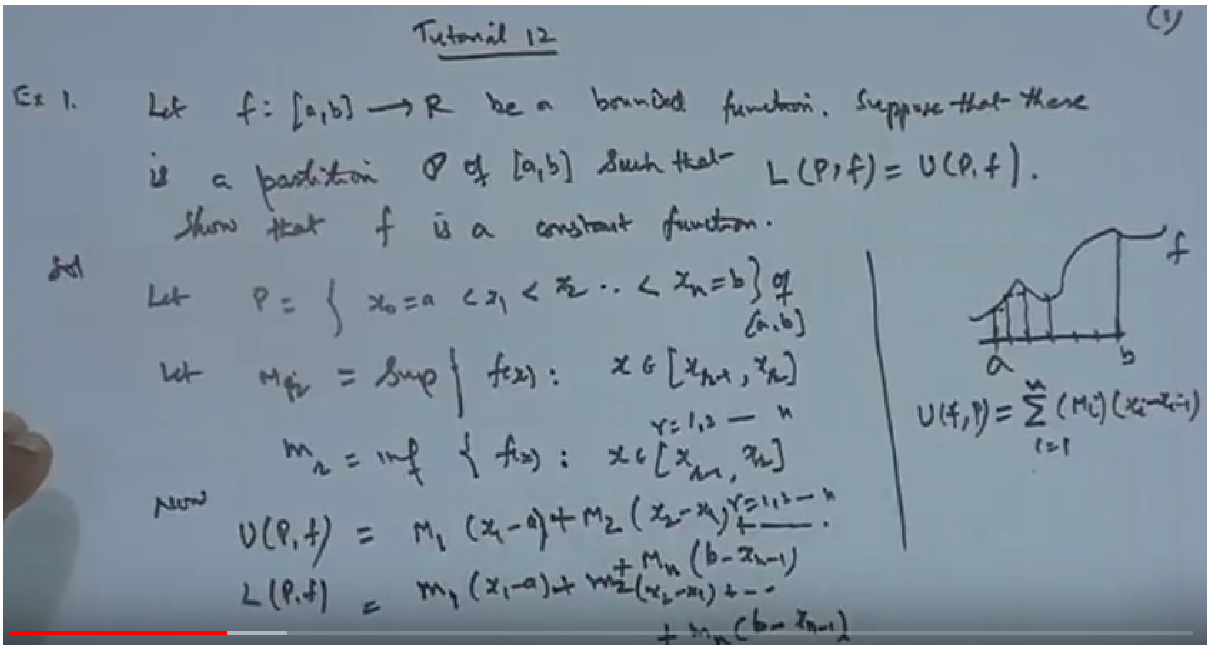
Model 12

Lecture 73

Tutorial - XII

Okay, so this is the last tutorial class, the problems are based on the Riemann's integration and related topics.

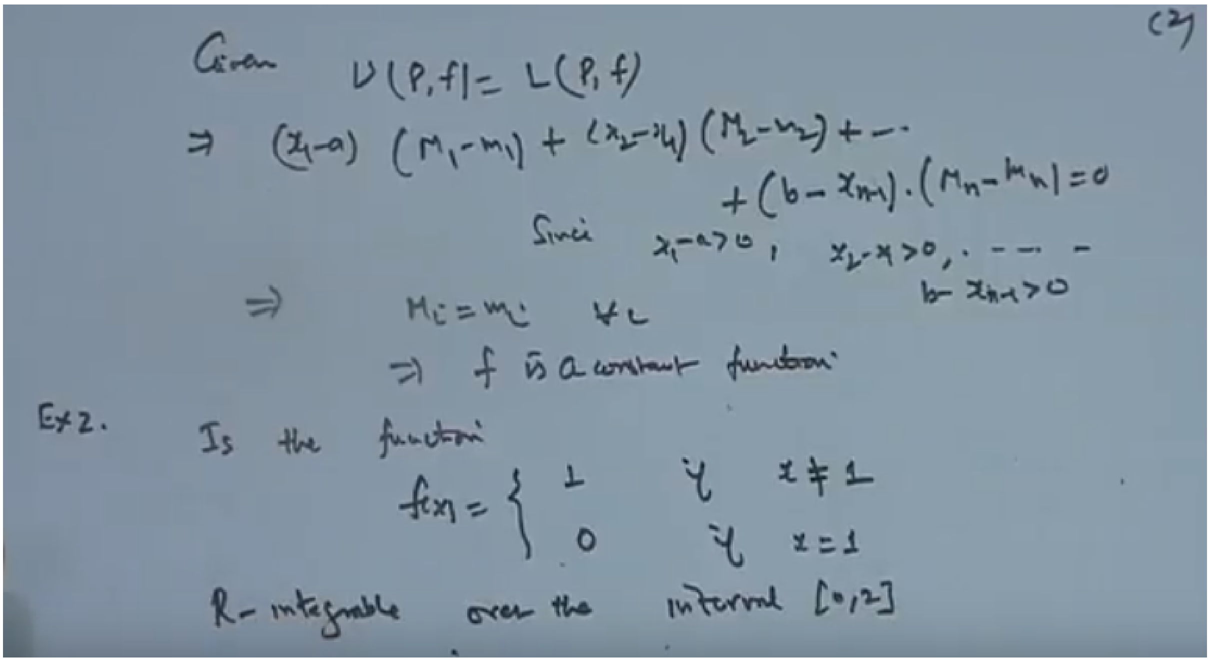
(Refer Slide Time: 00:26)



