## Model 2

## Lecture – 8

# Ordered Set, Least Upper Bound, Greatest Lower Bound of a Set

Hello, so we will discuss the least upper Bound, greatest amount of asset and compact sex today.

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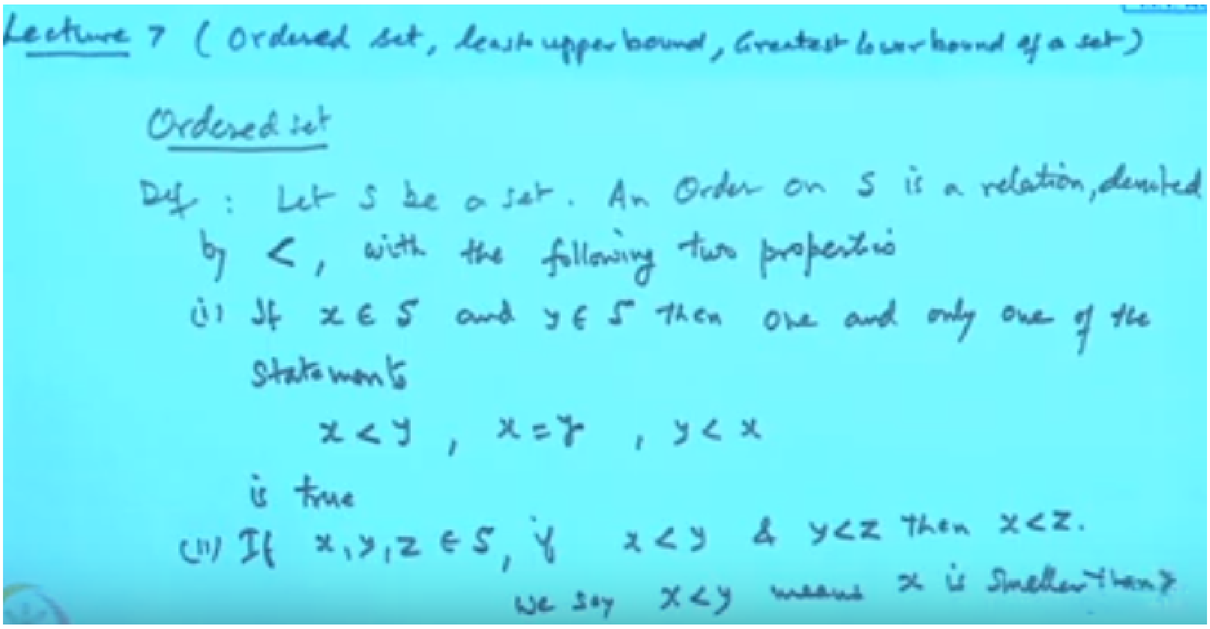


So this requires the concept of ordered set. ordered set first, we define like this, let s be a set S be a set an order on S and order on S is a relation is a relation denoted by sign this sign which we called net always a less than sign or a smaller sign with the following, with the following two properties. The first property is that if X is in S and by it's in S then one and only one only one of the following statement of the statement of these statements that is X is less than Y, X is equal to Y and Y is less than X is true is true. And second one is say if X, Y, Z they are the elements of s and if X is less than by and Y is less than Z, then X must be less than Z. so if a set s together with this operation which we call its together with this relation less than sign satisfy these two property then we say that this is an ordered on S. this is an order S. this sign less than sign we normally say we say X related to the means X is, is smaller than Y and the negation of this the negation of this

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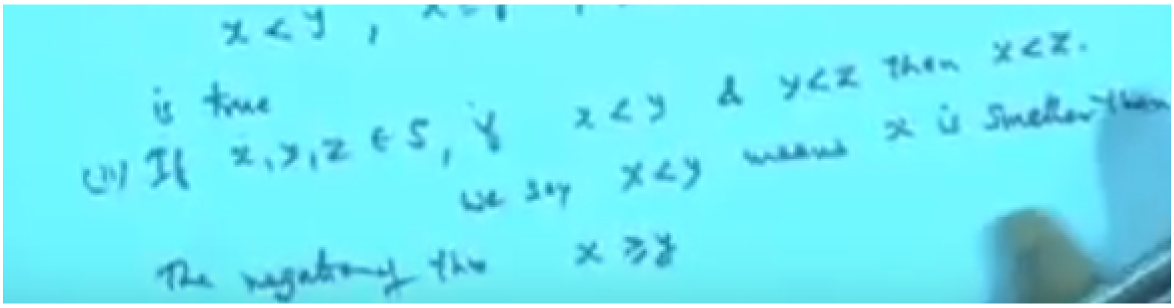


