Model 2

Lecture – 9

Ordered Set, Least Upper Bound, Greatest Lower Bound of a Set (Contd.)

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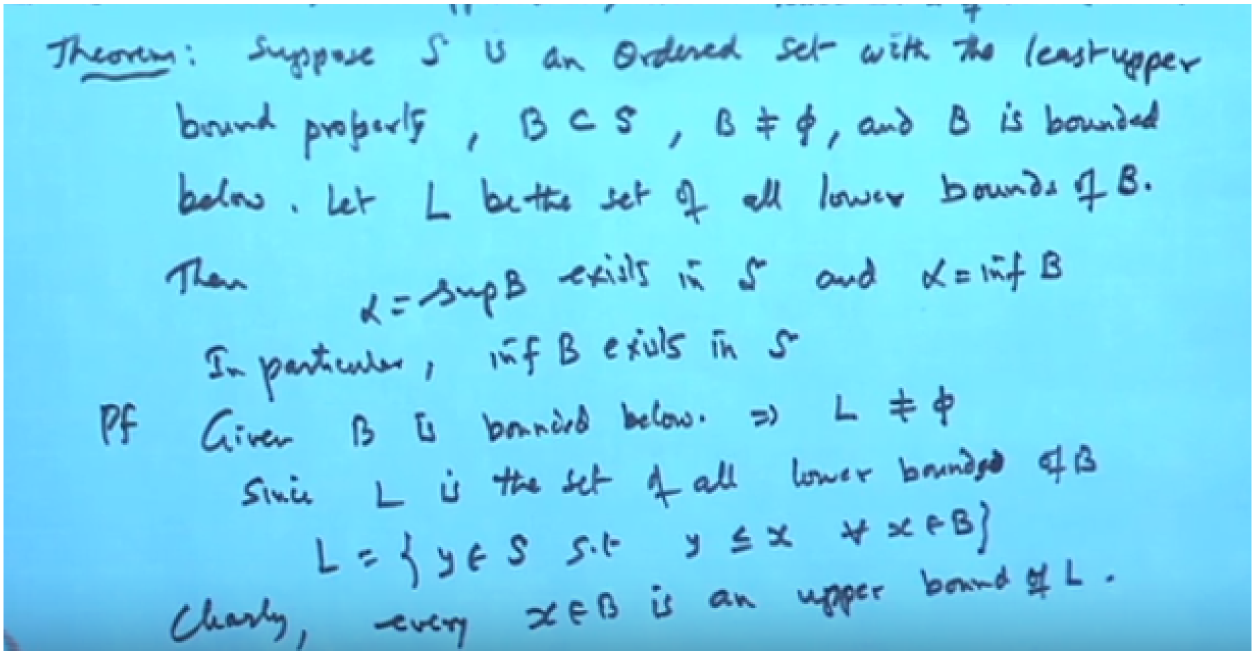

The theorem says, that if suppose, suppose, S is an ordered set, ordered set, with the least upper bound property, with the least upper bound, upper bound property. Suppose B is a non-empty subset of s, having and B is bounded below, bounded below. So s is an ordered proper, order set which has a least upper bound property and a set B has a property which is bounded below. Now this together will implies the least relation between the greatest lower bound property, Okay? So what it says is if B is bounded below, and let L be the set of all all lower bounds of B lower bounds of B. Then the supremum of B, that is the least upper bound of B, that is alpha will exist, exist in s and and this alpha will be the infimum value of B, that is it will the greatest lower bound for B. And in particular, infimum of this B exist in s. Okay?

Infimum of this. Let us see the proof of this. What is given is, s order set, which has a upper bound property, upper bound property means, if B if any set is there, which is a subset nonempty subset of this and if it has an upper bound then the supremum of this will exist in this. Now here we are assuming, that s as a upper bound property and a non-empty subset B, is bounded below. Then because of this upper bound property, in this condition, we will show that B will have the greatest lower bound and infimum of B will exist in s. That is what is. So since B is gone given; B is bounded below, this is given, Okay? And what is about our L? L is the set of all all bound of B. B is bounded below it is already given. It means there is a bond available. So L is non empty. So so this implies L is non empty, Okay? Now L is the set of all lower bound of B, so what is L? So clearly since L is the collection of, L is the set of all lower bound, bounds, lower bounds of B. So basically L consists of those bi, it means L is the set of those points, in s, bi and s, such that bi is less than equal to X, for every X belongs to B. Because L is the collection of a lower bound, so the Y is set at less than equal to X, then Y will be the lower bound for B and or such why we satisfy this condition, will come in the class L, Okay? And this will be a nonempty set, this one thing is clear. Now every X, so if we look the L, L is the collection of those points, which are no less than equal to X for FB, it means every point of B behaves as an upper bound for else. So clearly, clearly every X in B, is an upper bound, is an upper bound of L. So it means L is bounded above, thus L is bounded above, bounded above.

