the case we are discussing flow inside a tube but here we are discussing packing. So this has a lot of applications even though this is very elementary problem. So a lot of filtration processes they involve such geometries and also flow through some tissues okay.

So you have veins which are like pipes and then inside you have a lot of tissue involved. So this gives some insight okay. So let us look at the corresponding configuration and the governing equations.

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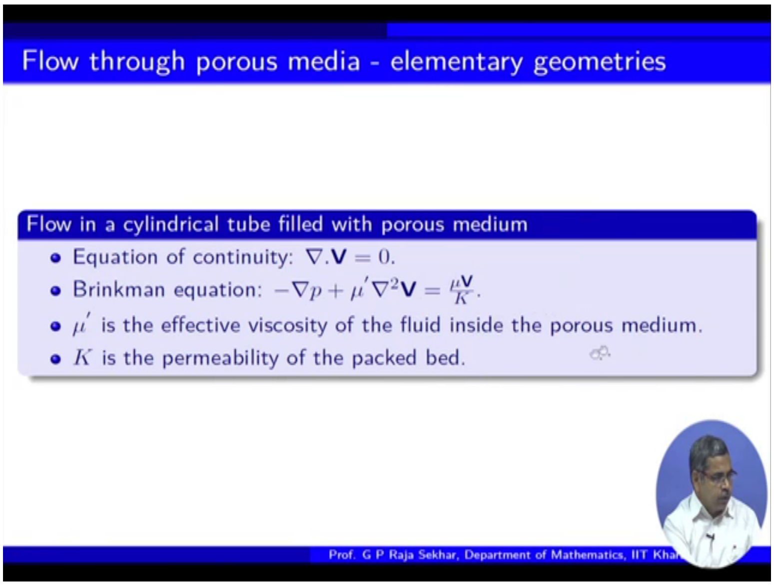

So these are the same. This is the equation of continuity, then Brinkman equation and we consider effective viscosity equals to the dynamic viscosity.

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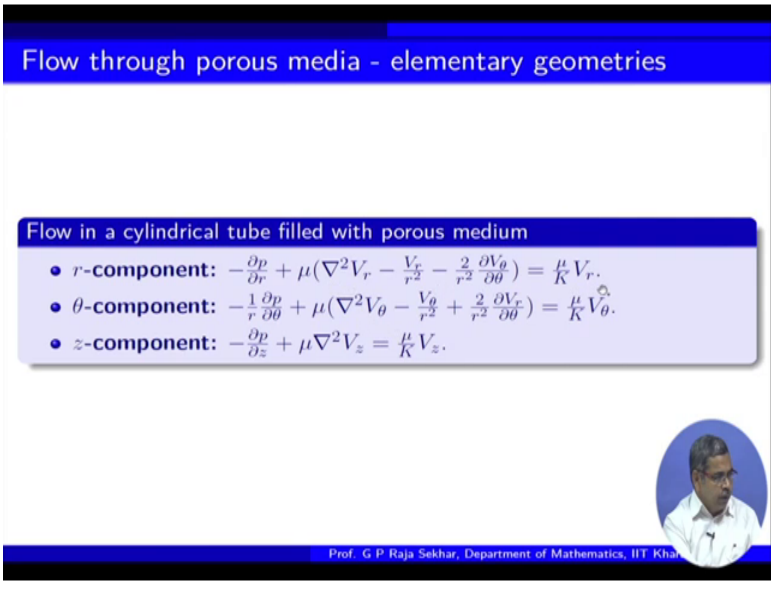

Now in a cylindrical coordinate system we have listed the corresponding component form okay. So the Laplacian on Vr this should be understood in terms of r, Theta, z coordinate system. And this is the damping term okay.