is X is greater than or equal to Y. this is the negation part of it, means X is greater than equal to Y, the negation of this will be X is deeply less than Y. so that will be the sign for this now obviously when we say the set of rational number or set of the real numbers then this order is defined. One can identify two real number or two rational numbers one can say which one is no smaller than the other or whether they are equal or whether one is greater than the other like this.

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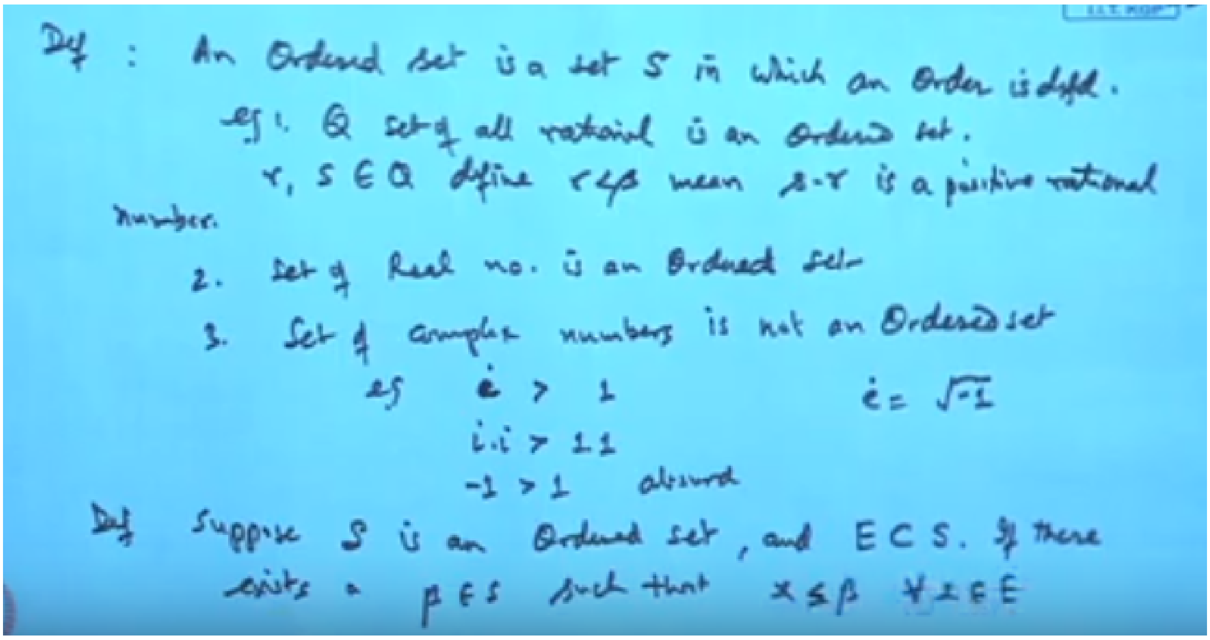


Then the set order set means, a set an order set S order set is a set S is a set s in which An order is defined. in which an order is defined. for example set rational number, set of all rational numbers is an ordered set, is an ordered set because if we take r and S, suppose these are two rational number and define the relation r is less than S means S minus R means s minus R is a positive rational is a positive rational number, is a positive rational number. Then obviously it was satisfied these two property which we stated so set of rational number is an Ordered set. Set of real number is an order set. Set of real number is an ordered set. Ordered set. However set of complex number. Set of complex number is not complex number is not an order set. We cannot introduce the order between the two elements of XC complex number set of all complex number because for example if suppose I take the complex number e and 1 okay and when we say e is greater than 1, ok e is the ideal element say square root of minus1. This is e or i, I will say i if you i, i this is a complex number so i is greater than 1 it means is the positive we are assuming. So i into i is still greater than 1 into 1, that is i square is minus 1 is greater than 1, which is a absorb, it means our ordering which you have introduced is not correct. Similarly when can show whether if i is less than 1 we can again lead a contradiction and like this, so we are unable to introduce the order in the over the set of all complex number that is the set of a complex number is not an ordered field. field we will discuss in

The algebra or some. Okay then. we are interested in particularly in defining the least upper bound and the greatest lower bound so let's see first what is an upper bound alone suppose S is an ordered set and E is an non-empty subset of S now if there exists some beta there exists as beta in S such that all the elements of E that is x belongs to E is less than or equal to beta and this is true for every X belongs to E.

