Modeling Transport Phenomena of Microparticles
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Lecture – 11
Solution of Arbitrary Stokes Flow

Hello! So far we have discussed about application of stoke stream function. For example flow past a circular cylinder and then flow past a spherical objects in case of access symmetry. So therefore we have introduced stream function and then obtained the corresponding linear operator equation and then attempted a separable solution and got some physical insights like how to compute drag force etc. okay.

But, in most of the physical situations one has to handle the 3-dimensional flow case where you have a spherical particles migrating in arbitrary flow. So there is no access symmetry but we would like to capture the complete flow structure okay. So which means we have to handle arbitrary case okay. So this can be done but involves some concepts on arbitrary solution because you have stokes equation.

So unless we have a solution in 3-dimensions we would not be able to apply. Because if you simply decompose into component form and then try to integrate it is no good okay. We would not be able to in fact proceed further from the component equations okay. So what is the structure? So the structure is one can nicely correlate some harmonic functions concept with stroke flows and then try to obtain a solution for stokes equations in arbitrary case; that is3-dimensions in terms of harmonic functions okay.

So let us see how we can do that.

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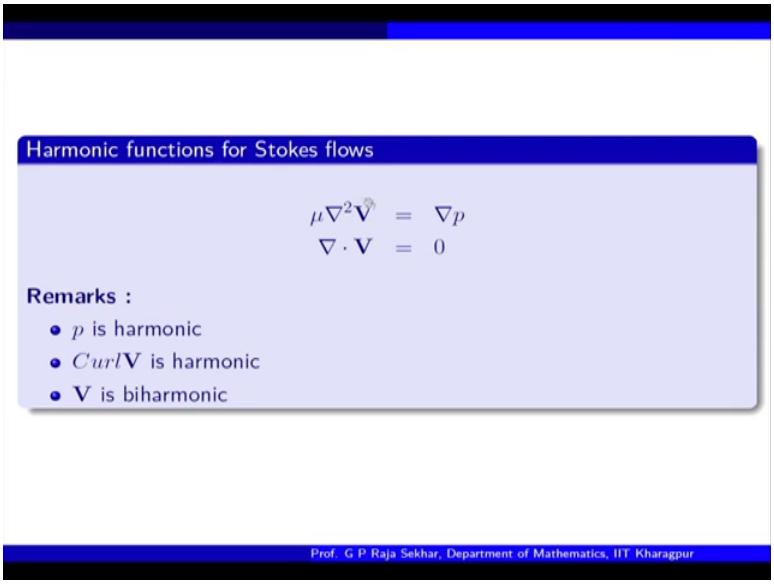

So let us consider stokes equations and with the corresponding equation of continuity. So we kept the title as harmonic functions for strokes flows because we can correlate nicely okay. You will see very much if you take divergence of this what happens this is 0. So we get pressure is harmonic. So that isthe first harmonic associated with stroke flow okay. So then suppose if somebody takes curl of this curl of gravity is 0, so then Del Power 2 and curl commute.

So we get Del Power 2 of curl V is zero, which means CurlV is harmonic. So that is the next okay. So then if you take laplacian of this since p is harmonic this is 0. So therefore we get Del4V is 0. That means V is bi-harmonic okay. So p is harmonic, CurlV is harmonic and the vector V Bar is bi-harmonic okay. So this involves three scalar harmonics because V in 3-dimensions naturally right okay.

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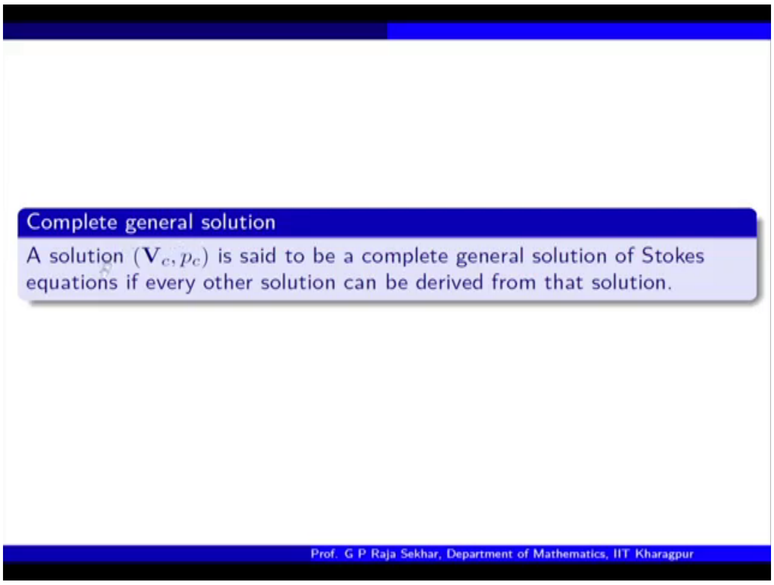

So now we are introducing the definition of complete general solution. So what is complete general solution is? You have the down equations you integrate and then obtain a solution such that any other solution can be obtained from this okay. So that isa complete general solution okay. So that means you have 2-D unidirectional cases, 2-dimensional and 3-dimensional particular solutions everything should be able to obtain by this solution, so then this solution is called complete general solution.

So that is the definition we are giving. This is said to be a complete general solution of stokes equations if every other solution can be derived from that solution okay. So now the question is do we have such solution? The answer is yes. But before we go to such solution so let us review the nice history are on solutions of stokes equations.

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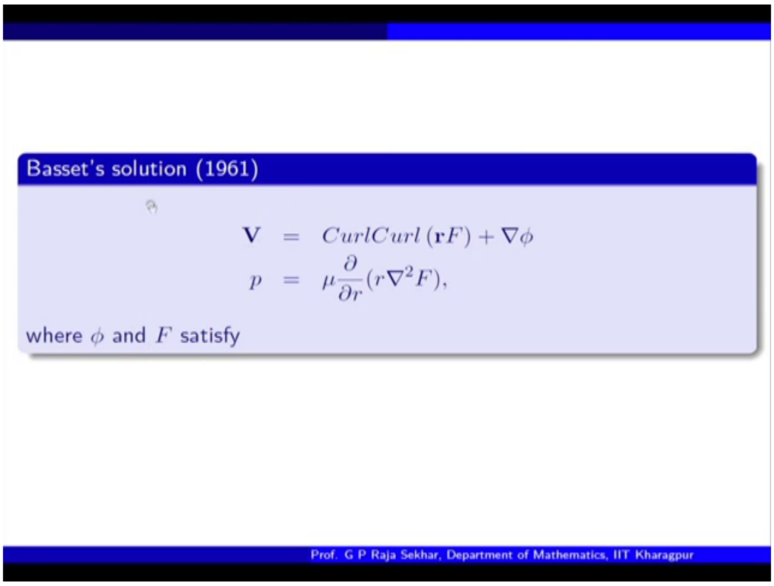

So Basset has given this solution velocity and pressure of stokes equations can be represented like this okay, provided Phi and F satisfy some equations okay. So what is the trick here is?

