Module 12

Lecture 72

Some more results on Riemann Stieltjes integral

Course

On

Introductory course in real analysis

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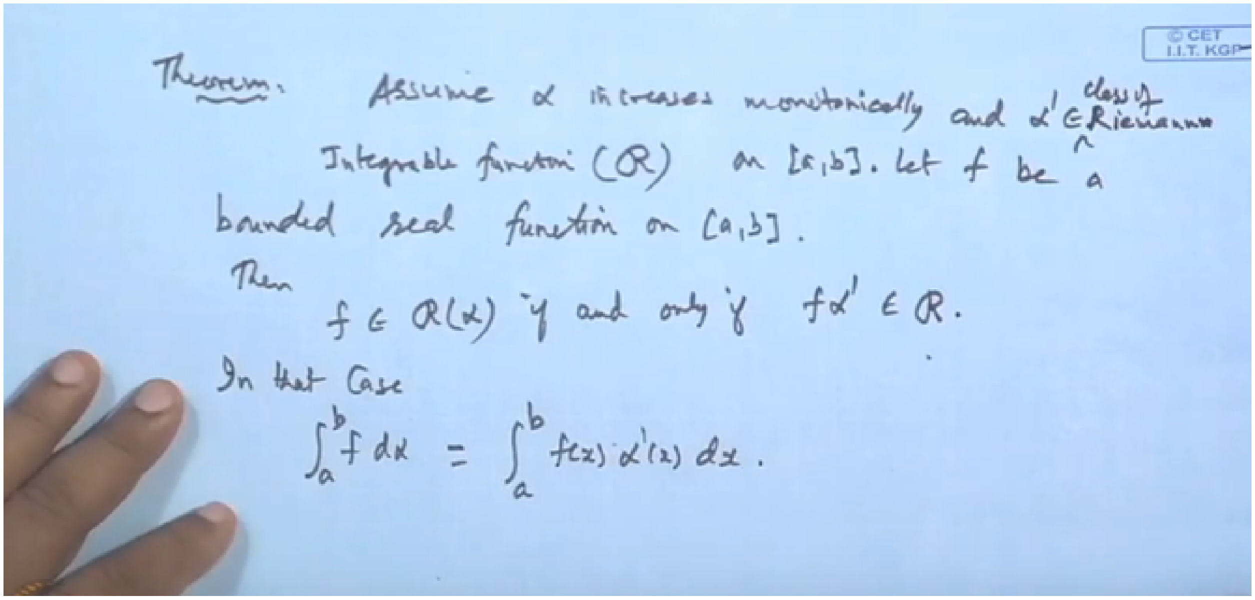

Next result is also in testing, we source the Riemann integral functions the integral can be calculated as if it is a day after integral under certain restriction. so assume, assume alpha increases alpha increases monotonically, monotonically and alpha dash, alpha dash is a Riemann integral function, on is a Riemann this you say is a Riemann integral function belongs to Riemann integral function. We denoted by this, okay. So, alpha dash belongs to the class of Riemann integral function, class of Riemann integral function this, on a b. Let F let F be a bounded, bounded real function, real function defined on the closed interval a b. Then the result says, F is in Riemann stresses integral with respect to alpha if and only if, only if the f into alpha dash, this product is Riemann integral function, this belongs to the class of Riemann integral functions on a b and in that case in, that case in that case the integral a to b FD alpha demonstrates integral of the function f is the same as a to be a to b FX into alpha dash X D X. as if it is just a definite integral when the function is can write this Riemann, okay.

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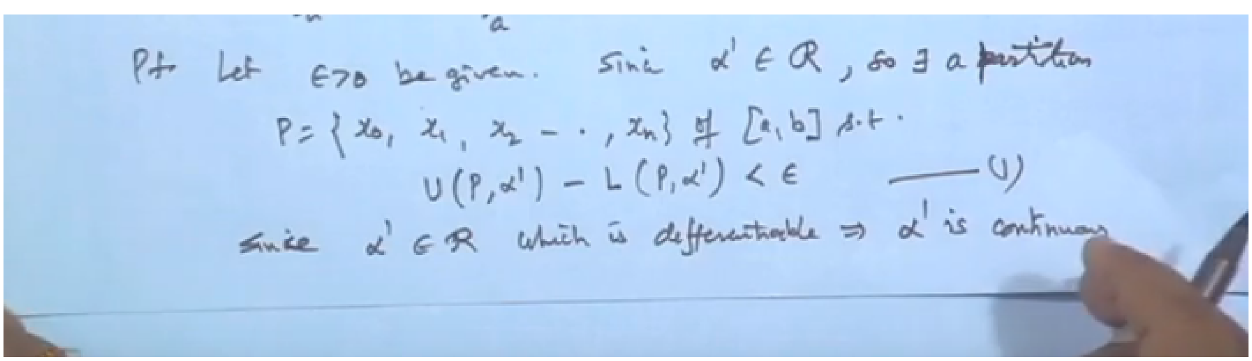