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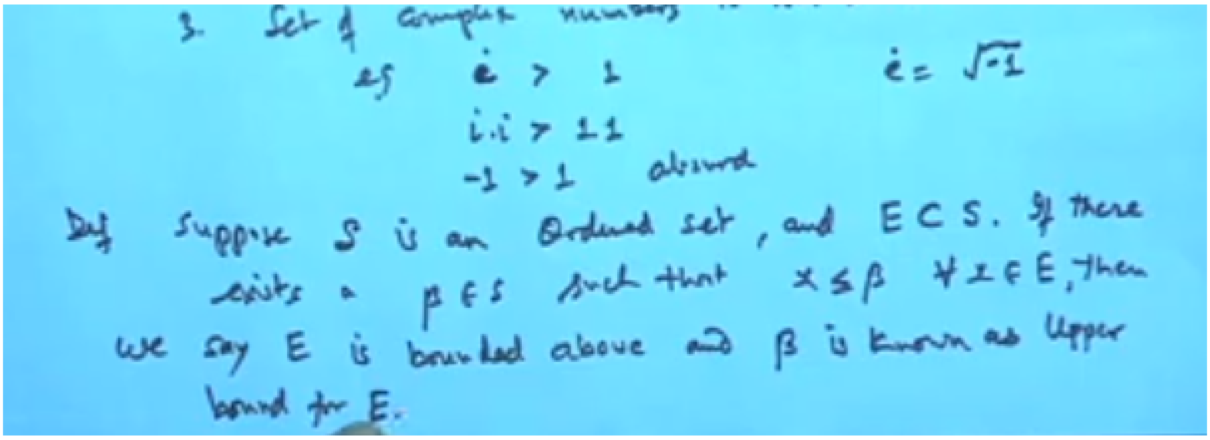


Then we say then we say E is then we say beta is bounded above is bounded above bounded above. okay and beta is an upper bound hole and beta is known as upper bound upper bound for E now obviously beta is an upper bound there are many infinitely many real numbers will be available which will be an upper bound for E. any number which is greater than beta will act as an upper bound for E okay so we are interested in the get list upper bound for it.

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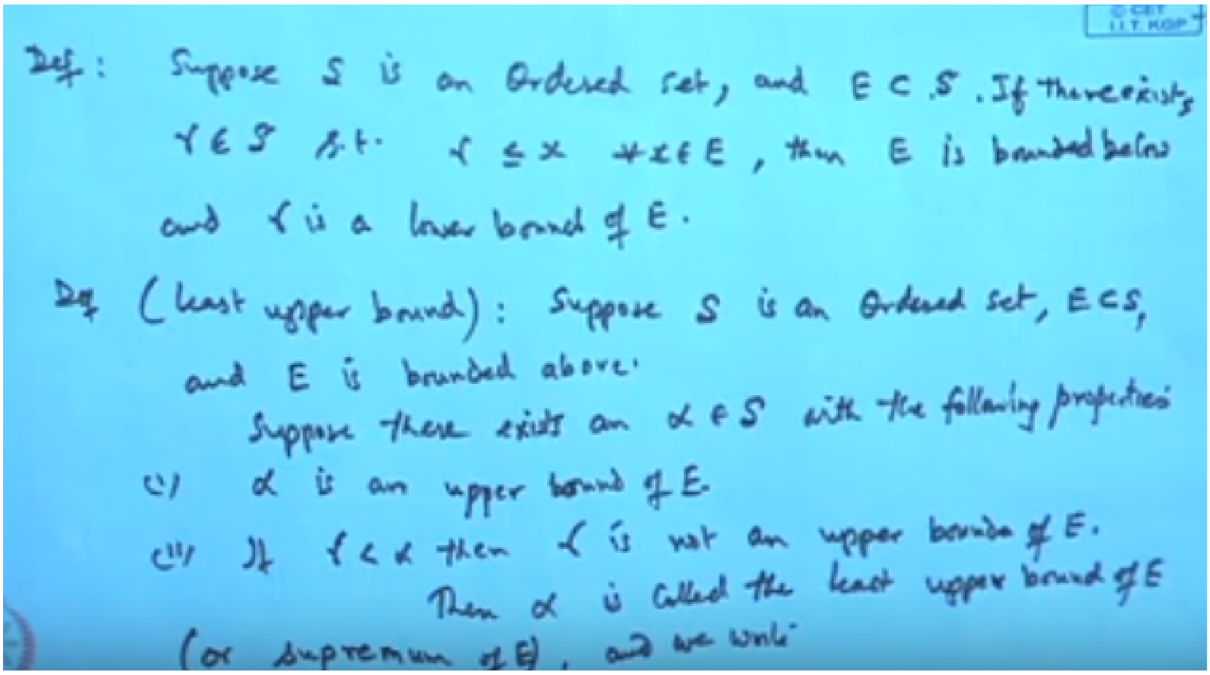