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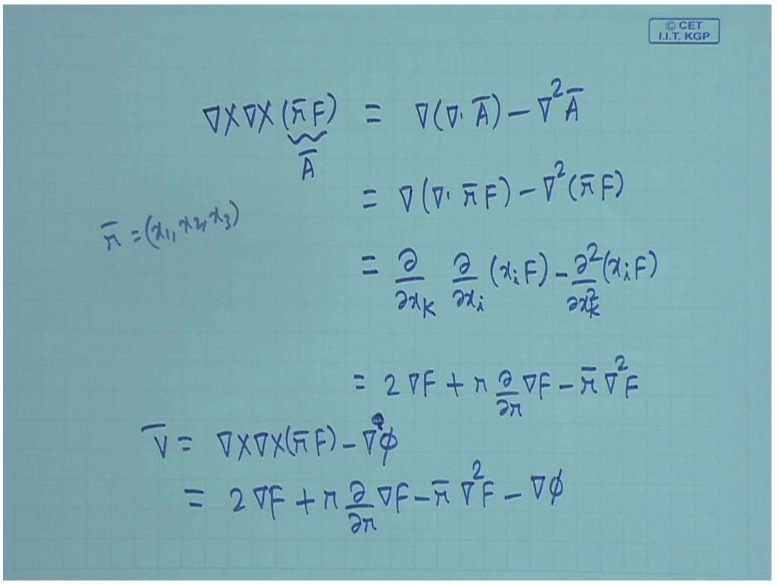

So the for example the first part I am writing Curl Curl rF so we have a vector identity, suppose if you call this some A bar, Curl Curl A bar is grad divergence of A bar minus Delta Power 2 A bar. So that means this is divergence of okay. So what is r? r bar is the position vector okay. So now simple calculations one can do okay, to show that so this information notation let us say one can write.

So this is xiF because the notation here r bar is x1, its the position vector. So this is nothing but this okay and for grad this is the summation notation and correspondingly. So doing algebra some little further expansions, we can show that this is equal to okay. So we one can show this. so I am purposefully not giving the calculation because it is a unless you do it you would not learn. So better you try this so we can show this.

So therefore overall V will be V that we have Curl Curl rF minus grad F. So this reduces to. Now we substitute this in the stokes equation okay and then choose p, this one then you will see that the remaining two quantities will be satisfied. In other words consider this we expand the way just now we have done, substitute in stokes equation. Then if F and Phi satisfy these equations, automatically p has to be this okay.

So that means V and p given by this represents solution of stokes equations in 3-dimensions provided Phi and F satisfy this equation okay. So now the question is whether this is a complete general solution because we have introduced.

But one can show that Basset solution is not complete but that itself is a difficult task. So only thing I can give a remark Basset solution is not a complete general solution.

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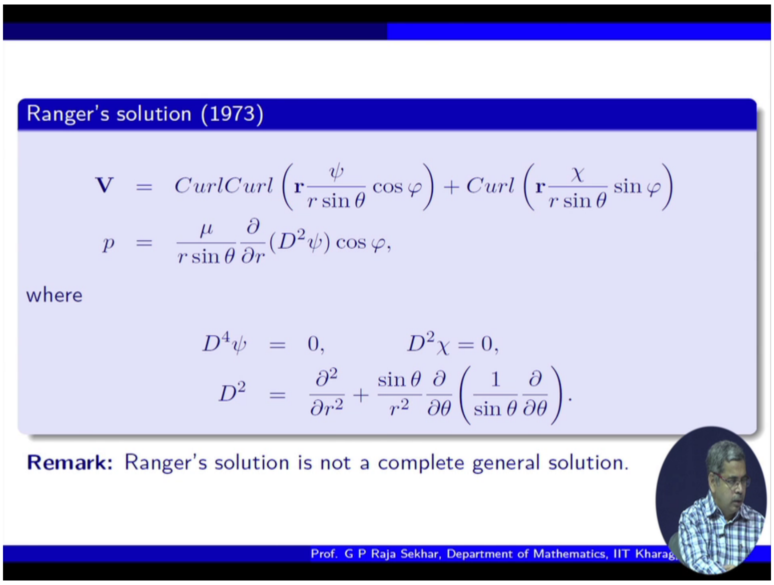

So the next solution is Ranger. If you see the CurlCurl structure somehow it is dominating right for stocks flows. So curl curl again the position vector so this time you will see a specific structure a scalar Phi by r Sin Theta Cos Phi. So in the earlier case these scalars F and Phi, they are harmonic and bi-harmonic.

So there is no restriction that means they are arbitrary. But here there is a restriction with respect to the structure of Phi. So naturally the corresponding expansion will lead to corresponding equations for Phi and Chi. Because two scalar quantities are involved okay. So is this represent arbitrary flow? So that isthe question and the one can very easily say that not. Because any arbitrary flow may not have this specific structure with respect to Cos Phi okay.

So therefore one can easily say that this is only a class of solutions okay. So therefore Ranger solution is not a complete general solution right. But the most interesting solution is suggested by Lamb. So that is the first solution which captures arbitrary stokes flaws okay. So what is the motivation?

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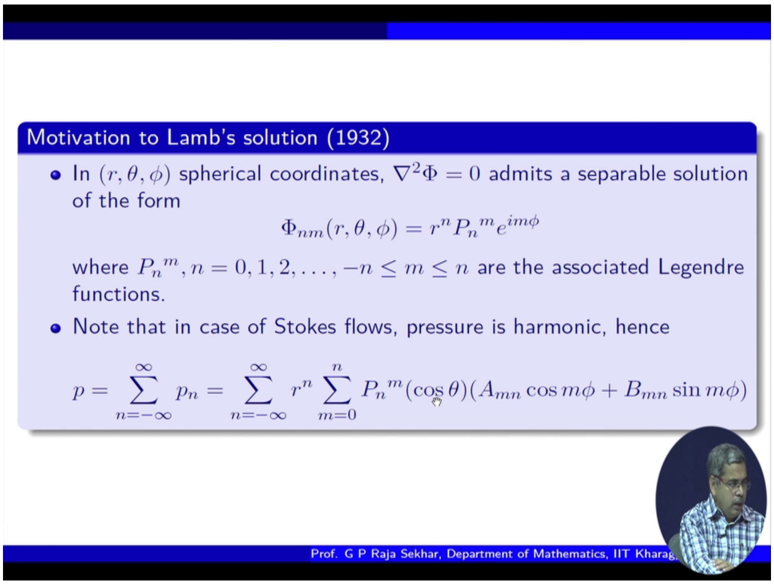

So motivation to Lamb's solution is in polar coordinates (r, Theta, Phi), if you take a Laplacian, one can see that using separation of variables it admits solutions of the form r power n Pnm eim Phi where Pnm are associated Legendre functions okay and these are involving Cos m Phi and Sin m Phi okay.