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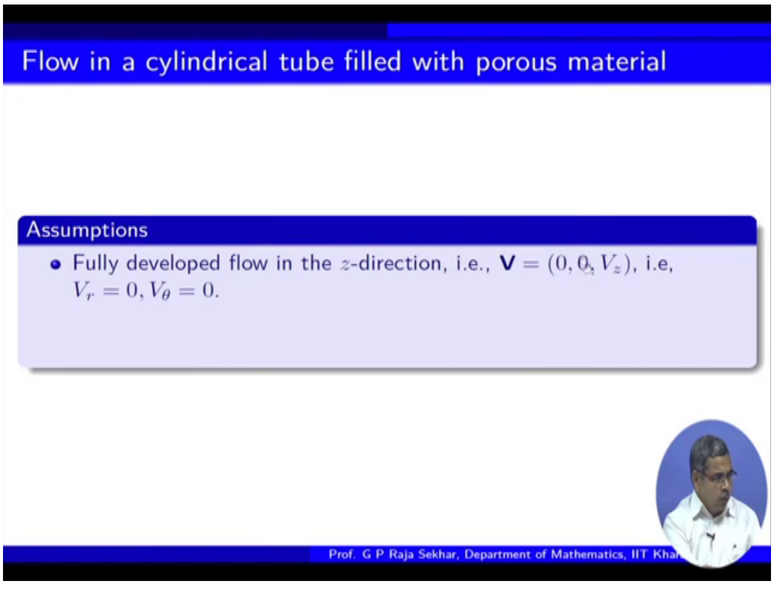

Okay so now we assume the fully developed assumption but is flow along the actual direction. So then with this assumption also you have axisymmetry.

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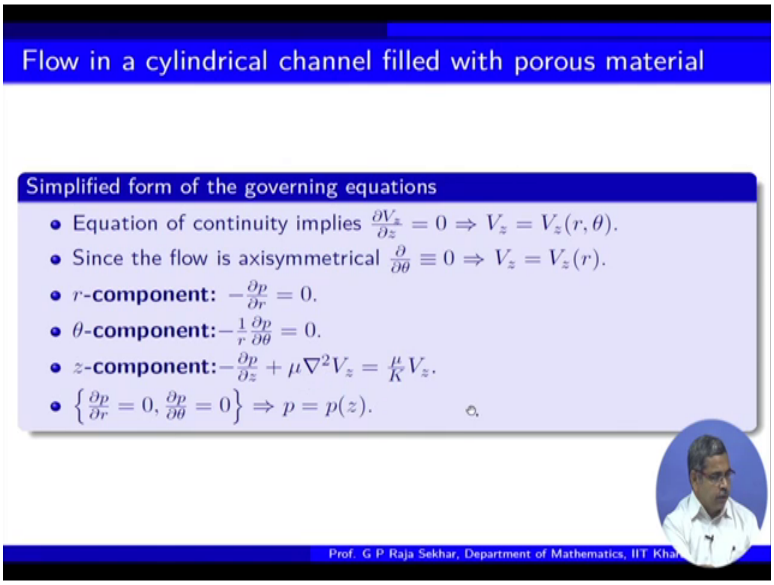

So we have seen for clear flow so it is a more or less similar analysis. So equation of continuity gives that Vz is function of r and Theta okay. So correspondingly axisymmetry brings it from r Theta to r alone okay. So that now r component, Theta component they ensure that p is independent of r and Theta okay. So this is the z component so essentially we have to solve this particular equation okay. So this two ensures that p is function of z alone okay.

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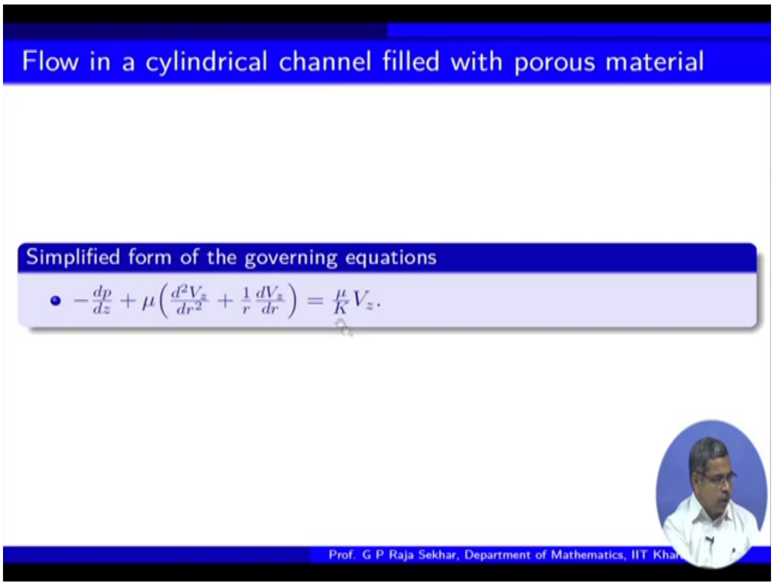

So now we have to solve this equation okay. So without this already we have obtained so now you have an additional term so that will change the structure of the solution. So like previous case that is what we have seen. So earlier we were getting only polynomial but in the Brinkman case for channel flow we got hyperbolic functions. Similarly for pipe flow without this we would have got a solution in a polynomials in terms of r.