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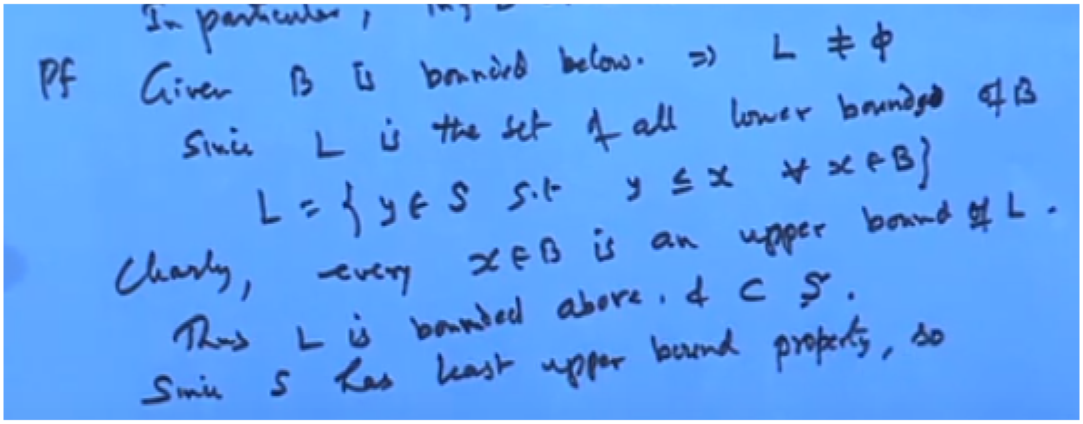

So L is a nonempty set, which is bounded above, it is a sub set of s. L, is a sub set of s, so bounded above and it is a sub set of s, is it not? As in or so we can apply the property, because s is an order set, having the least upper bound property, so by the property, since s has a, since s has a least upper bound property, so, so by this L will have a supremum value.

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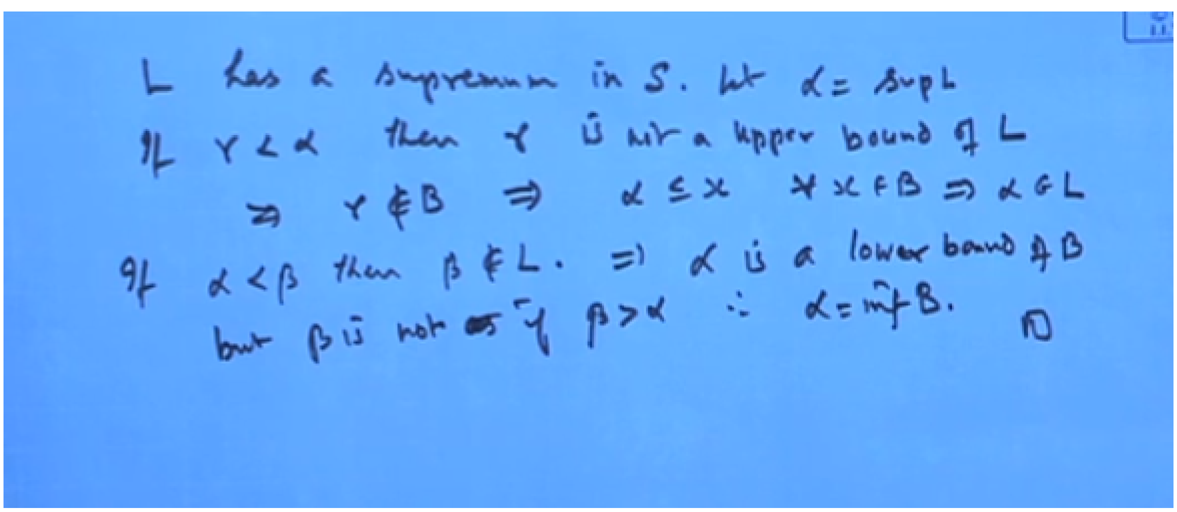