So the same test, let's see the proof of this, let epsilon greater than 0 be given be given, okay. Now given that since Alpha dash belongs to the Riemann integral functions over the interval a B so by definition there is a partition. So there exists a partition, or there is a partition P ,say X naught, X 1 ,X 2 ,xn of a B such that, the upper sum of the function alpha dash , minus lower sum of the function Alpha dash, with respect to the partition P is less by necessary and sufficient condition for a function to be the Riemann, integral is the difference between upper sum minus lower sum, will remain less there will be a partition where the difference fluctuation of the function will remain less than Epsilon, this is the fluctuation of the function ,okay. So let it be one, okay. Now since alpha dash is giving to be since alpha dash means is a different shape function alpha dash belongs to R, belongs to R, which is differentiable they exist so it is differentiable so it implies that alpha is continuous, alpha dash is continuous. Because if every continuous if the function is different it has to be continuous, so alpha dash will be a so that is a continuous function, so alpha will be a continuous function, otherwise we cannot talk about the derivative since referred a exist it means this source or this source that alpha is differentiable, that alpha is differentiable, so alpha must be continuous, that is the men, from here because this is given alpha days, alpha dash means derivative of alpha, so this exists ,so it must alpha must be a continuous otherwise the derivative cannot be if it is discontinue cannot be talk about the differentiability ,so alpha is continuous .

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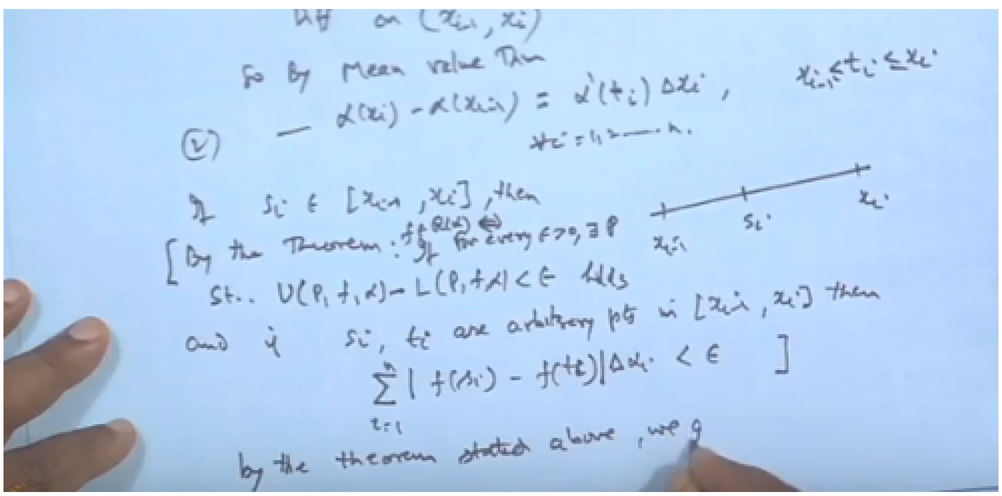

So alpha is continuous throughout the interval a B. So this is the interval a B and partitioning we are choosing the partition say X -1 and X I this is the partition so alpha is continuous on the partition, so alpha is continuous on the closed interval X I minus 1 X I for each I, it's differentiable on the open interval X I minus 1 X I, so by mean value theorem, Lagrange's mean value theorem, we say alpha xi minus Alpha X I minus 1 must be equal to the derivative of the function alpha dash at a point said t I Delta X I we have the TI lies between X I minus 1 to X I. Is it not? So there exists a point, now this may be closed also, ok. So TI below for some there exists, at some point TI we are this will exist by mean value theorem, so let it be this say number 2, ok .for each I and this is 2 for each I want to do now what is given, is that our alpha is in increasing monotonic function alpha dash is this and F be a bounded real function then f we found leave this, ok, so let us take the point suppose this is X I minus 1 here in the X I and Ti is somewhere so let us choose the point s I choose a point s I belongs to ,X I minus 1, to X I then let's apply that result what this result is which we have proved earlier the result is this one, the result says, if then by the result, by the theorem ,which we have proved earlier, the theorem each, what's theorem each the partition if u P alpha F alpha minus L P F alpha is less than Epsilon, if for every epsilon there exists a partition if for every epsilon R greater than 0 there exists a partition P, such that this holds, such that this holds, holds then and if SI and TI are arbitrary points, are arbitrarily points in the interval X I minus 1 to X I, then this sum Sigma I is equal to 1 to N mod of F s I minus f TI into delta alpha I , into Delta alpha I is less than Epsilon, this result we have already proved. So using this result because our function alpha I is given to be continuous function bounded sorry is giving satisfying this condition, okay. So for any partition P we can get this, so what we did apply the condition for us because, Alpha dash say one more thing is that this, so that if alpha is a function f it must be in our if and only if, okay. So, this source that and this, and this is a F in, in r F belongs to R alpha if and only if, this result is true. so what is the Alpha dash is in our Alpha dash is in R so we can get this result quickly, so by this year if s i be any point of this, then by this theorem by this theorem ,by the theorem stated above. We get, we get immediately Sigma.