So first problem is, let f, is a mapping form or a function from a,b to R, be a bounded function. Suppose that, suppose that, there is, there is a partition, P of the interval a, B, such that, such that, the L Pf, is equal to U Pf, the lower sum of the function f, over the partition P, is the same as the upper sum of the function f, over the partition P. then so that the function f is a constant function, so what is the upper sum and lower sum? the upper sum when we partition the interval a, B into sub intervals and this is the function f, which is continuous, define over this interval a B, then we get this the some intervals will the lower values and upper values, say small mi and capital MI, then upper sum, of the function f, over the partition P, or is nothing but summation of mi, into mod of Xi minus X i, minus 1 and i say 12, 12 n.., it's a partition following. similarly the lower sum mi, is replaced by the small mi, we get this, so this given the function f is a bounded function and there is some partition, for which the lower and upper sum are the same, therefore we have to show that F is a constant in that case the F must be a constant function. So let P be the given partition.

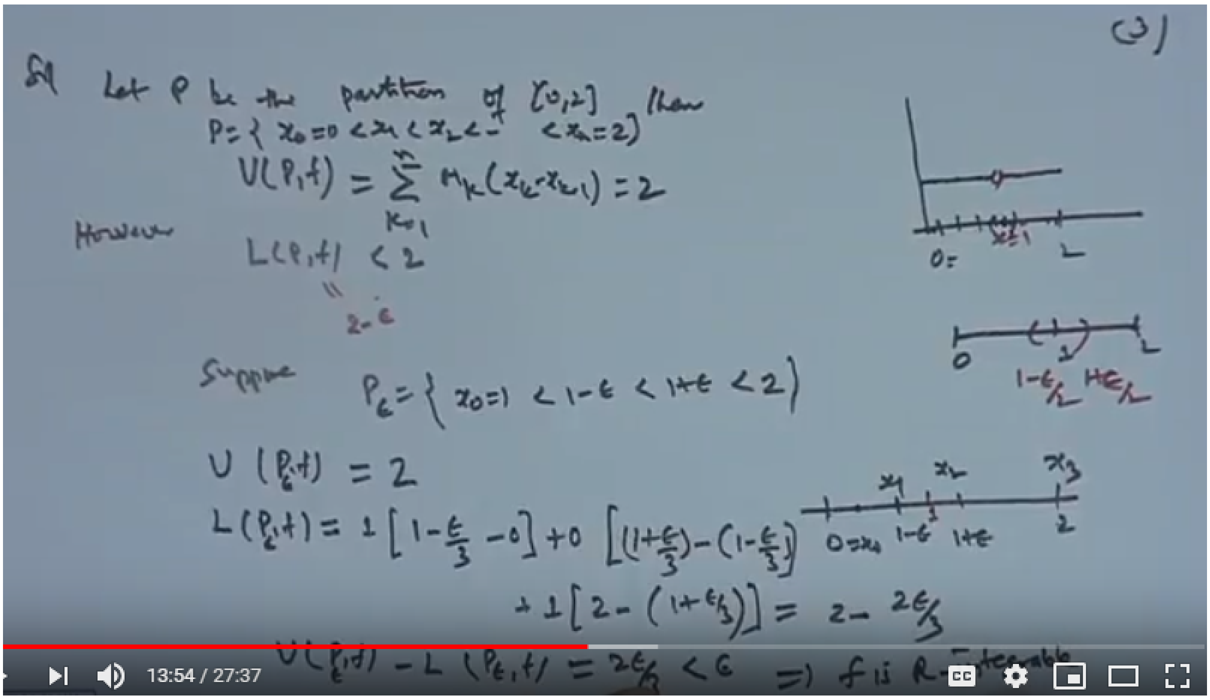
As X naught which is equal to a, less than x1, less than x2, less than say xn, which will be this is a partition of the interval of interval a, b, partition of. let Mi, Mi or M R either supremum of the function FX, when the X, belongs to the interval, xr minus 1 xr, xr when R is 1 to N and small mr, is the infimum of the value of the function FX, when X, belongs to the interval X R minus 1, XR, means some intervals, when r is 1 to n. so upper sum of the function f, over the partition P, is nothing but M1, X1 minus a, X 1 minus a plus M2, X2 minus X1, and so on, plus Mn, XB minus Xn minus 1 and the lower sum is, equal to small m1 a small m1 into X 1 minus M, plus a small M2 into X2 minus X1, and so on and it's more mn, B minus Xn minus 1, so this is the lower in a sum, it is given the low in a sum, is the same