So this motivated Lamb to have a generic arbitrary solution okay. So since we know that for pressure for stocks low pressure is harmonic, so Lamb expanded p in terms of harmonics because any harmonic function has this structure. So since pressure is harmonic for stock flow pressure is expanded see Pn then associate Legendre and to avoid the complex numbers this structure has been introduced okay.

So the solution Laplacian involves a double summation because if you see the separation process we get one index for the Phi and other for r and Theta. So therefore there is a double summation accordingly one can get this solution okay. So you can refer any standard mathematical methods book to see the separation of variable solution. So this is the motivation for Lamb. Pressure is harmonic therefore one can have this structure.

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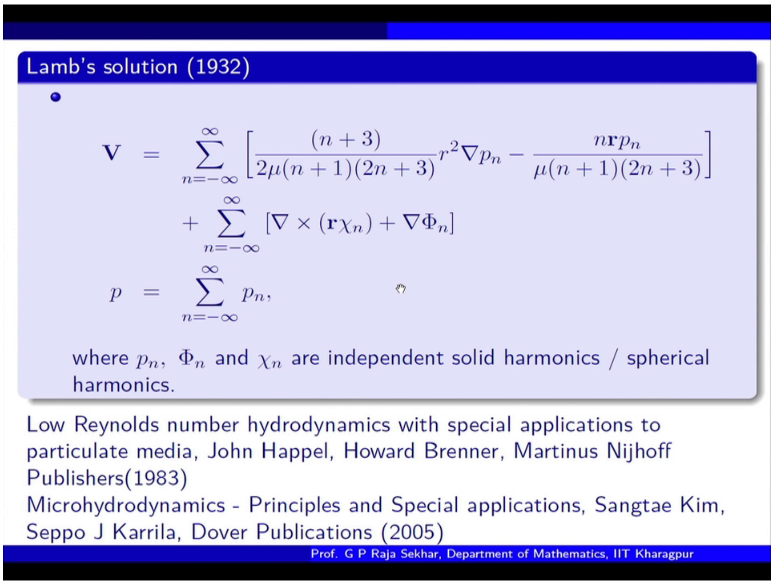

So with this motivation Lamb has suggested solution as follows. Velocity is given by P n R scalar harmonics. Then further introduced Chi n and Phi n are two more scalar harmonics. In total Lamb suggested a solution involving three independent scalar harmonics okay. So these two books give more details on the Lamb solution but even though these are three scalar harmonics, by virtue of the summation so depending on the problem.

So you get several of them okay. Now depending on note that summation is minus infinity to infinity so therefore we are going to have the positive indices and negative indices okay. So let us see how depending on situation these indices have to be controlled. So the advantage of Lamb solution is the vector problem is now being converted to solving three scalar harmonics which are given by.

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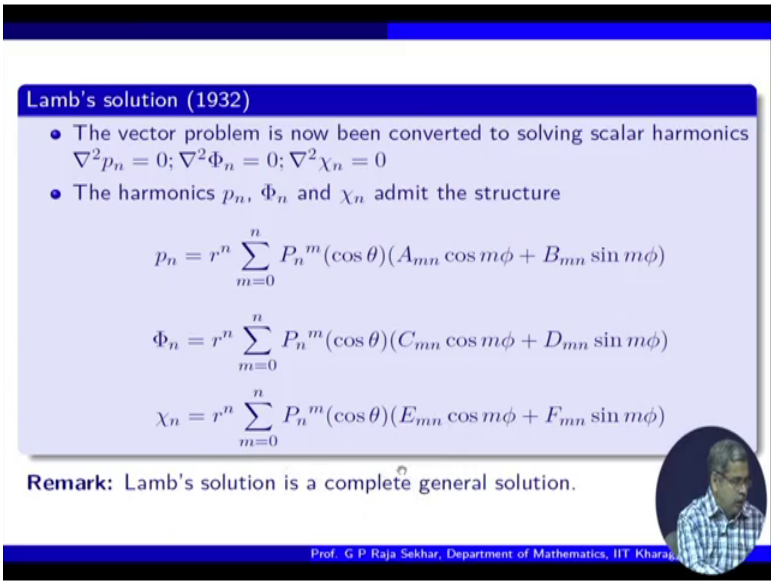

So this structure I have given already similar structure can be adopted for Phi n and Chi n because after all they are also harmonics but the orbital coefficients are changed corresponding to each of the harmonic. You see here Amn, Cmn, Dmn and Emn okay. These are the arbitrary coefficient. So one can handle 3-dimensional arbitrary flow using Lamb solution by using these three scalar harmonics okay.

So let us see more details about Lamb solution, in particular when it is exterior flow how to control the harmonics when it is interior flow how to control the harmonics okay. So one remark is Lamb solution is a complete general solution okay. So that means any given solution of stokes equation can be obtained from this okay.

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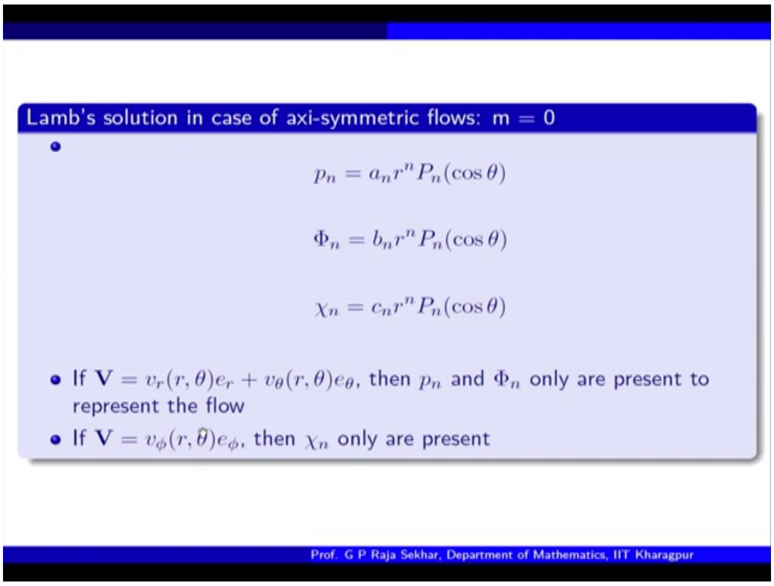

In case of axis symmetry you would not expect dependence of Phi, so therefore, m is 0 then we get this structure okay. So if velocity is having axisymmetry, then what we get is only pn and Phi n are present in the solution. We would not get Chi n. So what could be the reason? Let us see the structure of the Lamb solution okay. So the structure if you see this indicates rotation okay.