But here we are getting something we expect something else because of this.

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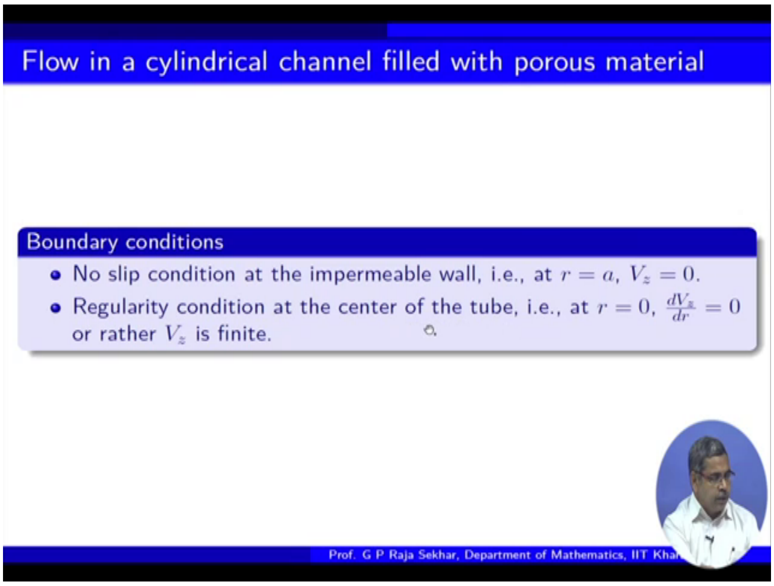

So we do the usual boundary value problem set up. So on r = a no slip condition and then regularity condition at the centre of the tube so we do not allow any singularities. So flow is bounded at r = 0. So therefore correspondingly we have this okay.

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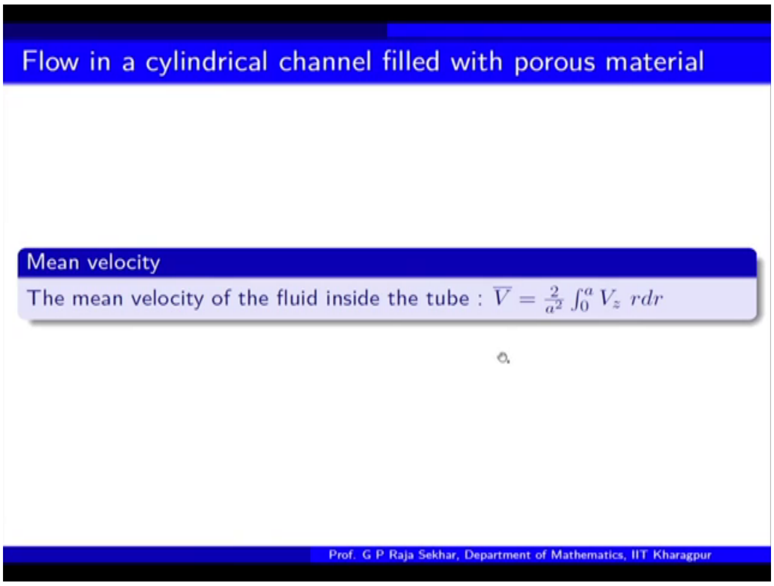

And this is the mean flow so this is a very so we would like to compute the mean flow okay. So what we are we are taking a cross-section and then we would like to compute the mean flow.