(Refer Slide Time: 05:12)



Given, given that lower sum and the upper Sum, is the same, over the partition P. So we get from here is, X1 minus a, into M1 minus small m1, plus X2 minus X1, into M2 minus small m2 and so on, plus b minus Xn minus 1, MN minus small mn is 0, but M1, Mi is a upper value, maximum value of the function, over the some intervals and Mi, M1 is the smallest value, so M1 minus, M1 and this X1 minus a, these are the partition X1, X2,.., xn, since these are X1 minus a, is greater than 0, X2 minus X1, is also strictly greater than 0 and so on, all these V minus xn, minus 1, is also greater than 0, so then the sum is, 0 implies, that each of mi, must miss call to small mi, for each i. it means the function f, has a maximum value and the minimum value, same throughout the any subinterval, so f is a constant function. f is a constant function. That's what we wanted to show it, okay. Next example is, is the function, is the function FX, defined as 1, if X is not equal to 1 and 0 if X, is equal to 1, is the function Riemann integrable? Is the function Riemann Integrable, over the interval 0 to 2, 0 to 2.

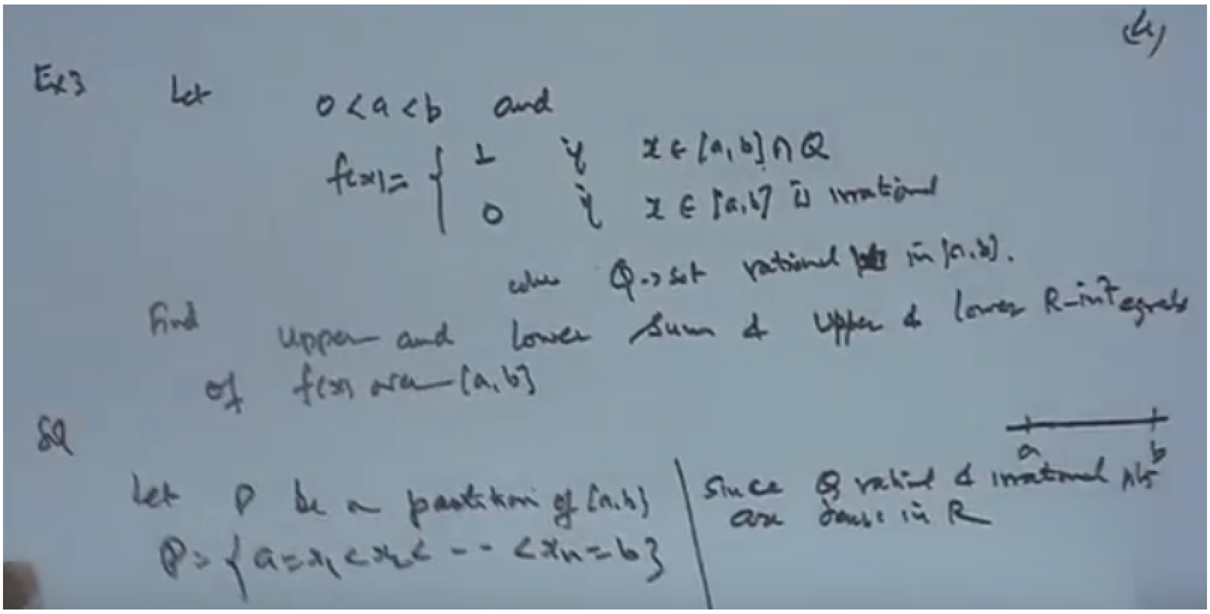
(Refer Slide Time: 07:40)