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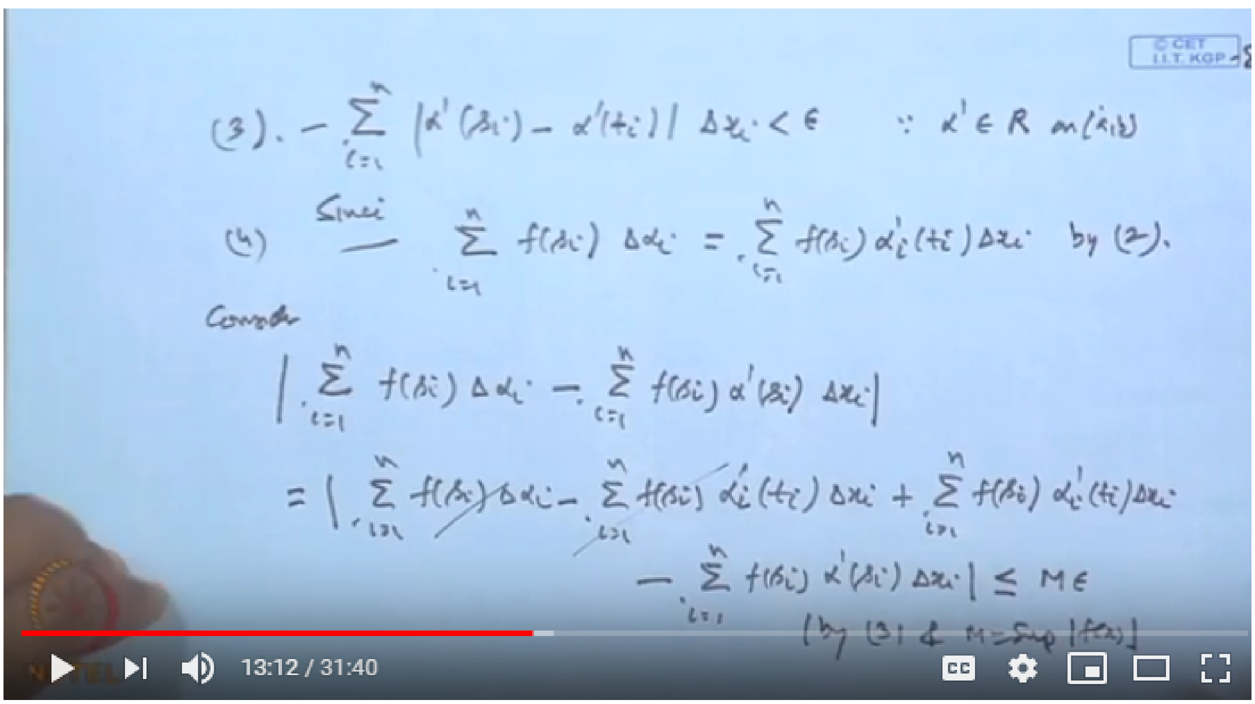