So this curl of R bar Chi n indicates a rotation, so if the flow quantities are independent of the azimuthal angle Phi, so then there are no rotations. Correspondingly our solution contains only pn and Phi n. Similarly if the velocities of this form that isonly swigling, so then the solution contains only Chin, no pn and Phin okay. So these are some quick remarks.

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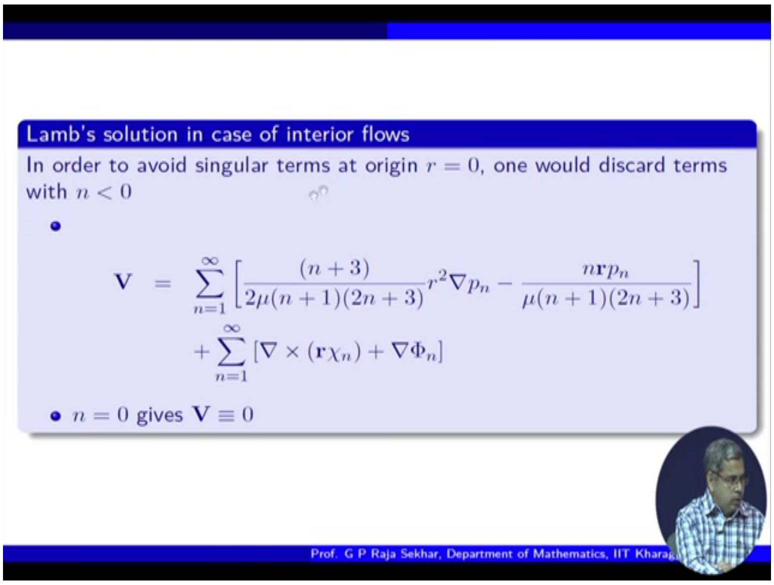

Now for the case of interior flow we avoid singularity at origin. So therefore, one would discard terms of a negative order because they give singularities. So correspondingly you see the summation is from n 1 to infinity and n equal to 0 in particular gives identically zero velocity okay. So this is a case of interior flows okay.

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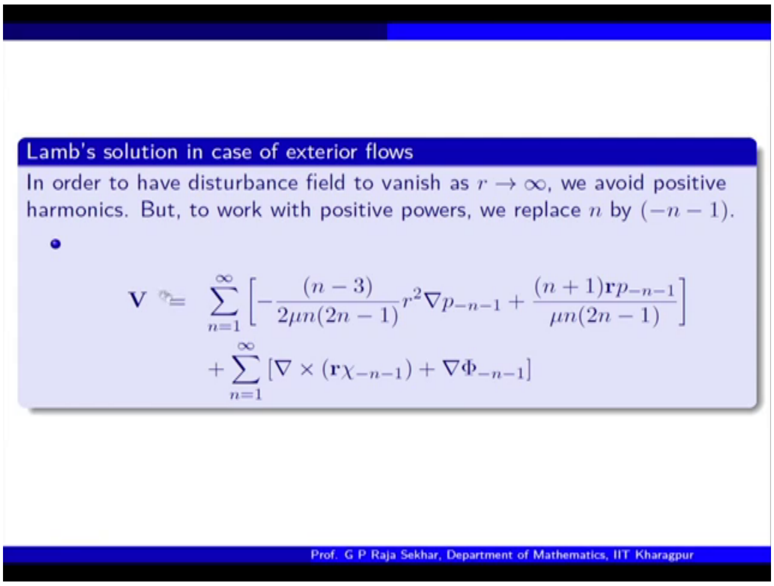

Now for the exterior flows. We expect for exterior flow the flow quantity, whatever disturbances created they are bounded as r goes to infinity okay. So we avoid positive harmonics in this case because Lamb solution contains minus infinity to infinity. So for interior we are avoiding negative for exterior we are avoiding positive but however for exterior we do a small trick.

Just to avoid our negative powers even in case of exterior we use this transformation so that the representation can be having a nice structure okay. So this is an adjustment. One can work with a simply throw positive powers and work with a negative index that is not a problem but just for convenience we have replaced n by this so that we have this structure okay. Now corresponding to this shift the arbitral coefficients for these will be changed okay.

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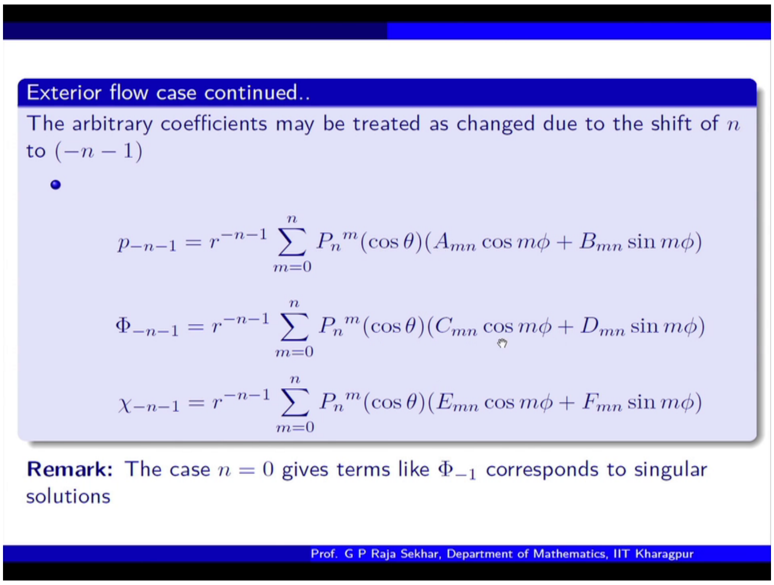

So therefore even though we kept the same notation they could be if somebody is following these coefficients and then do the index change n to minus n minus 1 then the coefficients will be changed. But what we are doing just it is a notation. You can have some tilde or bar whatever you wish okay. In any case for exterior flows the harmonics have the corresponding structure okay.