now the solution of this is, so what is given is the function FX, is given like this, over the interval 0 to 2, the function when X, is not equal to 1, it is 1, so the function is like this and here this is like this and when X equal to 1, this is 0, so this is X equal to 1, it is you know it means function has a discontinuity at the point X is equal to 1, function is not defined, at X equal 2, it is a sudden jump, at X equal to 1, ok so it is this continuous at this point now if P is a partition, let P be the partition, partition of the 0 1 interval, 0 to 2 interval, then the obviously the upper sum of the function f, over this partition, that is k equal to 1 to n, MK XK minus, XK minus 1, is 2, why? this P is a Partition, so partition P, is equal to X naught, which is equal to 0,less than x1, less than x2, less than xn, which is equal to 2. now in between there is a point 1 also, so we can have the partition like, now MK over any subinterval over any sub interval when you partition it, the upper value of the function or the supremum value of the function, over this partition, will always be 1, because when X, is not equal to 1, the value is coming to be 1 and here, also the value is coming to an except X, equal to 1, but when you take any interval any part is sub interval that even include the X equal to 1 the points are such where the function has a value 1, so upper some maximum value, mk, will all value one and the sum of these sub partition, is nothing but 2, therefore the upper sum will always be 2, however the lower some PF is strictly less than 2, why is it strictly less than 2?