Sigma I is equal to 1 to N alpha dash s I, minus alpha this t I into Delta X I is less than F signal is less, than F signal and because, because alpha dash is in R alpha is on a b , this is given and this, that's why? This is your disvalued, okay. So let it be condition 3, now since our this Sun Sigma I is equal to 1 to N F of s I Delta Alpha I this is nothing but what? Sigma I equal to 1 to N F of s I, Delta alpha I we have already proved it in the second, this is our Delta alpha is it not so, this Delta alpha I which is equal to, say Delta alpha I is nothing but alpha dash t I Delta X I, so by using the second, by using the second we can say this is equal to alpha dash, alpha dash i TI Delta X I by second, okay. then consider This, Sigma I is 1 to N F of s I Delta Alpha I minus Sigma I is 1 to N F of Si alpha dash s I into Delta X I, ok. Consider this thing ,now we have already shown this part is it, the Sigma f I Delta is this thing, so here is TI so what I say I suspect this thing TI and DI, so if I subtract TI in here so let it be 4, so using 4 consider this now this can be written as at smart Sigma I is, 1 to N F of s I Delta Phi minus Sigma I is 1 to N F of s I alpha I dash TI Delta X I plus Sigma I is 1 to N F of s I alpha I dash TI Delta X I minus Sigma I is 1 to N F of s I Alpha dash SI Delta X I, just innocent now, this part is exactly same as this so ,this will go to zero, now this is there but, because of this we get the earlier one result, Alpha dash I - this because of the three so ,apply the three condition so this is less than epsilon, and F is bounded so can you not say this is less than equal to M epsilon by third and M is supermom value of mod FX because of this, okay .So in particular we can say, this so, let it be so what we get is so ,so we get basically this result.

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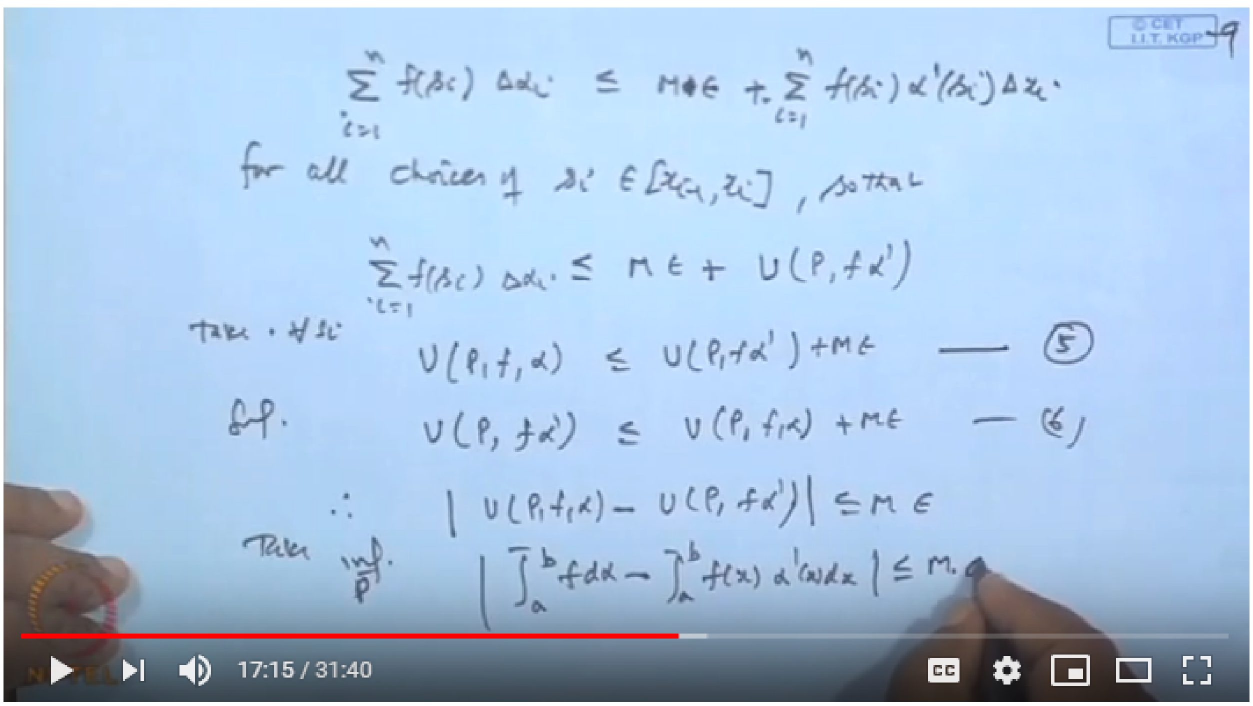