So L has a supremum value, supremum in s, in s, exist, supreme value will exist and let it be, let alpha is that supremum value of L, is alpha suppose, supremum value of this alpha, Okay?
Now if we choose gamma, if gamma is any number less than alpha, then, gamma is not an upper bound of L, upper bound of L. Because alpha is the least upper bound, so if we take any number lower than gamma, lower than alpha, then that cannot be have even a upper bound for it, otherwise gamma will be the least upper bound, Okay? So gamma if it is less than alpha, it cannot be an upper bound. Hence, what is our V? V is the set of those points, which are, for such that every point of this, is an upper bound for it. And here gamma is not coming in a per pound for L. So obviously gamma cannot be a point in B. Because all the points of B must be an upper bound, is an upper bound which we have shown, but gamma is not an upper bound of L therefore gamma cannot be a point of B. Okay? So what this follows, it implies that, it implies that, alpha is less than equal to X, alpha is less than for every X, belongs to B. Because any number less than alpha cannot be a point of B. So alpha will be the least number and then alpha will be less than equal to X. So this source that alpha belongs to L, Okay?
That is what alpha belongs to L. That is the supremum will exist and it is in L. Now if alpha, any number which is less than Alpha, any number beta which is greater than alpha, then beta cannot be nil, because alpha is the least upper bound and all the beta is greater than, so again it will not be in L. So once it is not in L, alpha is not in L, beta then what happened? That this beta, which is greater than alpha, in other words, that alpha will be the infimum value of B. Because then beta will be then in which, beta will be in B. So it is a alpha, beta. So this source alpha is a lower bound, lower bound of B, is a lower bound for B. But beta is not, what beta is not, as if beta is greater than alpha. It will not be lower bound for this, Okay?
Because, it is the least. Therefore, alpha will be the infimum of B and that proves that it is awesome. ? Okay? So that shows now. Having proved this thing we will come back again to the sets, where we are discussing the open sets and closed sets etcetera. We are the supremum value and infimum value will be required. So we need basically, we wanted to show that result, we are the supremum concept and infimum concept is required. So that is why all these things were taking.

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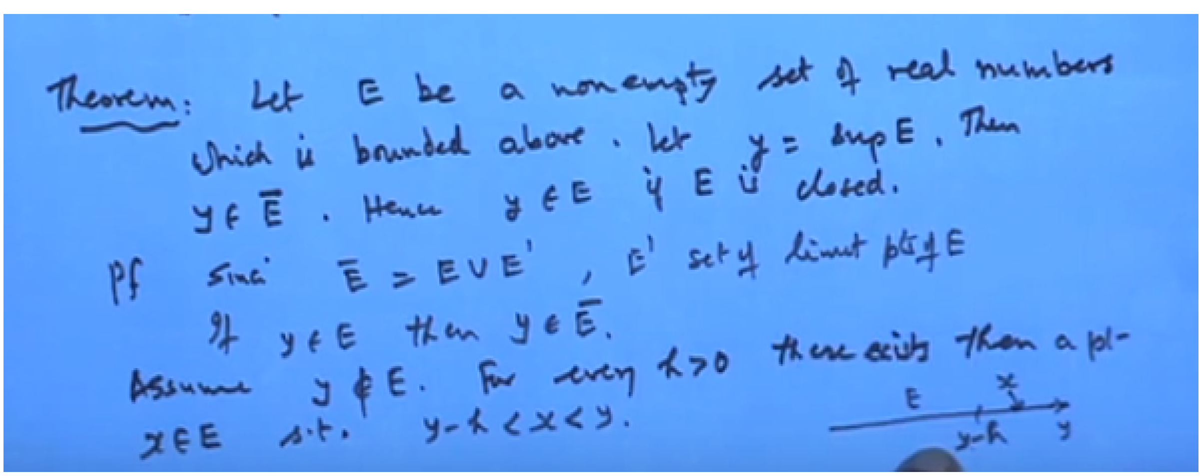

So this result, we wanted to show the result is let E be a non empty, non-empty set of real numbers, non-empty set of real number, which is bounded above, which is bounded above. Let Y is the supremum value of E, that least upper bound for e. Then the result says that Y will be a point of closure of e. Closure of means set e, together with this limit point, is the closure set. Okay?