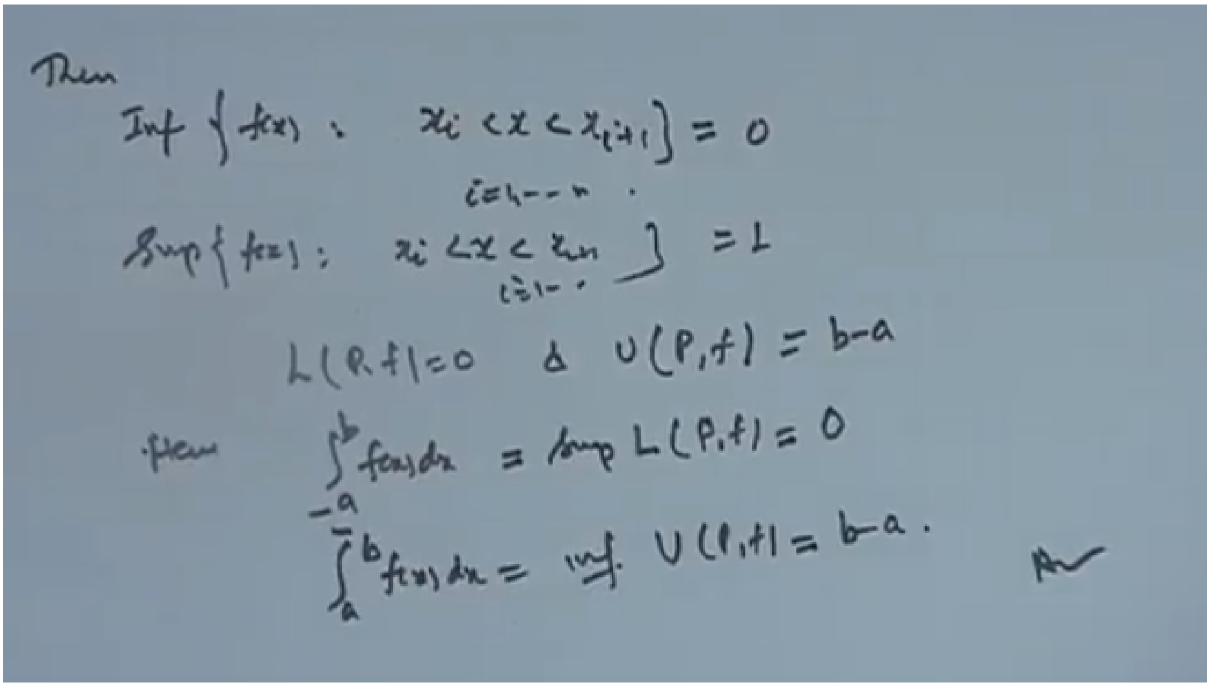
Because there is a sub interval, say I take the sub interval ,0 2 1, is this I choose the sub interval like this, just a 1 minus epsilon, 1 plus Epsilon, having the length said 2 Epsilon. now in this sub interval, at least this point is 0, at least this point is 0, so lower sum, will be 0, in that case, because the meaning of Epsilon, so 2 epsilon length, is subtracted from here, so this will be 2 minus, 2 epsilon, 2 minus epsilon. therefore this will be that is we can say it is 2 minus F epsilon or 2 minus, if we take this epsilon by 2, minus, so it is strictly less than 2. so this is clear from here. Now let us choose the, we wanted this integral, whether it is riemann integral or not so let us pick up the partition suppose P is this partition which is value x naught is 1 then 1 minus epsilon then 1 plus epsilon R and then - it means we are partitioning this 0 to 2 as X naught then you are taking 1 minus xn as X 1 then X 2 1 plus epsilon R and say X 3. so 1 is the point somewhere lying here this is 1, okay so let us compute the upper sum and the lower sum, so upper sum, will always be a upper sum of this any over any partition will all will be 2, but what about the lower sum? lower sum of these, is over the partition with respect to this say P epsilon, so I take the P epsilon partition, the partition, so lower some is in between 0 to 1, minus f epsilon, the value of the function is 1, so it will be 1, into 1 minus F final by 3, minus 0, then over this interval it is 0, lower sum is 0, so it is 0 times of 1, plus F epsilon by 3, minus 1 minus F epsilon by 3 and over this interval again it is 1 and then 2 minus 1, plus epsilon by 3, so that is the we get and finally what we are getting is this is 0, this is 1 minus epsilon by 3 and then plus 2, so 3 minus that is 1, this is 2, 1, so 1 minus so overall this sums comes out to be 2 minus, 2 epsilon by 3, so upper sum is this, lower sum of this, therefore what is the difference? PF minus lower, PF epsilon minus lower P epsilon F, this is also P epsilon, is nothing but 2 F epsilon by 3, which is strictly less than epsilon and this is the nest sufficient conditions, for a function f, to be Riemann integral that if for a given epsilon if certain F epsilon the upper sum minus lower sum is strictly less than epsilon then the function must Riemann integral, integral so this sort f is Riemann integral, okay, so that is it.

(Refer Slide Time: 14:10)