Sigma I is equal to 1 to N F of s I Delta alpha I is less than equal to M plus F, sorry M into F sine of M epsilon plus Sigma I is 1 to N F of s I, f of Si alpha days alpha days that is what? He's saying is it not f of s I F of Si alpha dash I pay for this, this thing this part in F of s I you have taken outside so what you are getting is F of s I Delta alpha - this part so - this part is less than epsilon so this thing is, this thing is less than equal to this Plus this so f of Si alpha – s I Delta X I this is true, so which we get from here so we get this thing. Now this is true for what, for every I and s so if I replace this by the upper sum of this so for all choices, of s I which is in X I minus 1/2 X I this result is whole so, it take this, upper bound for this so once you take the upper bound so, that we get what? This is less than equal to M epsilon plus, the upper bound of the function f alpha dash, because this is the F alpha dash function so, replace this by its maximum value so you are getting the upper bound for this ,and which is less than equal to I greater than equal to 1 to N F of s I Delta alpha I. now, take the upper bound for this so, not take the upper bound left-hand so, take upper replace by this maximum value so ,you are getting this ,this is true for every s I. so, we get the upper bound of this if alpha is less than equal to upper bound of F Alpha dash plus M epsilon, let it be say, fourth quiz fifth. Now, in a similar way we can also use this thing, if I take this is less than epsilon so, this part is less than equal to this, greater than equal to this minus Epsilon so, from similarly we can say, similarly same argument we can say, that, upper sum of P and F Alpha dash is less than equal to upper sum of P if alpha plus M F son in a similar way, from the same annuity therefore, the difference of this, upper sum of P F alpha, minus upper sum of P f alpha dash , Alpha dash each less than equal to M epsilon, okay .clear, now if we take this, this is two. So what you get take the infimum value of this, take the infimum so take infimum over also partition p. so, this would lead to the integral a to b upper sum, FD alpha this, will lead to the integral A to B evil f X alpha this XD X and this entire thing is less than or equal to M epsilon.

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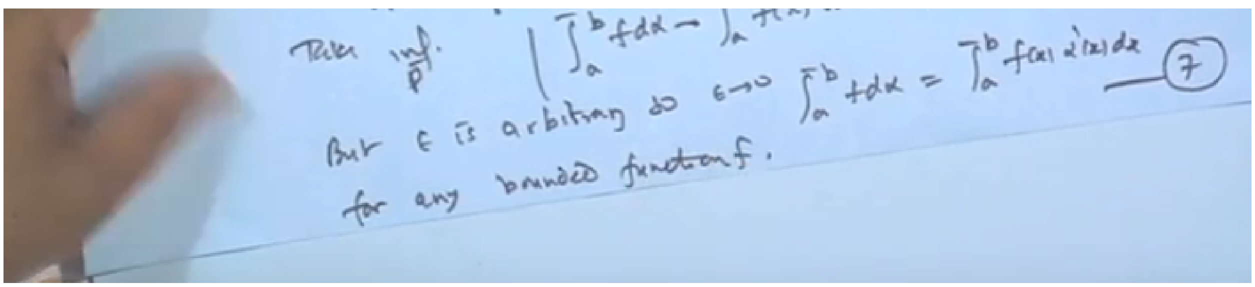

But epsilon is arbitrary, but epsilon is arbitrary small number so, as epsilon goes to 0, we get integral a to b VAR FD alpha is nothing but the to b upper sum of this, upper sum of FX alpha dash x DX, Okay. Let 7, similarly for any bounded Function, this is true for any bounded Function f.

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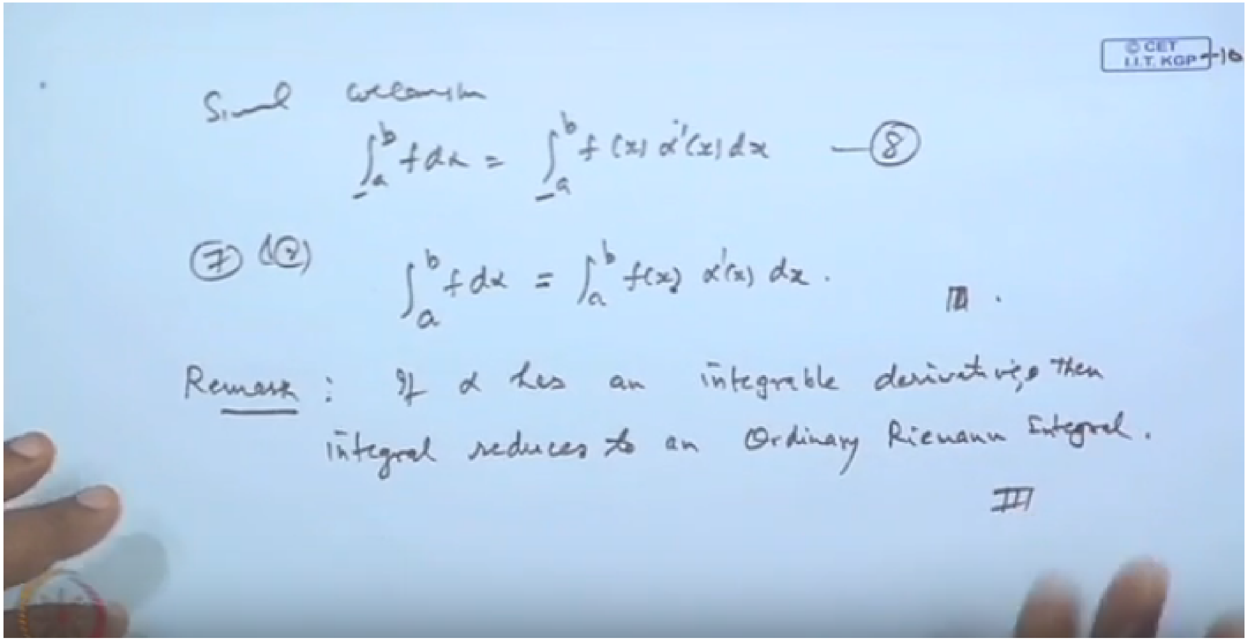