And the case n = 0 gives terms like this which are singular solution okay. So we do not pay much attention of this. It is enough to have basic understanding of the exterior flow and interior flow and then when we discussed about singularities towards the end of may be in another 2-3 lectures, so then you get some idea about singular solutions okay. So this is about the Lamb solution.

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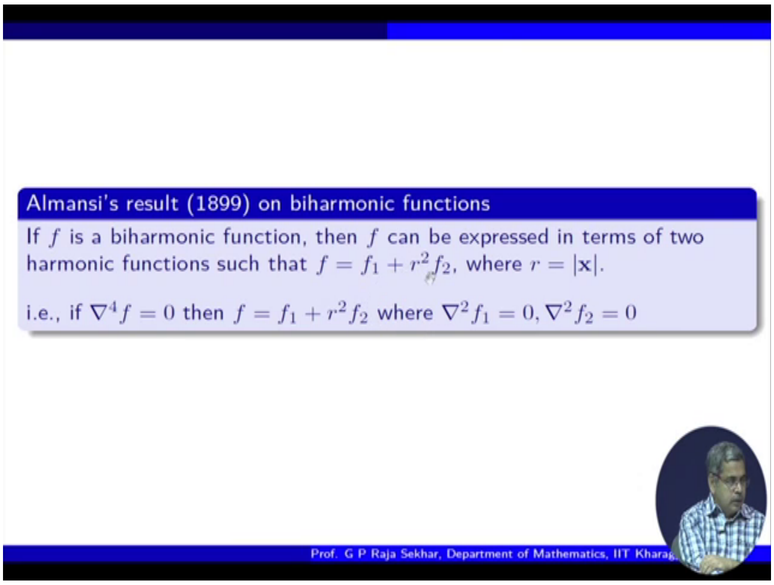

Now as I indicated the harmonic function has a vital role. So Lamb's solution is complete which is in terms of three harmonic functions. So the natural question is why only three harmonics are sufficient to capture complete general solution okay. So let us see the corresponding arguments.

We recall a classical result due to Almansi, which indicates that if f is a bi-harmonic function, then f can be expressed in terms of two harmonic functions such that f is given f1 + r Power 2 f2 where r is the position where r is the magnitude of the position vector okay. So which indicates any bi-harmonic function can be expressed as combination of two harmonic functions okay.

So that isthe summary of Almansi's result. So using this let us analyze the velocity vector which is a bi-harmonic okay.

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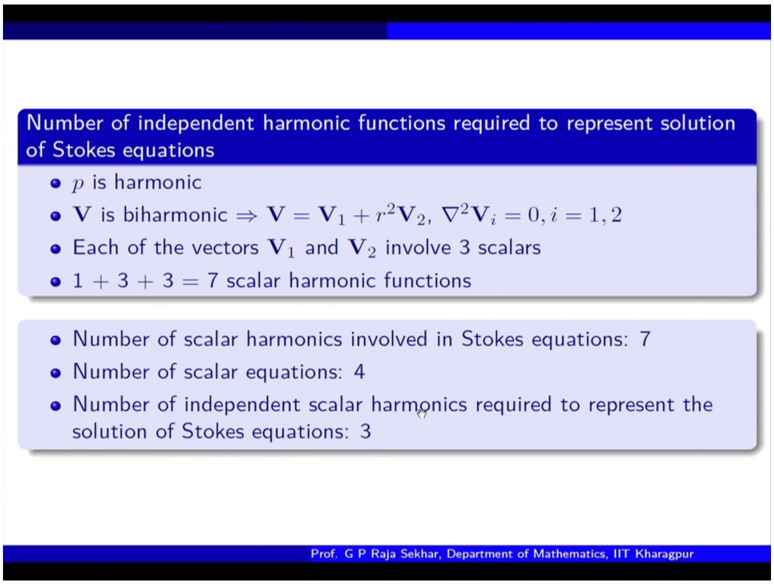

So we have already remarked earlier p is harmonic and since V is bi-harmonic using Almansi result V can be decomposed into two harmonics okay. V1 V2 where each V1 V2 is harmonic. Then each V1 V2 involve three scalar because in 3-dimensions.

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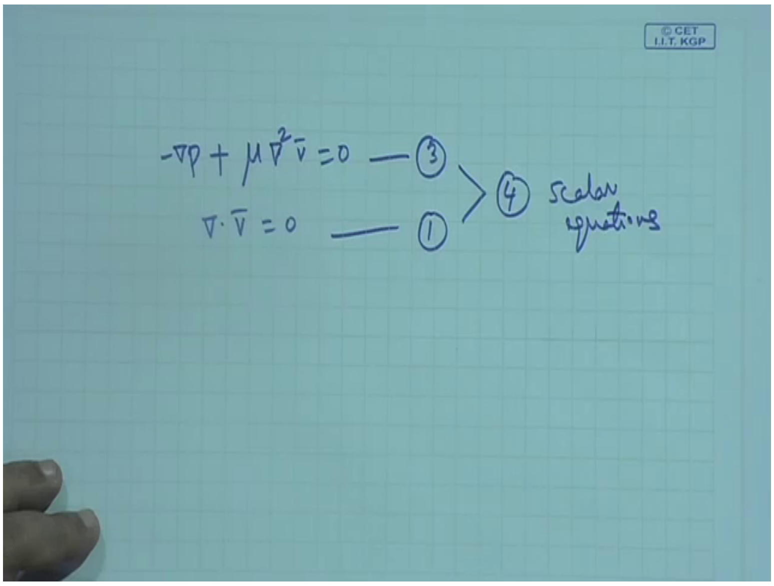

So therefore if you count the number of harmonic functions involved, they are how many? One. Then for V1 you have three for V2 you have 3, total seven. 1 for 3, V2 three, so total seven scalar harmonic functions okay. Then what are the governing equations? So this is one scalar equation and this is 3 so total four scalar equations okay.

So we have number of scalar equations four then number of harmonics involved are seven. Therefore the total number of independent harmonics that are required to capture Stokes flow are 7 - 4 that will be 3. So that is the reason Lamb could give solution which is a complete general solution and it involves only three scalar harmonics okay. Which means one can represent stokes equations in 3-dimensional and arbitrary solution in terms of three scalar harmonics. So this is an important point.

Okay so Lamb solution is there so then now our problem is resolved. One can use a Lamb solution and get solution for say arbitrary Stokes flow past a sphere or maybe other situations. But if you pay close attention to Lamb solution it involves the summation of these harmonics. So it is slightly complicated and the structure may give some challenges. So this motivated people to think for better representations okay. So I am going to give you quickly one such representation which has some additional advantages okay.