The same case happens if it is lower bound we define the lower bound in a similar way. Suppose S is an ordered suppose S is an ordered set, auto-set and E is a subset of S nonempty subset of S if there exists if there exists number say gamma belongs to S such that gamma is less than equal to x for every x belongs to E, then we say E is bounded below and gamma is the lower bound is an lower bound is a lower bound of E. so there again there will be many lower bound available as soon as you take any number less than gamma which will also behave as a lower bound, so we will be interested in knowing what will be the greatest lower bound of the set E. so we introduce the concept of the upper bound and lower bound as follow. This is the concept of least upper bound. Suppose S is an ordered set, S is an ordered set and E be a non-empty subset of S and also assume is bounded above E bounded above, now suppose there exists suppose there exists an alpha belongs to S with the following property alpha is an upper bound is an upper bound of E, of E this is the first property and second one is if I take a number slightly lower than alpha then it should not behave as an upper bound and second is if gamma is any number less than alpha, then gamma is not an upper bound, a upper bound of E. Ok so easy bounded above and alpha is such a number which is an upper bound of this but if we take a number slightly lower than alpha then that number will not behave in upon it means alpha is the least upper bound for E, then alpha is called then alpha is called the least upper bound upper bound of E. and we denote this thing is and also or some we also say it is a supremum, supremum of E ok and we write is we write that least upper bound of set E each alpha all it's the same as when we say supermom of E.

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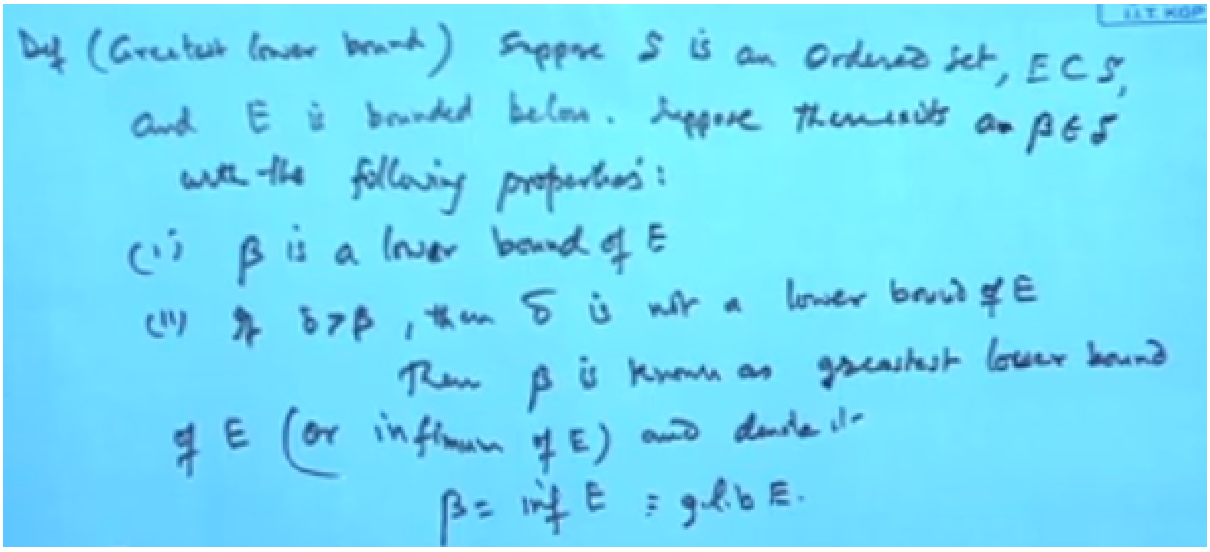