similarly we can prove the lower integrals, similarly we can so, similarly, we can show that a bar f say, a bar of this part, that is e bar of f FD alpha is the same as a lower bar b FX alpha dash X D X. so, a 10-7 in it implies if, f is given what is given is that function alpha Dash is in this and f is given to be a Riemann stretch integral then this left hand side are equal therefore these two are equal. And we get integral a to b FD alpha is the same as a to b FX, FX alpha dash x, DX, DX okay. Bi degree so, this improves the result completely, ok. So, this shows that remark you can say, the remark is that what they, if so what if alpha has an integral derivatives, integrative derivatives then the integral reduce, then integral reduces, integral reduces to an , to an ordinary Riemann integral hence this can be easily computed. So, this is what we are getting, okay. So, this almost we have completed. Now, we will give a slight just a concept of what is our measure? We have discussed already earlier, already earlier but, let us see some in slightly in detail, what is the measure and what do you mean by the almost every real functions and like this.

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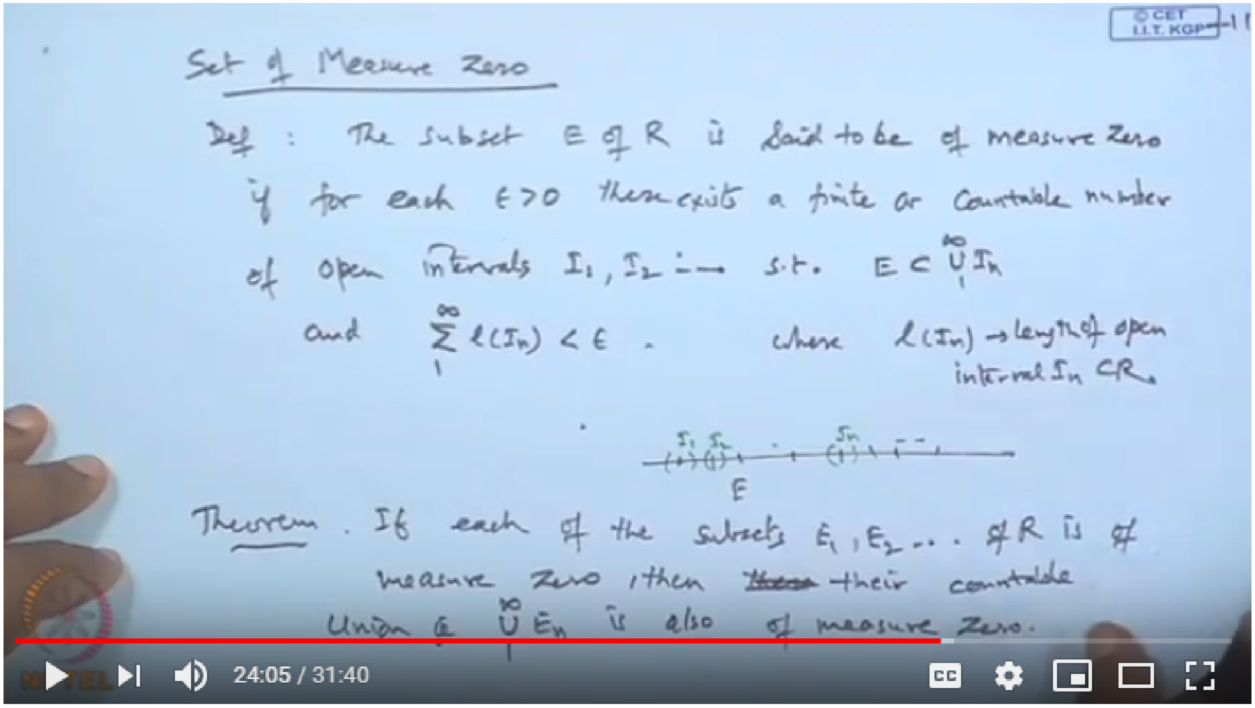