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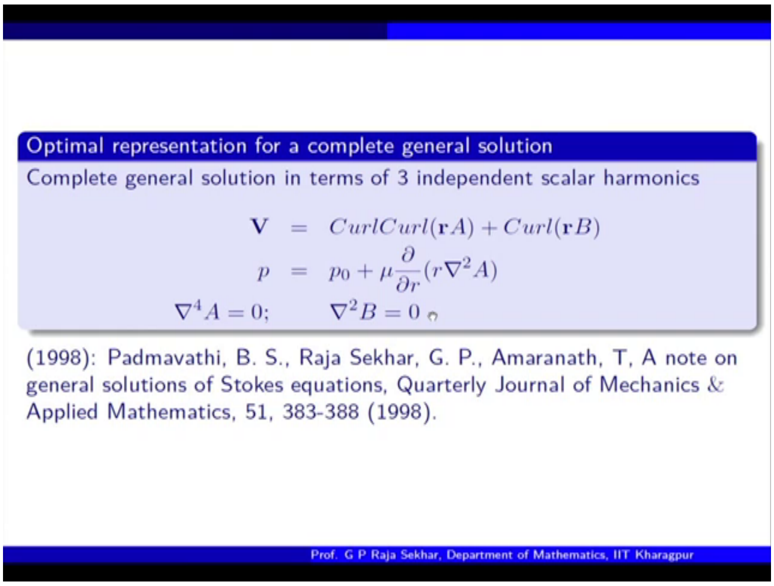

So optimal representation when I say a representation for solution of stokes equations involving three scalar harmonics. So Lamb solution is one such but here we are going to see without any summation a representation which involves purely three scalar quantities. So velocity is Curl Curl r bar A plus Curl r bar B. So r is position vector, A and B are two scalars and the pressure is given by this up to a constant p0 okay.

Then V and p given by this will be solution of stokes equations provided A is bi-harmonic and B is harmonic okay. So the proof for can be obtained by in this article given by these people okay. So this was started much before but the completeness that this representation is a complete general solution of stokes equation is given in this. Otherwise earlier this representation was used by the group of T Amaranath, Nigam Palaniappan, and Padmavathi. But the completeness has been shown in this article okay.

So now we have one representation which gives solution of arbitrary stokes equations which involve three harmonics. Why? You have B which is harmonic and A is bi-harmonic due to Almansi result A can be represented in terms of two harmonics. So total three harmonics involved. Now immediate question is what is the advantage okay? So what is the advantage of this particular representation?

You will see this representation has a lot of advantages in particular handling circle geometries okay. So how let us see. So V given by Carl Carl r A plus curl r B if we decompose in spherical r Theta Phi coordinates, we get this relation.

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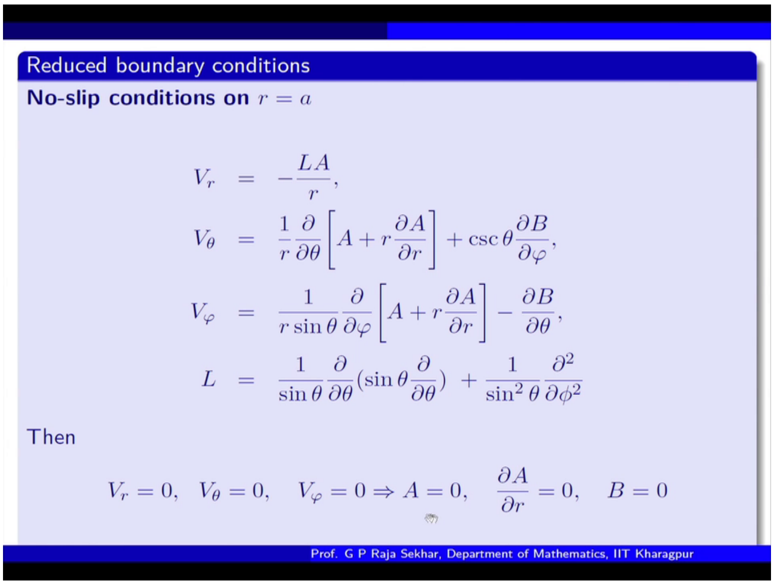

This is a radial and the two tangential velocities are related to the vectors like this okay. Now suppose our aim is to satisfy no slip conditions, Then Vr should be zero okay. What is L here? L is an elliptic operator which is independent of r. So normal velocity is LA where L is this operator and V theta involves some Theta derivatives and Phi derivative, V Phi involves Phi derivative here and Theta derivative here.

So we are going to discuss the power of this representation. Suppose you decompose the representation in terms of Vr, V Theta, V Phi you will be puzzled how we got this. Simple expressing Curl Curl rA rB in r Theta Phi polar coordinates. If we do this okay, we get this representation. Now suppose we want to have no slip condition Vr is 0 V Theta is 0 V Phi is 0. These reduced to okay. These reduced to A0 Dow A/Dow r 0 and B 0.

So that means the corresponding boundary conditions in terms of velocity can be reduced in terms of these scalars okay. So the proof is involved little bit because L is elliptic operator. So when we say Vr is 0 LA is 0 okay.

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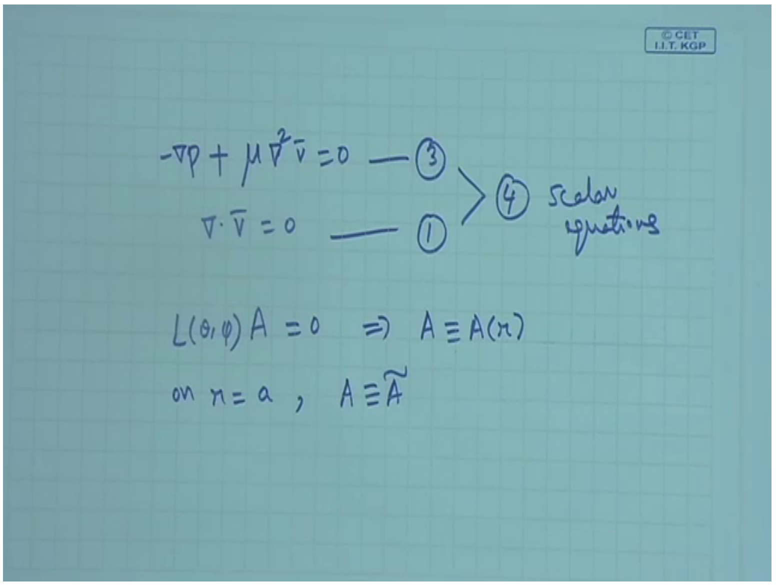

So I can just give you the L is function of Theta and Phi. So when LA is 0 this implies A can be function of r okay. But we are using boundary condition on r equals to A. So on r = A, function of r will be constant okay. So on r equal to A by virtue of this inference, A can be taken to be some constant and then constant can be considered as zero by giving some usual arguments.