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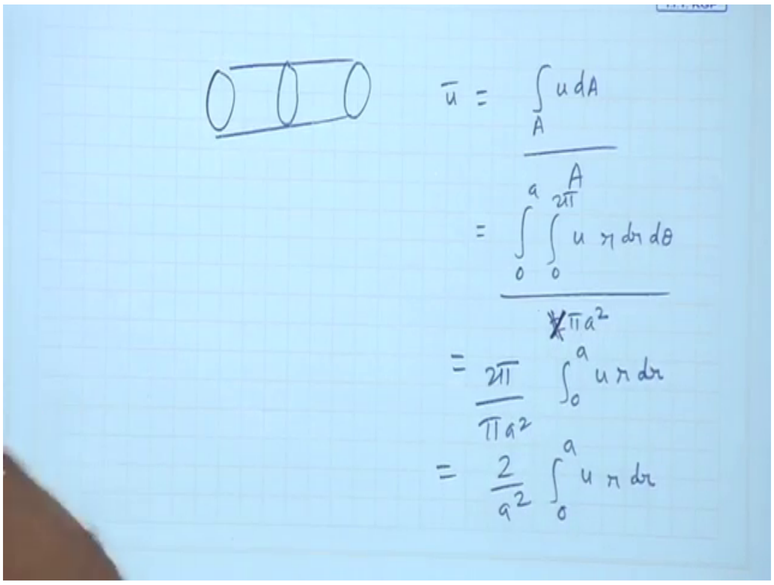

So that is nothing but so u along this area divided by area okay. so corresponding areal element what we get is so this is ur dr d Theta. So Theta is 0 to 2Pi, r is 0 A okay. And then correspondingly so this is 2PiA power 2 okay. So once you integrate so you get 2Pi get 2Pi will be getting cancelled okay. This is the area okay. So once we integrate so we get 2Pi by Pia power 2 integral 0 to a ur dr. So this will be 2 by a square ur dr okay.

So therefore we get this is the mean velocity okay. So again we non dimensionalize then we get this equation so with this no slip and the regularity condition and this is a normalized volumetric flow balance okay.

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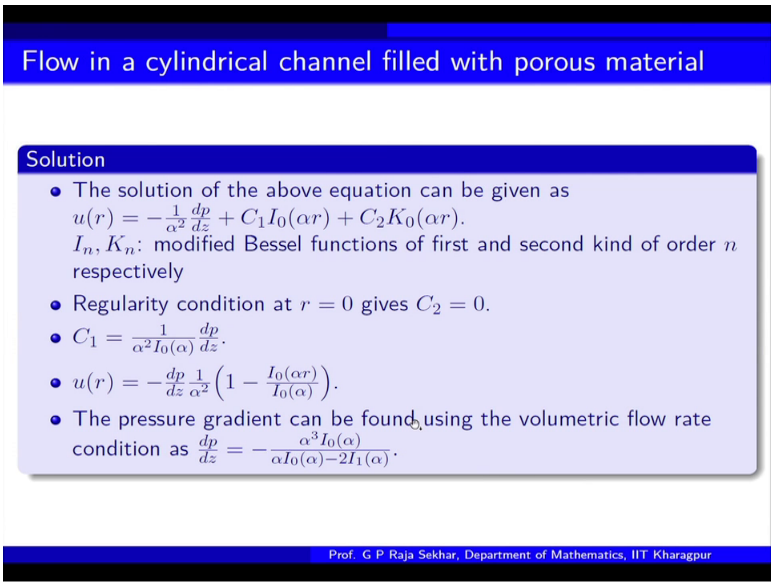

So as I indicated the solution is no longer polynomial in r rather we are getting something interesting. So this involves a Bessel functions. So these are modified Bessel functions of first and second kind of order n. But here we are getting of order 0 okay. So these are the modified Bessel functions. So then we use regularity condition and the boundary condition. So if you see so these are this is bounded at origin and this is bounded at infinity.

So since our regularity condition requires flow should be bounded at origin so therefore this coefficient has to be killed. Because of this is bounded at infinity this is bound later origin okay. So therefore our requirement is flow should be bounded at origin because interior flow so this coefficient must be forced to be 0. So that we have this contribution and using a no-slip we can evaluate and this is the corresponding solution okay.

So again using volumetric balance so one can determine the pressure gradient okay.