Hence y belongs to e, if e is closed set. Okay? So obviously when e is closed set, e Y equal to e. So this result, second part follows immediately, then nothing. The first part we wanted to so first, so that e is a non-empty set of the real number, which is bounded above and supremum of e is b suppose Y. Then Y will be a point in Eval. Now since Eval is a, basically eval is the union of e and e days. Where e days' it is it the set of all limits point, set of limits points of e, connection of all the limits points of e denoted by E days'. Now if Y belongs to e, if Y belongs to e, y belongs to e, then obviously Y will be the element of eval. Because it is a union of this and Y bar, so, nothing to prove. So let us suppose Y is not in E, but Y is a limit point of e, assume we will show then, Y is an element of eval. So assume Y is not in e, but we wanted to shoe Y is in an e closure. It means y must be a limit point of e, so that we wanted to prove. So let us seek for, every H greater than 0. There exist, there exist, there exist then a point, there exist then a point, say X belongs to e, there exist a point x belongs to e, such that, such that, y minus H, less than X, less than y horse.

Why here this is our Set, say by E, this is the set e, the point by is not in E, it is outside of it. Then we can find for each H greater than 0, we can find at least some point, which lies in the interval, say, this is y, in between y minus s2 y. This point X will always be available.

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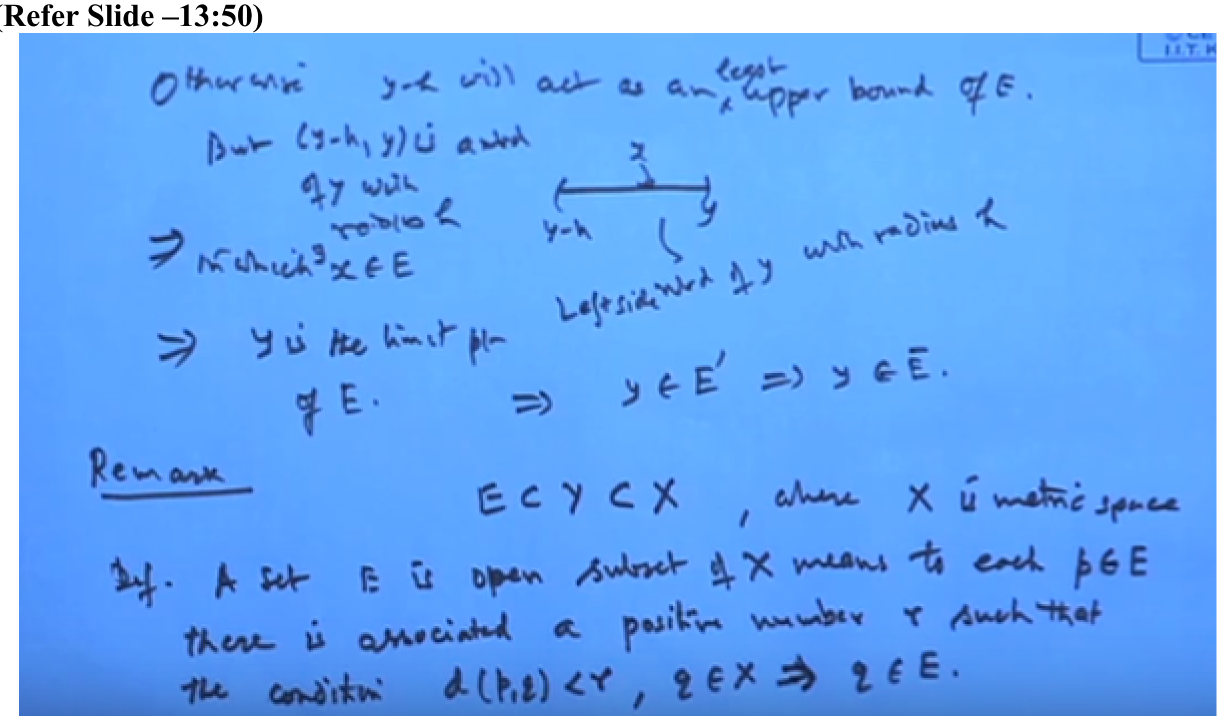

Otherwise if it is not so, then y minus s, will behave as a y minus H will behave as a upper bound for e. If it is not so and this is true, because otherwise, Y minus H, will act as an upper bound, least upper bound, for ,e as a at least upper bound of e, which is not true. Because Y is given to be the upper bound. So as soon as you take a number slightly lower than this, then this number must be available. It means in between y minus N and y, one can always get a at least one number of X e, which is available. But what is the y minus y n? But this interval y, y minus H, is it not in real x-lite? So is it not a neighborhood of Y? With radius say H? It is the left hand neighbourhood, left side, left hand left side neighborhood of Y and H.