So, let me just say, a few concepts set of major 0, set of major 0, set of major zero, okay. Let us see, we define that, the subset e of r, e of r real line, subset e of R is said to be, is said to be of major 0, of major 0. if for each for each epsilon greater than 0 there exist, there exist a finite or countable number of, number of open intervals I 1, I 2 and so, on such that, they're countable union covers e, countable Union cover e and the length of this, length of I N 1 to infinity is less than Epsilon, we are L denotes the length of the interval, length of open interval I n of r clear. So, what I mean, where that if suppose the subset is there, suppose these are the points in e, these are the set e, and if we are able is this set is said to be measure 0, if we encloses these points by means of an open interval I 1, I 2 say, I n and so on such that, length of these open intervals is less than epsilon. such that, countable union of this cover C means all the points of V belongs to the countable union of I n but, the sum of their length cannot is not ,exceeding by epsilon then we say, the set E has a major 0, or is said to be major o .Now, that's what we can say length of token and there is one result which will be used for the result says, if each of the subsets, each of the subsets e 1, e 2 and so on of r age of major 0, is of major 0. Then, their countable union, then they are countable Union that is union of I e n, n is 1 to infinity is also of major zero.

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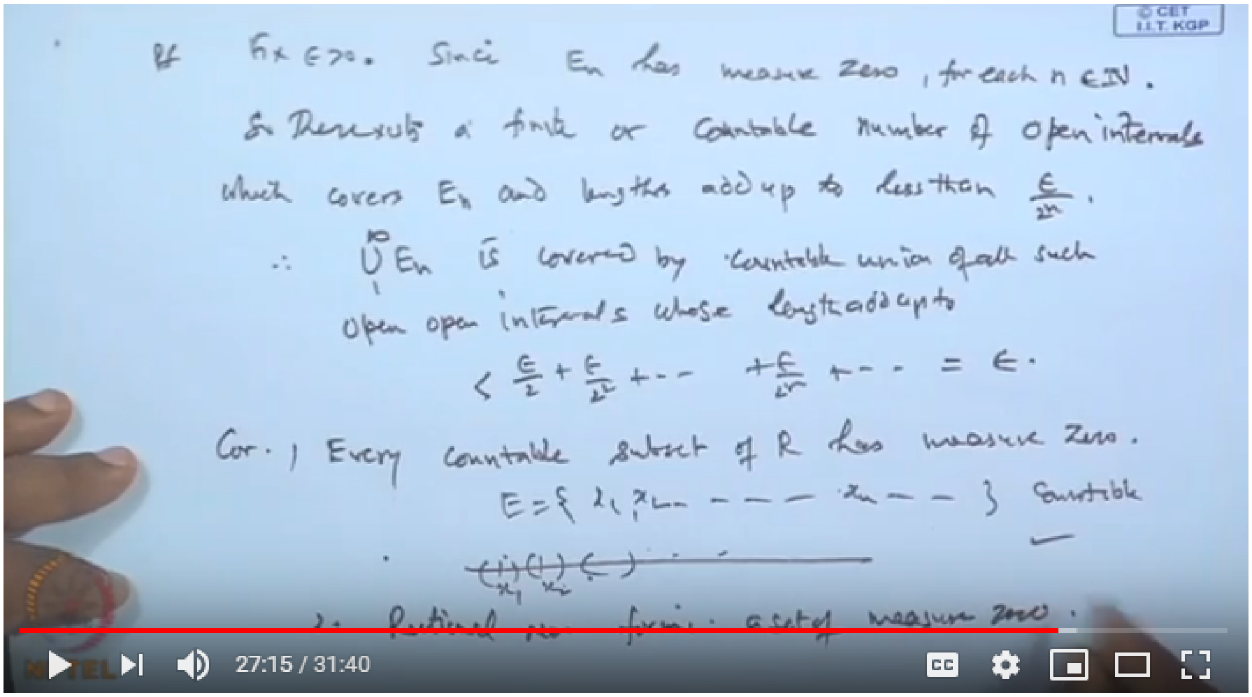