In the similar way we define the greatest lower bound. in a similar way we can introduce the concept of the gate test lower bound of E. so what we assume is suppose S is an ordered set S is an ordered set, E is a subset non-empty subsets of S and E is bounded below bounded below. suppose there exists and say beta belongs to S, a beta belongs to S with the following property with the following properties, the first is beta is a lower bound is a lower bound of E, is a lower bound of E. second one is if a number if I choose if a number say delta which is greater than beta, then delta is not and is not a lower bound of, is not a lower bound of E. okay so if we take any number delta slightly well then there is will not be lower bound. then this beta, then beta is known as, there is known as greatest lower bound of E or we can also say it is the infimum, infimum of the set E and we denote this as denote it as beta is the infimum of the set E, or is the same is the greatest lower bound of E. so this is the way we define the greatest lower bound and upper in the least upper bound greatest one and least one.

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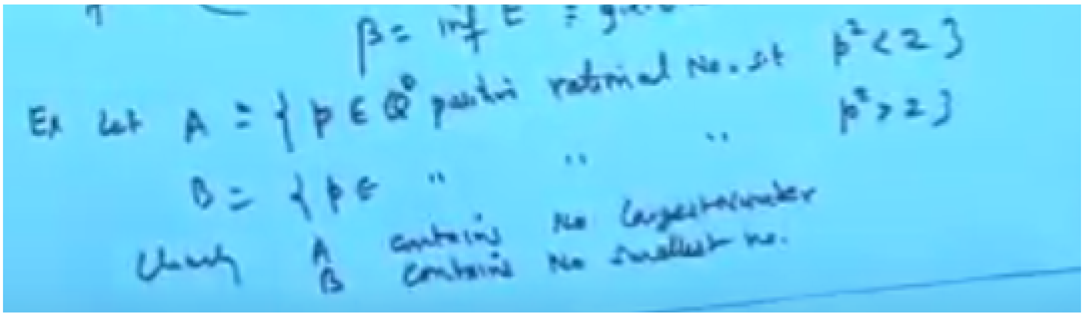


Now let us take an Example, suppose I take the set a let A be the A be the set of all positive rational number P, is positive rational number, so it is positive rational numbers, such that P square is less than 2 and let B is the set of all positive rational numbers such that P square is greater than 2, P square is greater than 2. now if we look that A and B is the set of all positive rational number whose square is less than 2 and B is the set of all positive rational more square is greater than 2. obviously A is bounded above in fact all the elements of B will be the upper bound will be head will act as an upper bound for A, but A does not have the least upper bound because we can't get the rational number which is for which we can say is a really a number which is a least upper bound for A. similarly for the B, if you look the all the elements of B satisfy this condition then it has a lower bound and all the elements of A behaved as a lower bound of this plus all rational number which are negative or zero, will behave is a lower bound for ease but neither A nor B has an upper bound and the greatest lower bound okay. So this will show that a contains no largest number and B is clearly, A Contains no largest number, while the B and B contain no smallest number. Okay? so in this case the greatest lower bound A has no greatest we had no greatest lower bound and has no largest number.

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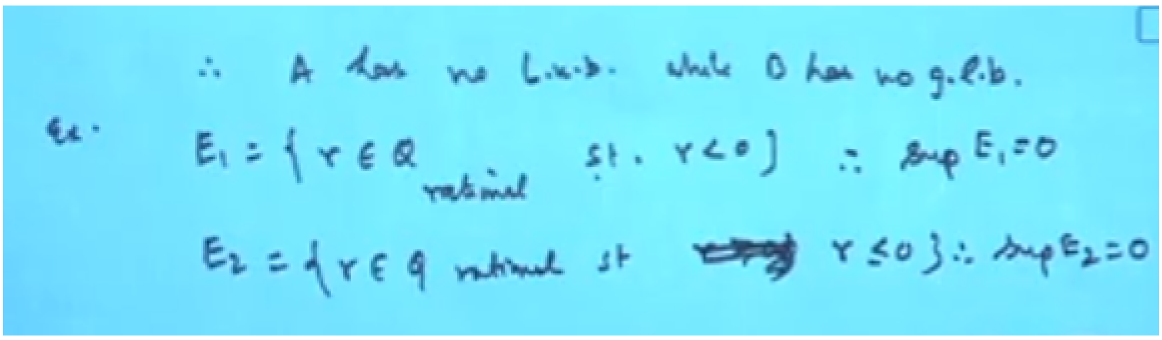