So in that sense when we say normal velocity is zero on r = A, which is surface of the sphere we get A = 0 okay. Similarly you make this equal to 0 so these indicates this is 0 and this is 0 and the same inference will come from these two okay. So this also can be obtained with the simple arguments because if you see A, A is a independent harmonic, I mean bi-harmonic.

So B is a independent harmonic. So therefore, when these are 0 so you have to make individual quantities of this 0 and this 0 and those give you the corresponding inference okay. So this representation is very much useful for spherical geometries and one can use this and solve flow past a sphere very easily okay.

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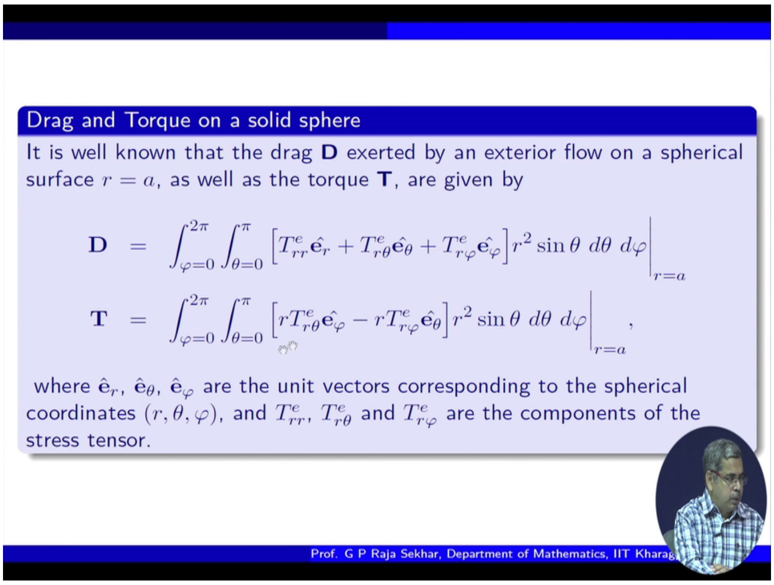

So before we do that I give you additional advantage of this representation. For a flow past a sphere we discussed drag can be obtained by integrating the stress tensor. Of course there we consider stream function so there is no dependence of Phi. But here it is arbitrary case so the corresponding drag and torque can be obtained where these are the stress components okay.

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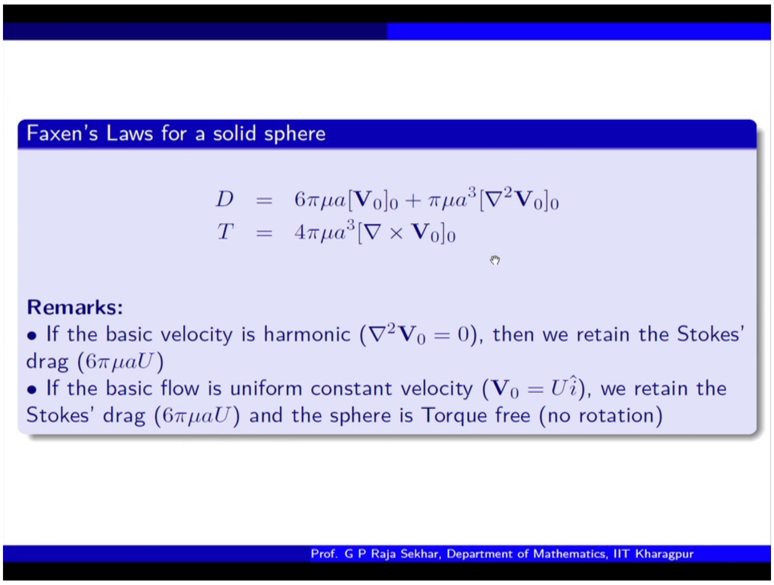

But Faxen has given very nice representation of a drag and torque okay. So this is very established literature so therefore, I am not giving you the derivation. So derivation involves it cannot be done lecture. So any book on stokes's flows contains a discussion about Faxen laws. For example the books that I mentioned, one is by Apple and Benner. With that I have shown in the references that also contains discussion about Faxen law okay.

So Faxen law indicates a compact expression for drag and torque in terms of basic velocity. So if V0 is the basic ambient velocity, drag and torque can be obtained by evaluating at origin. So what is the notation?

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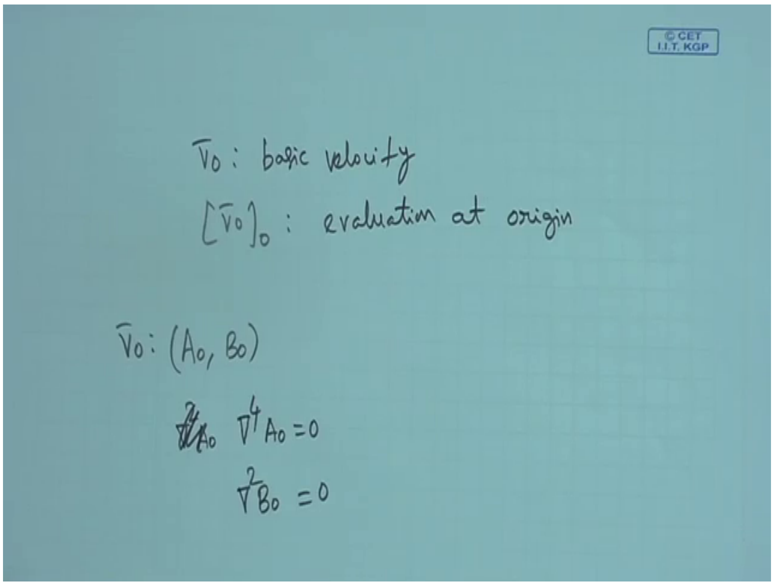

So the notation is V0 represents basic velocity. This indicates evaluation at origin. So in this case say r = 0. So instead of handling this one can consider these compact form okay and try to compute a drag and torque okay. How? Suppose we know let us say the basic flow V0 is known simply calculate these quantities V0 evaluate at origin, and then a Laplacian of V0 evaluated origin Curl of V0 at origin, then you get to drag and torque.

Did you get it okay? So now some remarks if the basic velocity is harmonica then this term is 0. So what we get is simply the stokes drag D reduced to this which the stokes drag okay, which is this. Suppose if the basic flow is uniform constant velocity, so then we retain stokes drag but if it is constant Laplacian will be zero, curl will be 0, therefore there would not be any rotations, so the torque is zero okay. So these are some remarks.