So there exists a neighborhood of Y, which includes the point of X. It means every neighborhood of this Y, will include at least some point of e. So this shows that, but this is the neighborhood, is a neighborhood of y, with radius H, in which the point X is in E, in which there exists a point X in E. So this source that Y is the limit point of e. Because the definition of the limit point of the set a means, every neighborhood around the point y, every neighborhood to buy, how small radius maybe, must include the points of E. And this is true here, that if we take any neighborhood of y, at least one point X is available. The otherwise if it is not available, th en it will contradict to the fact, that Y is the supreme value of e. So if it is a limit point, then y must be the point in the closure. A days hence, by using closure of this set. So this proves the result.

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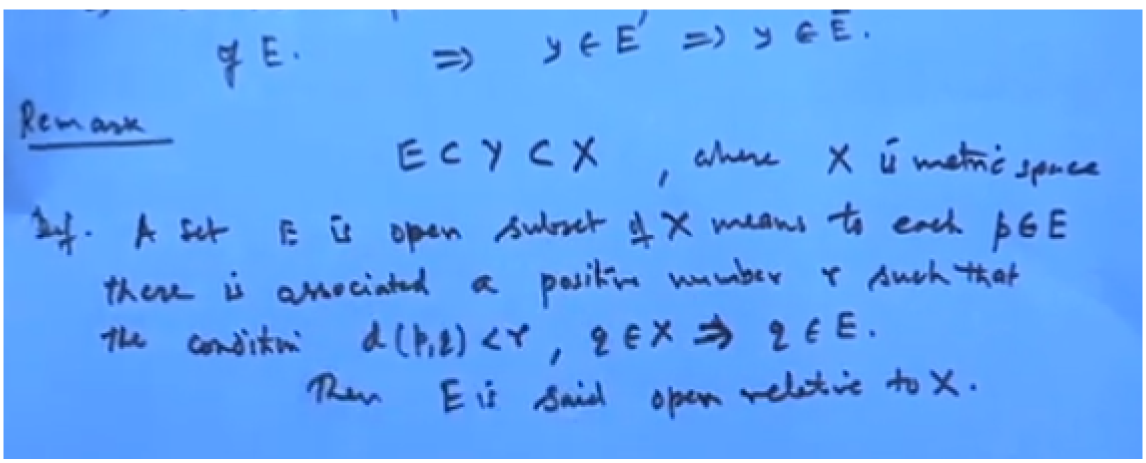

Now remark we can see. We know, if E is a subset of y, is a subset of X, suppose, Where X is a metric space. Then we have seen that set may be open in X, may be open in y and may not remain open in X. This we have seen just like a open interval A B, which we have seen, it is open in r1, but it is not open in r2. So in case of the open set or closed set, the space which encloses the set is important.

Where the set is he is open. So that is why we can introduce the concept of, an opposite relative to the space, relative to y all relative to X. So we introduce here to definition, that one is, a set E is open, set E is open, is open, subset of X, means to each P belongs to e, belongs to e, that is associated, associated, a positive number, a positive number r, such that such that the condition, such that the condition, D of P Q, is less than R, where Q belongs to X, imply that, implies that, Q Genie. Then we say e is open with respect to X. The open subset of X means, then e is said to be open, open relative to X, relative to X. Similarly we can say y, since y is also a metric space, under the same metric topology day, so e may also be open, with respect to Y.

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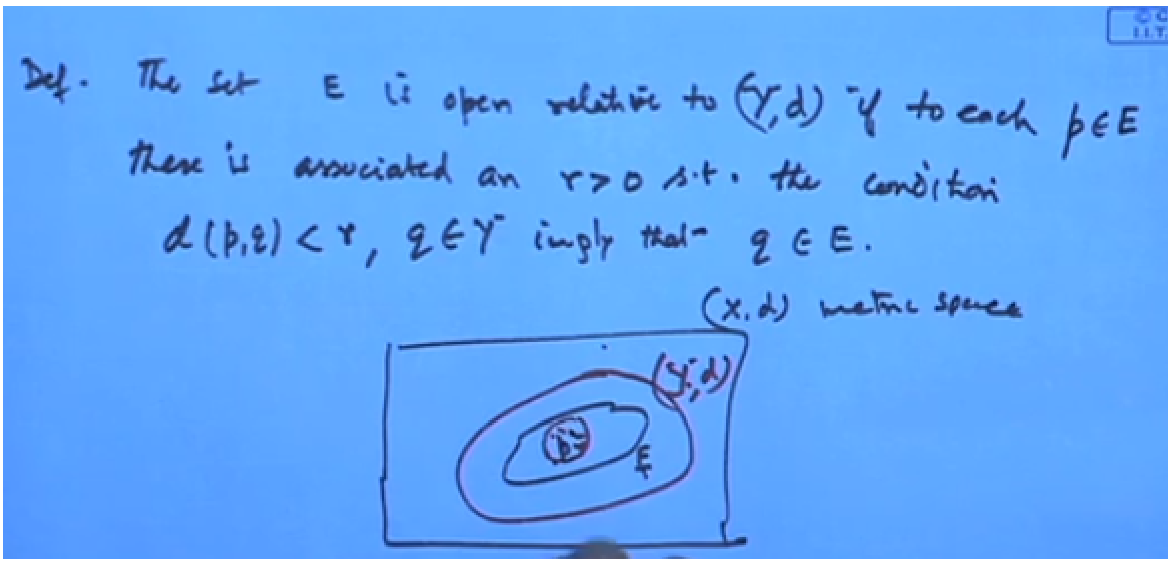