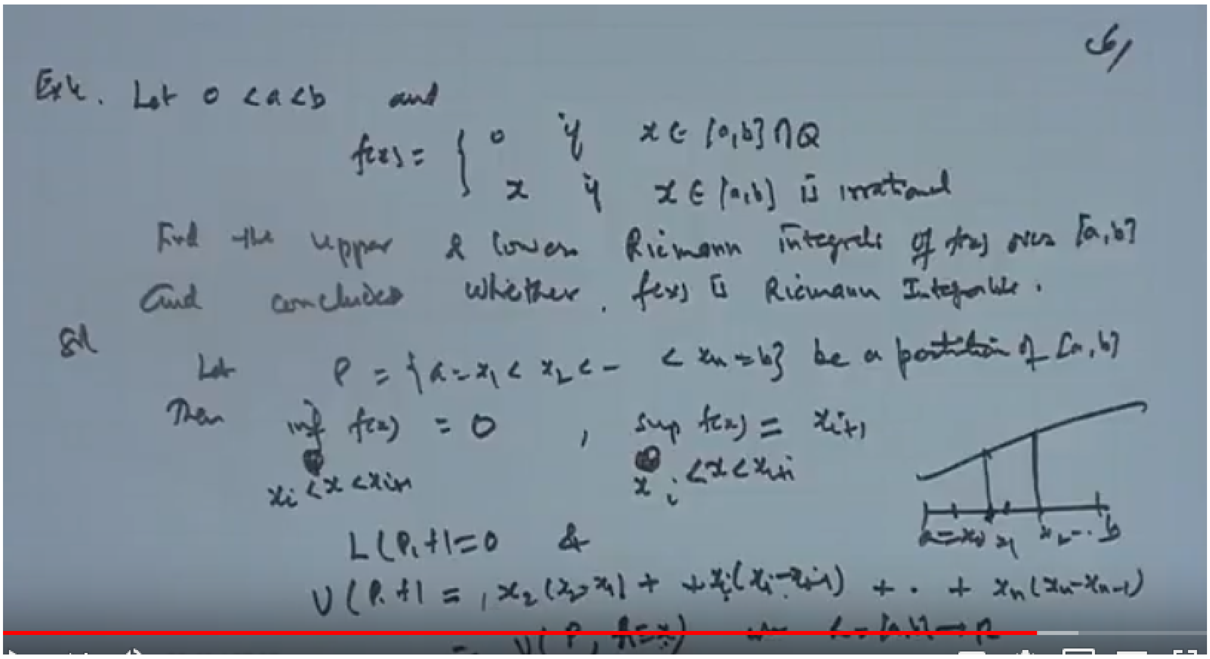
the next let zero, let zero less than a, less than B and FX, is defined as one, if X belongs to the closed interval a, b intersection Q, where Q, is the rational number G set of rational points, if X belongs, to the interval a B, a is irrational points, so Q, is set of rational number, in the interval say a, B, in the rational points, between say points in the interval. now find the upper and lower sum, lowers Riemann, find the upper and lower sum and Riemann integral of and upper and lower Riemann integral Riemann integrals of the function FX, over the interval a B. so function is giving to be one when X, is the rational numbers, in fact the intersection of a b, means set of all rational number in the interval a B, so at the rational point in the interval a B, the function attains the value 1 well for a rational point the function attains the value 0. so since rational any rational, rational and rational points R dense in all, it means whatever the sub interval you choose, there will be infinitely many rational point as well in front of mainly is rational point are available for the it means the function will attain what the value is 1 and 0, over any summit of us. ok so let us take a partitioning, let P, be a partition be a partition of the interval a B, suppose this partition is say a is x1, less than x2 and less than say xn, which is B, ok then what will is the infimum and value and the supreme value of the function?

(Refer Slide Time: 17:05)