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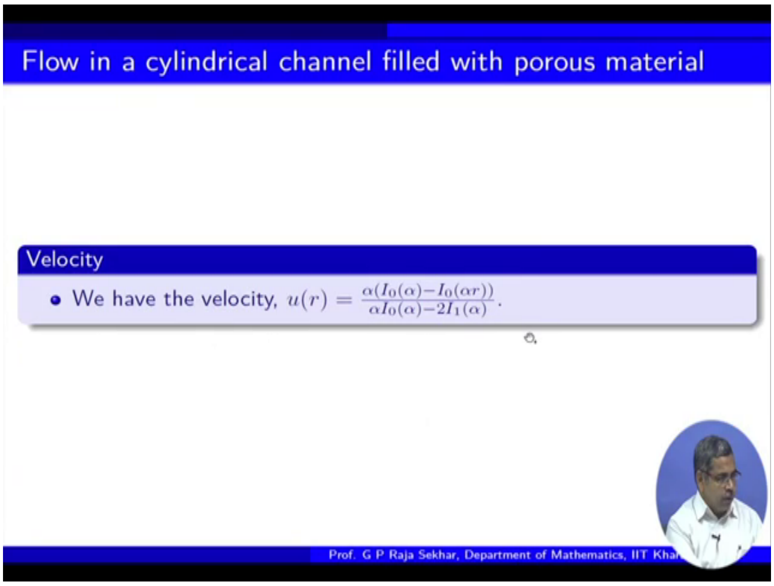

So with this we have the velocity. So now you may ask again whether we get the corresponding limiting cases, of course we get. So for the plane parallel plate channel case I have shown you the corresponding asymptotic expansion of a hyperbolic functions with a large Darcy case. So one good exercise here is you consider large Darcy case that is alpha small.

And then expand these modified Bessel functions so this is of order of 0 and 1 and then take the limiting value to see that this solution indeed agrees with the Hagen Poiseuille flow okay. So that is a good exercise one can do. So now similar to the parallel plate case we have already explained so similar phenomena is happening here. So this is a low Darcy number and this is a high Darcy number.

So correspondingly the velocity is getting adjusted and then correspondingly the total volume flux remains constant okay. So the viscous effects are more prevalent near the boundary and the maximum velocity is at the centre of the tube okay. So you might be wondering we are showing so we are showing only the portion right. So only this portion is plotted so that is what you are seeing in this plots okay.

So this is a competition with varying the Darcy number and with the large Darcy number you have a complete agreement with the Hagen Poiseuille flow. This is a Hagen Poiseuille flows create for solution okay. So corresponding analytical solution you should take the limiting case and then check it.

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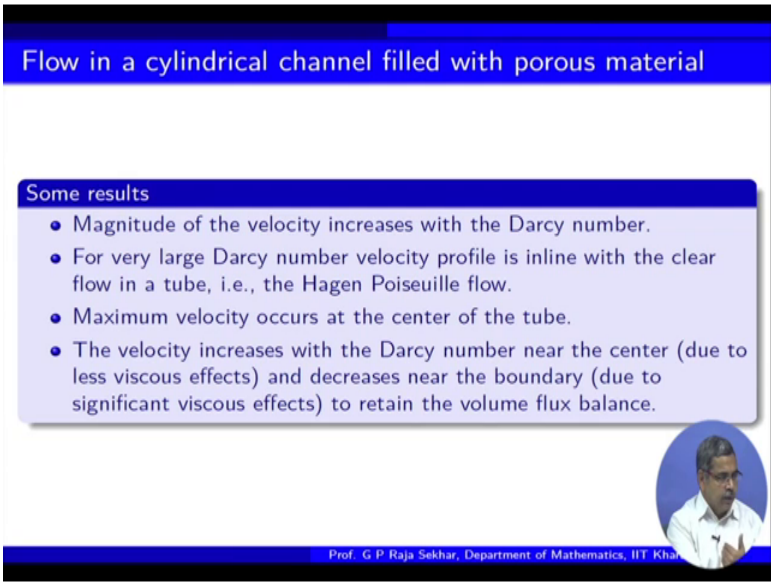

So again results magnitude of velocity increases with the Darcy number and the solution is agreeing with the Hagen Poiseuille flow in case of a larger Darcy number this we have seen. And a maximum velocity at the centre of the tube and the velocity increases at the centre and then decreases at the boundary. And the total volume flux balance is readjusting the velocity profiles with varying Darcy number okay.

So this gives some insights about the elementary geometries how the corresponding physical insights are differing compared to clear flow and then porous packed plate or tube okay. So there are lots of applications where instead of complete porous you can have partially filled porous and then these are typically having lot of applications.

For example if you take a glycocalyx layer in a human body or any animal body. So glycocalyx layers are where you have a clear flow and then a porous layer coating. So it is like a tube. So then you have various blood cells migrating in it. So these are like one can discuss microparticles transport inside channel etc. So before we really discuss such application so in coming lecture we discussed about a composite channels. That is partially porous and partially clear flow okay. So until then thank you!