Then we say define, we define that, the set E is open, the set E is open, open, relative to Y, relative to Y, by D, by D, is a metric Space, by D ,if to each. if to each here also X D we will, write XD, is a metric space. Okay? Relative to X y, relative to Y, if to each P E E, belongs to e, there is, there is associated, there is associated an R, greater than 0, such that, such that, the condition, condition D of P Q, is less than R, Q belongs to capital Y, imply that, imply that, Q is in E. Then we say it is, it means what? Suppose we have this set X D, which is a metric space, a set e, this is a set e, we say it is open in X. Means, that if we take any point P in e, then one can always find out neighborhood around the point P, or a ball, centered at P suitable radius say R, such that all the points inside this, is a point of E, is a point of e. Then we say that E open. In all the points Q, Q, which are of X, all the point Q, which are in X. If they are, they have all the point of E, then we say, it is a open. It means that every point is an interior point, with respect to X T. But when you say, say this is our bi, bi is a subset of X, so it is also a metric space, with respect to D. Then we say that E is open, with respect to Y. Now here, when you draw the

neighborhood around the P, then the point Q which you are choosing, will, must be a point of y. Okay? Because you are not getting the nes, of course the point of Y, is also the point of X, but there may be some point ,which all in X but not in y. So this relation, when the distance of P Q, is less than R and Q belongs to Y, implies if Q is in E, then we say e is open in bi. So as if, there is no X, only e is a subset of bi and E will be the open set in bi, every point of P, is an interior point, with repect to y. That is all. Then we say e is open relative to Y. Similarly e is open.
Now this has been shown, that a set e, may be open with respect to Y, subset, may not be open with respect to the large set, and that the examples we have seen. However, in case of the compact set, we will see this result, this restriction is not there. So that is friendlier, than our open set or closet. So that is the model. Okay? So we will see that.

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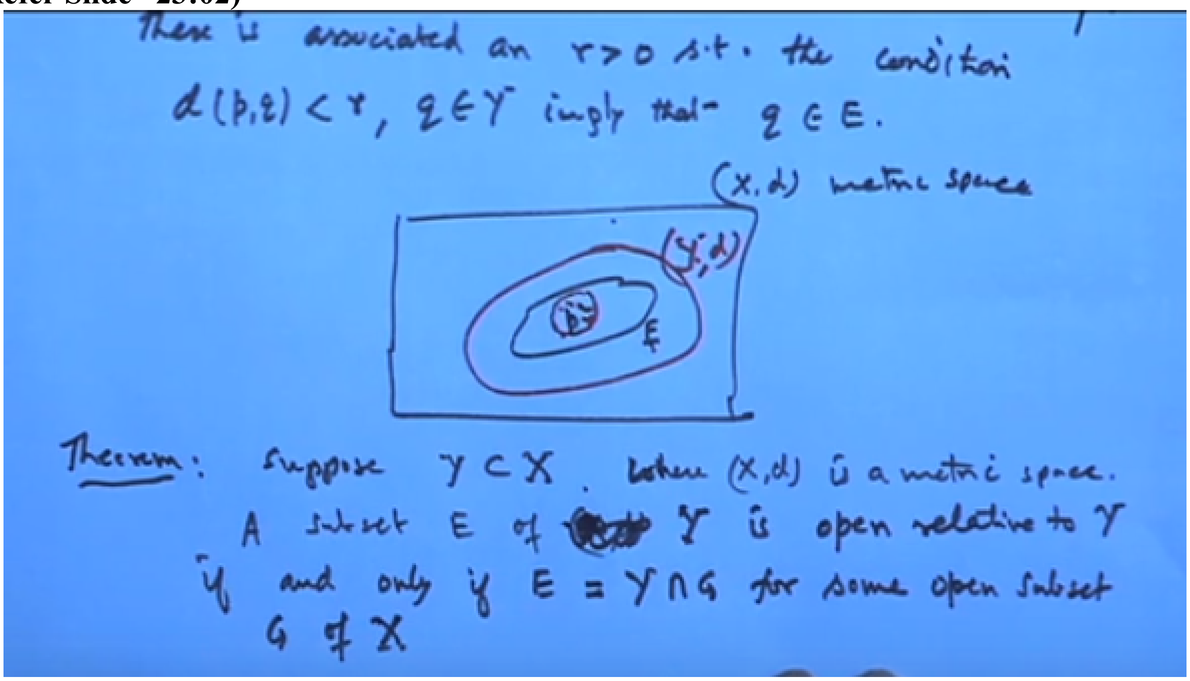