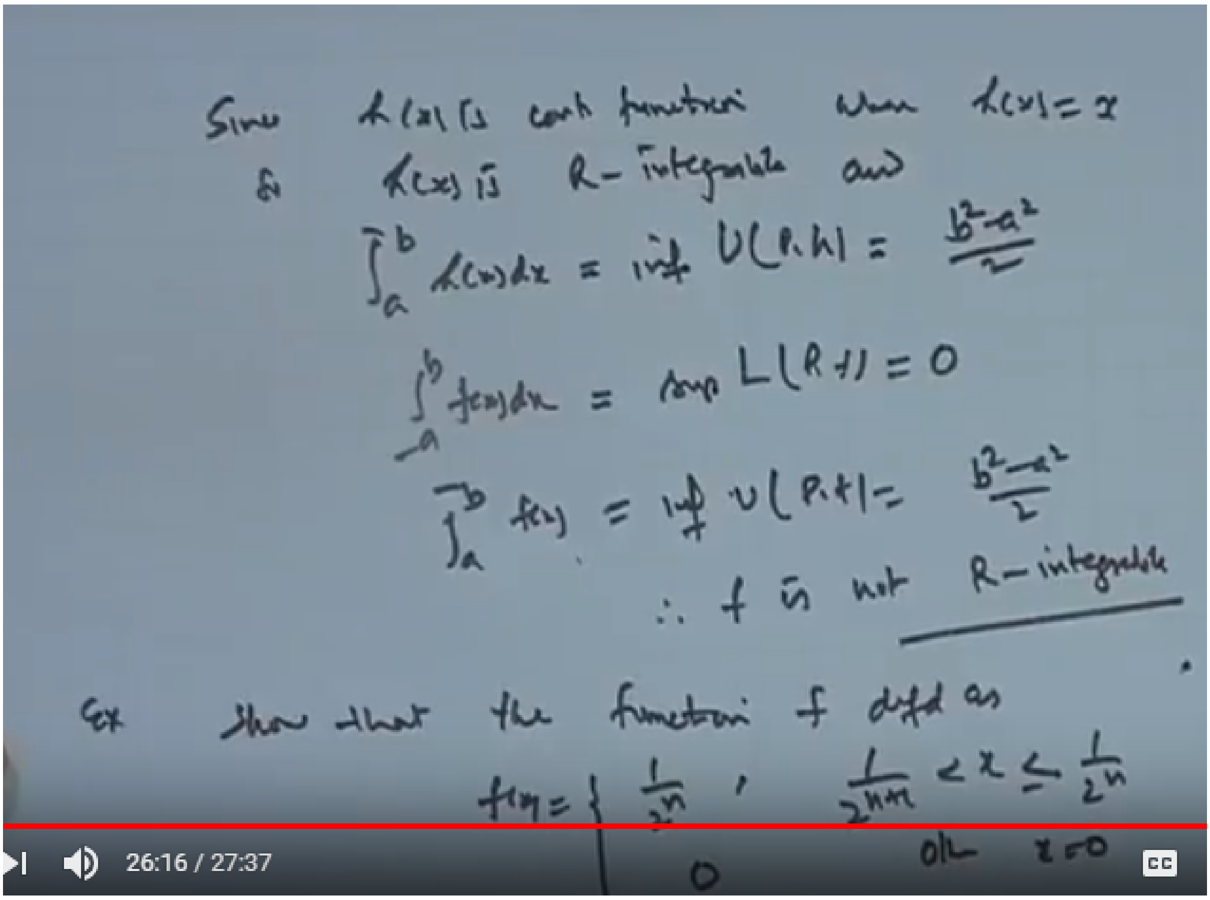
Then infimum of, infimum of the function, FX, when X belongs to the interval Xi, minus, this and Xi plus 1, when i vary from 1 to infinity, i is equal to 1 to N and so on. What is the infimum value? now the function attains the value 0, as well as 1, when X is irrational, the value will be 0, when X is rational, the value 1, FX will have 0 and 1 both, so infimum is only 0 and the supremum will be 1. Supremum will be 1. One two three is one, so that. So therefore the lower sum, of the function will be zero and upper sum of the function over the partition P, will be equal to what the maximum value will be one, the total length of the interval is B minus a, so it will b minus a. okay now hence the lower integral, Riemann integral aFX, DX, this will be equal to supremum value of the lower sum, all the partition over partition that is zero, and upper riemann integral of this function is the infimum value of the upper sum, over the partition and that is B minus a. so this is the answer for it, okay, clear?

(Refer Slide Time: 19:05)