And this shows therefore we can say A has no least upper bound least upper bound where, the B has no greatest lower bound. That's obviously okay let's take another example. Suppose I take the set of rational numbers, let E1 be the set of all rational number r belongs to Q, Such that r is strictly less than 0 and E2 with the set of all rational number Q rationales', such that r is greater than equal to 0. Okay r is sorry less than equal to 0 same such that r is less than equal to 0. Suppose I take this thing, then the set of rational number which are less than

Zero so obviously it has an upper bound 0 here also has upper bound 0. It means the supermom value of E so therefore supermom of E1 will be 0, supermom of E2 will also be 0, but you see that supermom value of even does not belongs to even well the supreme value of E2 belong stay s so it's not necessary when we say the least upper bound or get a lower bound then it's not required it's not necessary that that supremum value will be a point of set. It may or may not be the point of set that we have of shortly. Okay similarly but both are having the same okay? similarly.

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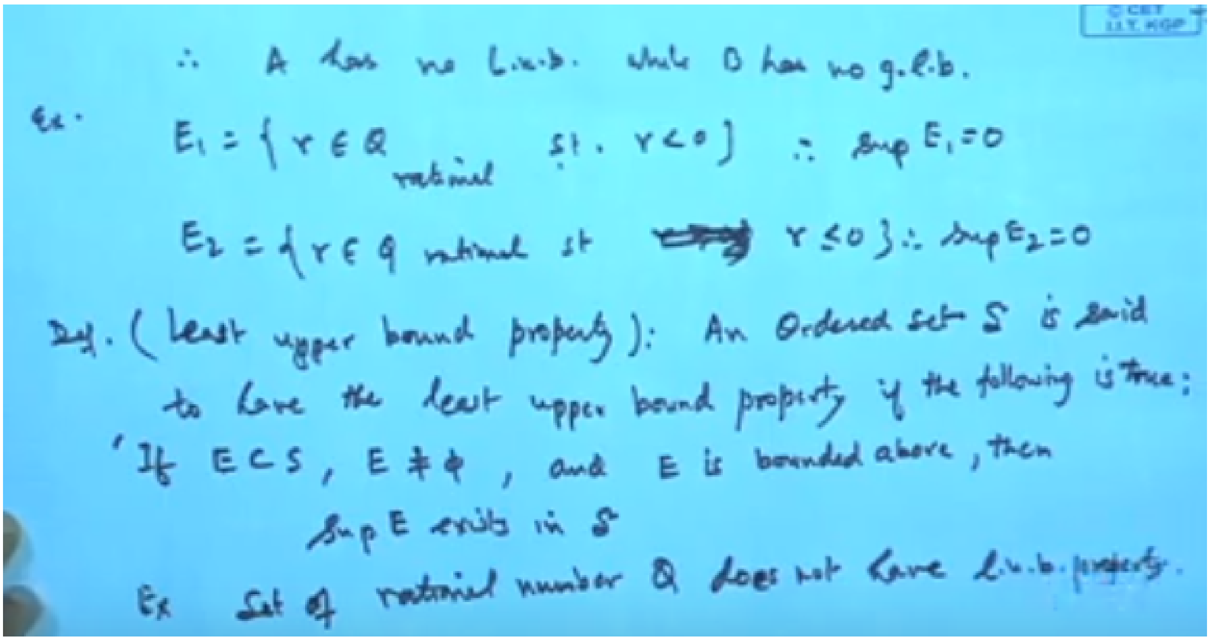


Now another interesting property which is in connection with the order set, is this is known as the, least upper bound property. Least upper bound property, what is the least upper bound property? An ordered set, an ordered set S is said to is said to have said to have the least upper bound, least upper bound property if the following is to is true. falling is to what therefore if E is a non-empty subset of S is not empty, is a non-empty subset of S and E is bounded above. Bounded above, then the supremum of E that the least upper bound of E will exist and exists in S, exist in , so this is the least upper bound property of a set. An order set S is said to have a least upper bound property, if the following is true. That is if we take any subset non-empty subset of which is bounded above, then Supremum may exist in S then we say this set S is a least upper bound property. if for any set this supremum does not exist then the set will not have a least upper bound property. for example set of rational numbers set of rational numbers that is which is noted by Q, does not have does not have least upper bound property, bound property.