Before going for the compact set definition, we have one more results that result shows, what will be the form of the open sets, in the relative case. Suppose Y is a, non-empty subset of X, X D, be a metric space, where XD, is a metric space, let XD, be a metric and bi be a non-empty subset of X, a sub set E of X, a sub set E of XD, of XD, is open, a subset a of XD, a subset e of bi, I am sorry. So I will subset e of YD, YD, let it YD, a subset e of bi a subset a of Y, is open is open, relative to Y, relative to Y, if and only, if, if and only if, if and only if e can be expressed as or E can be written as, Y intersects and G for some open subset, G of X. So what this result says is, let XD be a metric space and Y is a non-empty subset of X. So bi under the same metric D, will also be a metric space. And suppose E is a subset of Y, then we say a subset a of Y will be open with respect to Y or relative to Y, if e can be expressed in to this form, for some open set G of X ,if and only. That is if e is of this form, then e will be a open set, subset of bi and if E is open, then it can be expression to this form. So vice versa. Let us see the proof of this.

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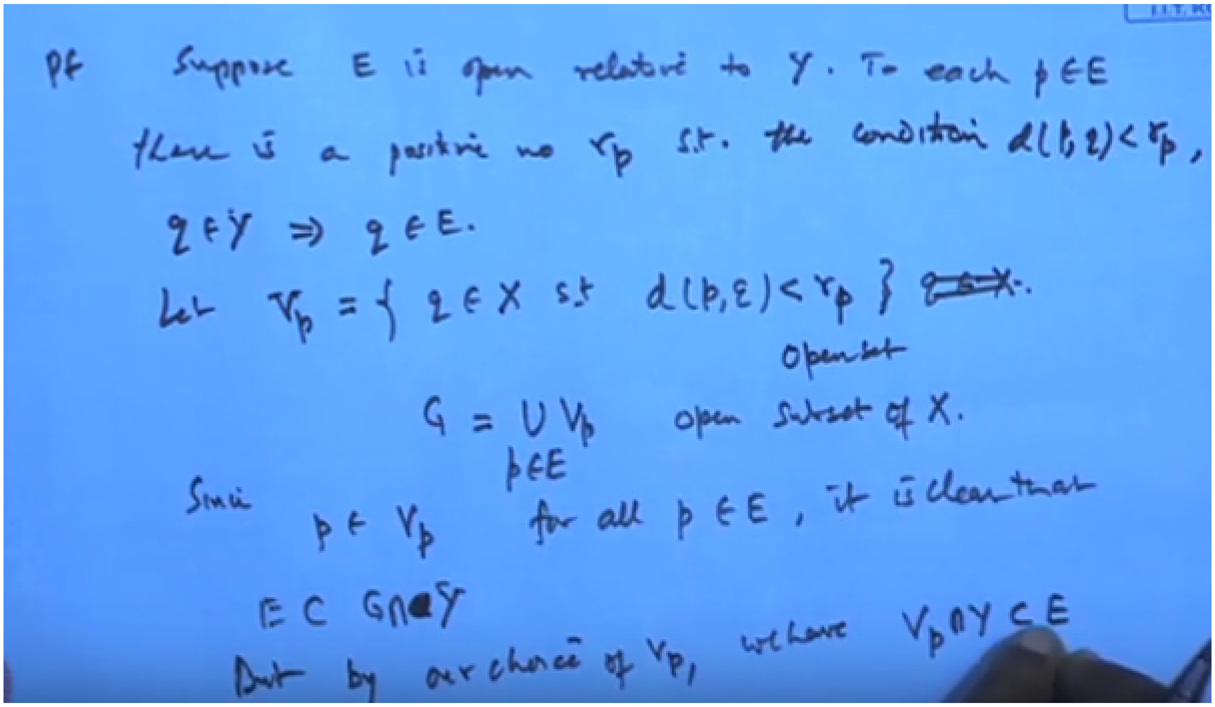