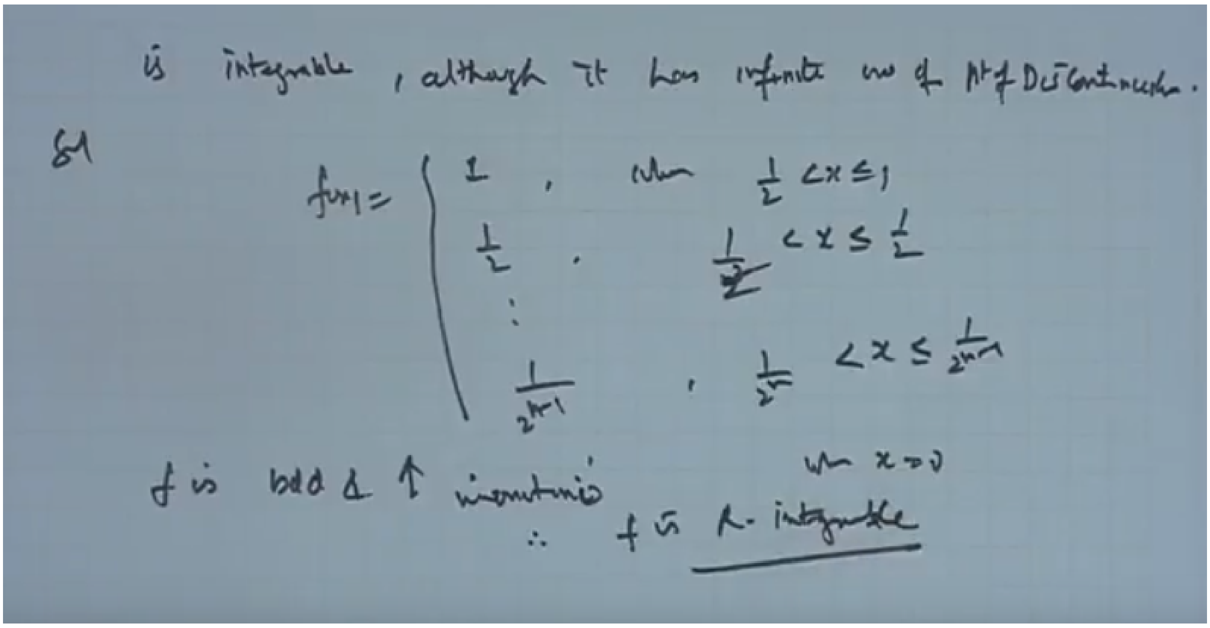
the next example is say for let zero less than, a less than B, and FX, is defined as, FX is defined at zero, if, if X belongs to the rational X, is the rational number over the interval a B, and X if X is, in the interval will be what it is in irrational points, irrational. Then find the upper, upper and lower Riemann integrals, Riemann integrals, of the function FX, over the closed interval, a, b and concludes and conclude, conclude whether, whether FX, is Riemann integrable. so this problem is parallel to our previous one, a slightly difference in there because we are taking X, so earlier when we are taking something constant etcetera, it were not coming to be a riemann integral, but in this case you will see it comes out to be a riemann integral function. So let P be the partition of this in say a is x1, less than x2, less than xn, b, be a partition of the closed interval, a B, then infimum value will be infimum of FX, over this partition p, if I take the infimum of P, then this is 0, because of the 1 point, and the supremum of the over partition p, that will come out to be what? xi plus 1, why it is so? because this is our interval a B, when we are taking the partition of this thing and so on, so suppose I choose this partition, over this x1 to x2, okay the function attends both the value 0, as well as X, so X is the largest value at this point, otherwise at this point it is 0, similarly for this interval it is the largest value and at this point it is 0, so it has a very 0 and then sorry rational and irrational and among rational irrational point it has a value X2 and then X1, but X2 is greater than X1, so therefore every sub-intervals supremum over this interval Xi, into Xi plus 1 is Xi, real this infimum also you can take xi to xi plus 1, so this is 0, this is Xi plus 1, therefore the lower some, of this will always be 0, well the upper sum of these nothing but what the first is X naught, X1 so it is equal 2X minus X1 and then continue these, so we get Xi minus Xi minus 1, into Xi and last is xn, xn xn minus 1, because of this so this is nothing but the upper sum of the P and the function f, is identically x. so you can say H, is identically x function. Where the H is the a B, to R, in fact it looks like that this is the functional value, so FX equal to H, when H is X, so the functional value is x2, over the interval X2 to X1 over Xi minus 1 the maximum value is Xi and like this so it is a function X, therefore this mapping upper sum will be, so basically what we get it that

(Refer Slide Time: 23:51)



Since HX is a continuous function, is a continuous function, where the HX, is equal to X and n is Riemann integral. so HX is riemann integrable function, every continuous function over the closed interval riemann integrable and the upper riemann integral sum, of this HX, DX is equal to what the infimum of upper sum of P, H and that is equal to, b square, minus a square. okay upper sum and then for the lower sum also we get this thing, so up, riemann integrable function in sorry F is riemann integrable function, H is riemann integrable function, H is continuous, in and upper sum is this, now when infimum, is taken over all partition, the lower sum of this FX, DX, not HX, this is the supremum of L, PF which is zero. so upper integral of FX, DX, this is equal to infimum of U, PF and that will be equal to b square, minus a square, but upper and lower sum dissolves, therefore F is not riemann integrable, f is not, h is riemann integrable, because they behaviour is this one, okay so we are getting this, now let us take the last one result, that way show that, show that, function f, the function f, define as, FX is 1 by 2 to the power n, when X is lying between 1 by 2n, plus 1, 2, 1by 2N and 0, otherwise X is 0, Ok, x is 0.

(Refer Slide Time: 26:23)



So when u is Riemann Integrable, is Riemann integral is Integral although it has an infinite number of point this, although it has in finite number of point of discontinuities, this can the reason is this because when we open this FX, the FX is coming to be 1, when this lying this equal to ½, when X lying between 1 by 2 square, less than or equal to half and continue 1 by 2 n minus 1, when X lying between 1 by 2 n, less than equal to 2 n minus 1 and 0 when x is 0, so f is bounded an increasing function, therefore monotonic increasing, therefore f is Riemann integrable, is Riemann integrable ok so that's all.

Thank you.