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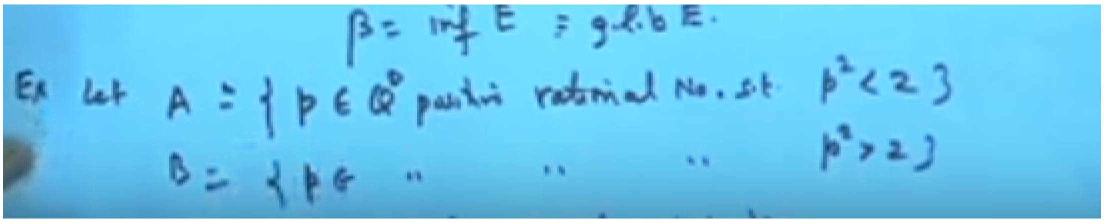


X` and that we have seen already with this example because basically A and B these are the two subsets of the rational numbers and neither the A nor B has an upper bound is it not? No other agent does not have a upper bound B does not have the lower bound for it so basically the set of rational number you can say does not have a least upper bound property. now there is a relation between the greatest lower bound least upper bound and the least upper bound property in fact it is shown that if the set is having the least upper bound property then it must have a greatest lower bound property also and that can be judged in the next theorem the relation between the nation between the least upper bound greatest lower bound and this.

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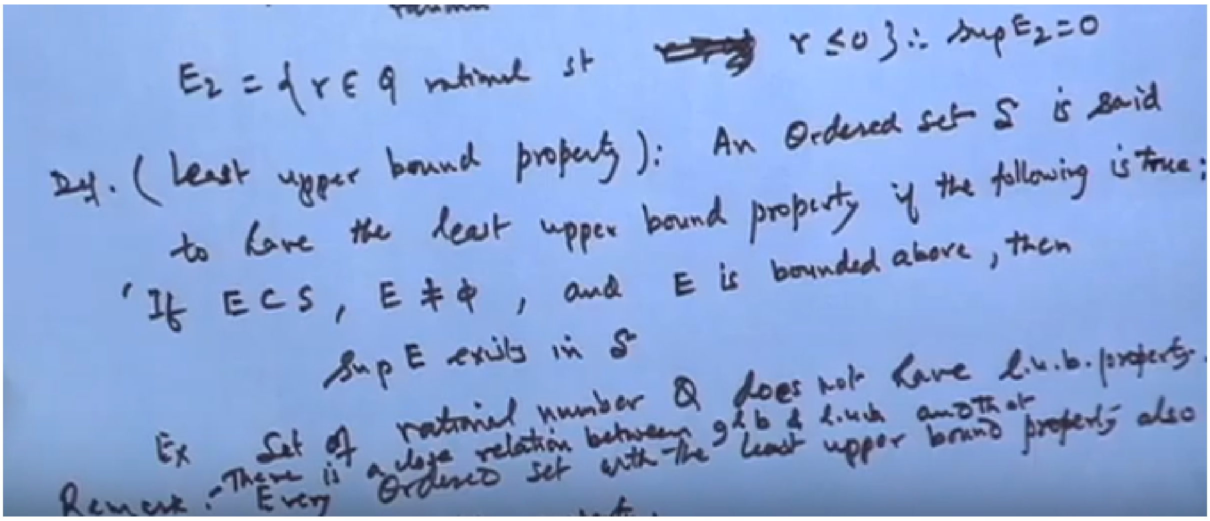


So, theorem says or you can before this, you can write the remark. I would write the remark here it's every Order set, there is a relation between the gate result one that every order set Order set with the least upper bound property, upper bound properties. With the Least of also has, there is there the remark is, there is a close relation close relation between greatest lower bound and the least upper bound and that and that this that every order set with the least upper bound property also has the has the greatest lower bound property.
This can be seen with the help of this result.