Okay, suppose E is open related to Y, E is open, relative to Y, Y. We wanted to show each will be of this one. So by the definition of the relative to Y means, to each P belongs to e, there is a, there is a positive number, positive number, say, our r P, such that, such that the condition, condition D of P Q, less than r P, where the Q belongs to Y, implies, implies that Q will be in E. This is by definition, when e is open, relative to Y, Okay? Now let us consider, B P, as the collection of all such Q, belongs to X, such that, distance from P Q is less than our R, R P, less than R P, where Q is in the elements of bi, let us Q is in X, Q is in X, so this is already there. Let us see. Now obviously this is a Neighborhood, so once in neighbourhood, it is an open set, it is an open set. And G, if I take the union of all these B P, where the P belongs to set E, then this collection of the open set, will also be able so. So it is an open set, is an open subset of X. Clear? Nothing to. Now since P is in the neighborhood VP, which centered P and radius say R P, P is in this, for all P, belongs to e. This is by construction.

We, because P is the center of this Neighborhood. So it is clear that, then it is clear, that e will be contained in G, which is G intersection, G intersection bi. Because this B P, P is a set in E and all the points in E, belongs to B P and B, G is the union of VP, so E every point of E is in bi, as well as in G, as well in G. So it is intersection of this thing, is obvious. By our choice, but, by our choice of B P, we can say, we can say, we have that VP intersection bi, is a subset of e. By our choice means, because we have B P, constructed like this way, set of all such that this is there is a positive such, that this one is, so when it restrict Q to Y?

Then all these points basically, they are the points common to e, intersection with this and contained in E. So by our choice, because this will be the set, to each P there is a project, because E is open, because E is an open set. So this entire thing is available in e. Because E is giving to be an open set, relative to this. So this is the way our choice. Therefore, four hours or for every P, so that, so that G intersection bi, take the union of this. G intersection by is contained in E. And hence e will be equal to G, intersection Y. So one result is complete, Converse the, just one more.

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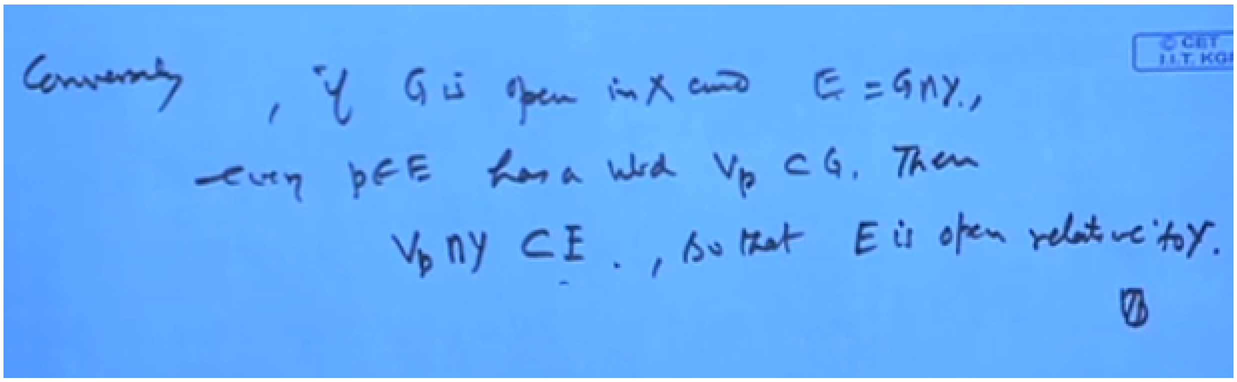

Conversely, if G is open, G is open in X and E each of the form, G intersection bi, then every P, belongs to e, every P belongs to e, has a neighborhood, V P, this is totally contained in G. Because G is open and e of this form, so e will also be, for any P belongs to e means, it will be in g and g is open, so neighborhood must be available in G. Then the G, Y intersection bi, neighborhood intersection bi, will be contained in E. Okay? B because P is in there and G is this form, so intersection will be available in e. So that E is open, so that E is open elative to Y, bi relative to Y. And that is prove the results.

Okay. Thank you very much. Thanks.