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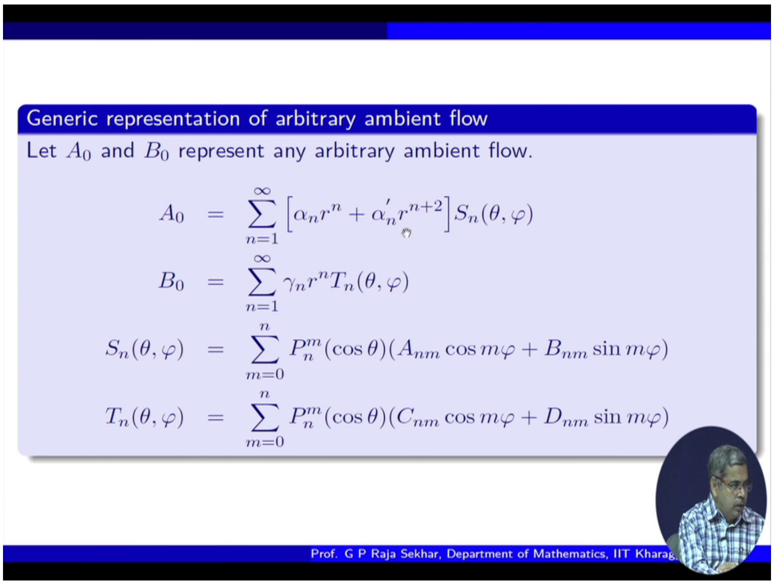

So now I give you a little quick look at the representation. A0 is bi-harmonic okay V0 is harmonic, so what we are considering is let us take some basic flow V0 instead of handling V0 we handy A0 and B0. And what is this? A0 is bi-harmonic and B0 is harmonic okay. So these are the solution of a harmonic bi-harmonic and harmonic at r goes to infinity okay.

Because any solution of B0 can be represented as some anr power n + bn/r power n+1 and some spiritual harmonics okay. But what we are considering is this is represents basic flow. Which means this should be bounded at infinity. So therefore, while considering B0 I have neglected this term. This is shown. Similarly A0 which is basic flow but bi-harmonic, it is this where S n(Theta, Phi) and T n(Theta, Phi) are the surface harmonics representing this okay.

So these are the surface harmonics okay. So if this is the scalars, our problem can be solved in terms of the scalar. How?

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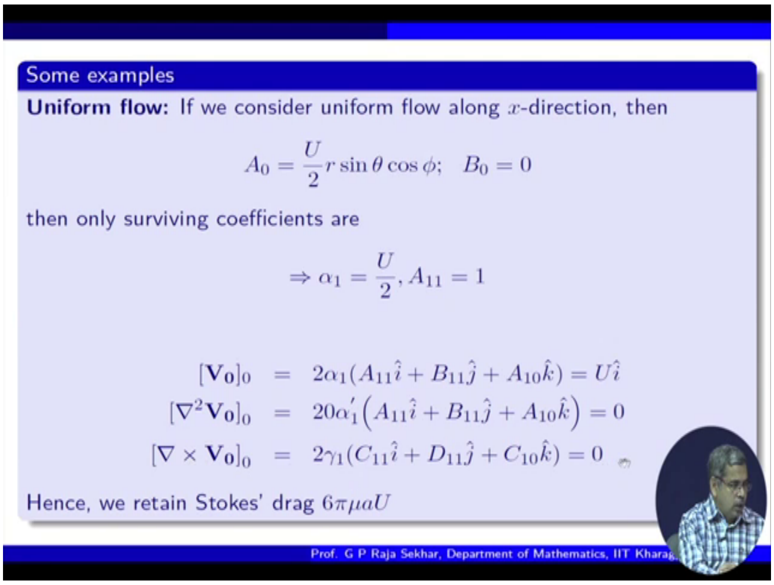

Let me show you an example for uniform flow is A0 and B0 can be given by this structure okay. So how one can get the corresponding A0, B0? So that involves so this will be the material will be supplied as a supplementary so that you learn how to get it. So for uniform flow A0 is B0 is this.

So we compared with this one. Then one can realize quickly that α1 is this and A11 is this. Then the basic velocity is this and Laplacian is 0, curl V is 0. So therefore we can get the corresponding the stokes drag okay.

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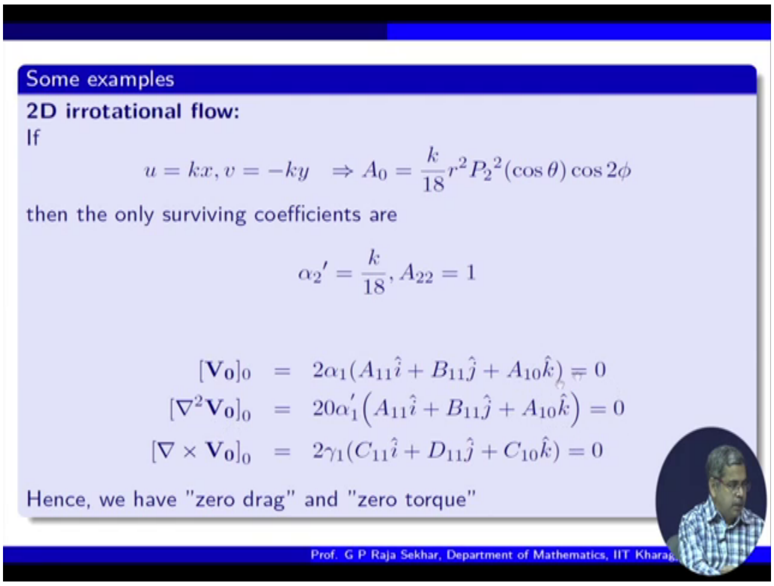

So similarly there is let us say another velocity field. Suppose you have velocity like this then one can get the corresponding A0 okay. Then compare this A0 with the general structure, and then we get this.

So correspondingly these can be computed and we see that the 0 drag and 0 torque. So I know you would not get complete a picture of how to integrate this A0 because it is not a very straightforward. So the corresponding integration will be provided in a supplementary material so that you learn how to compute the A0 B0 for a given flow and then using that one can solve strokes flow past spherical particles using this arbitrary solution okay.

So two important arbitral solutions we have learnt. So one is a Lamb solution other is the solution which is given by the group of Amaranath okay. So these are very useful and a lot of literature has come up using both Lamb solution as well as the solution in terms of three independent scalars okay. So this will give you how to deal with arbitrary flows okay. Thank you!