And this is very easy to prove, but suppose I fix up proof of this simple suppose I fix up epsilon greater than zero, okay. Since en has a major 0, as major 0 so, by definition there are the contact for each n, for each n belongs to say I is the natural number or some and, and natural number, then since en is worse so there exist, so there exist there exist a finite or countable, or countable connections countable number of open intervals, intervals which covers en, which covers en and whose length, length add up is less than epsilon, a length add up to less than say, epsilon over two n, okay, therefore the countable union event, therefore countable union of en 1 to infinity is covered ,is covered by countable union countable union of these intervals off or such open intervals, of interval whose length, whose length add up to what? Epsilon by 2, epsilon say first term epsilon to square, epsilon by 2 n, and so on so, if I add this becomes less than Epsilon. So, up to less than this number, hence it is it Ok. So, this proves as a corollary, we can say, every countable subset of measure of r has measure 0. Every countable subset because, why the reason is suppose is e is the countable sets we can arrange in the form of the sequence like this, this is a countable set. So, each 1, X 1, X 2 ,X n we can enclose it by means of a countable number of n intervals and some of this will be each term is less than epsilon double union of this symbol also be less than Epsilon so, this is counting. Rational numbers, rational numbers forms a set of measure zero. Because, it is count relation numbers are countable measure zero .so, this was dismal.

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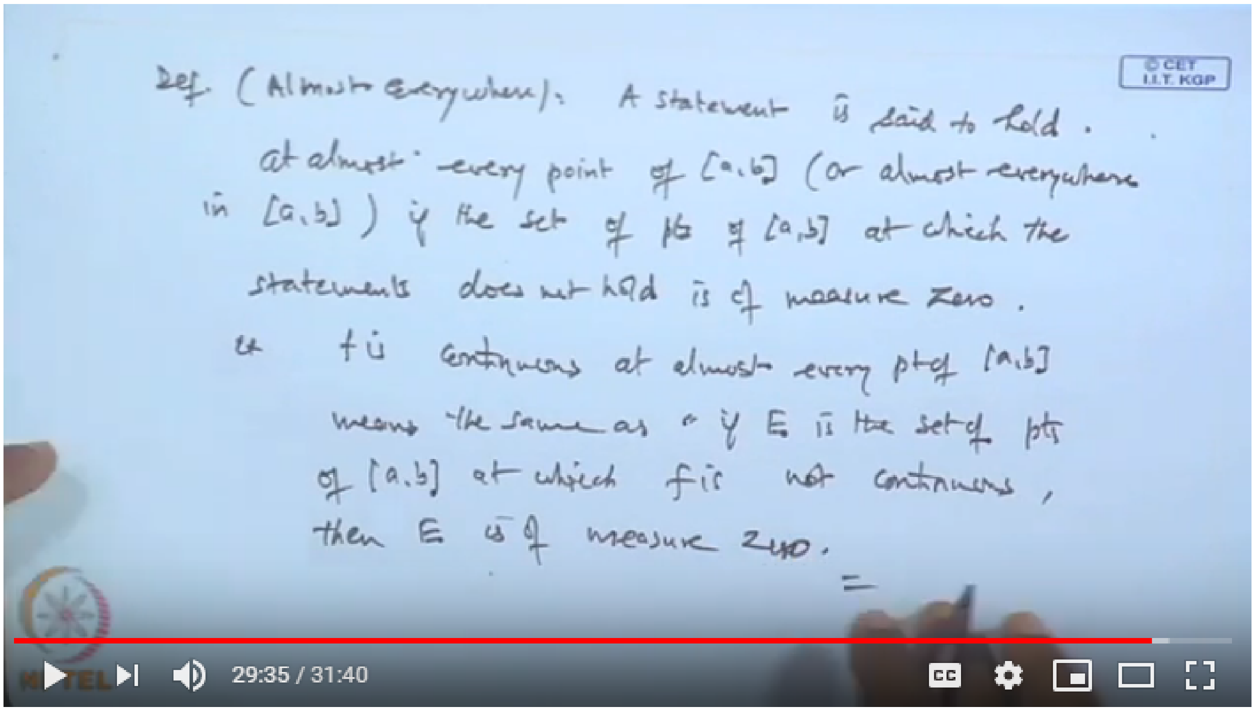

Now, they say last one which is really important almost everywhere, a statement is almost, almost everywhere a statement, a statement is set to hold, hold it almost, it almost every point of the interval a B ,a B or almost everywhere, everywhere in the open in the interval a B if the set of points, set of points of a B, set of points of a B at which, at which the statements ,statements does not hold, does not hold each of measure zero. So, that's the definition so thus we say, for example if we say F is continuous, F is continuous. At most, at almost, at almost every point of the interval a B means, means the same age, age if E is the set of points of a B, set of points of a B at which, F is not continuous, F is not continuous then e, then e is of measure zero. So, that's what is reason F is continuous almost everywhere is.

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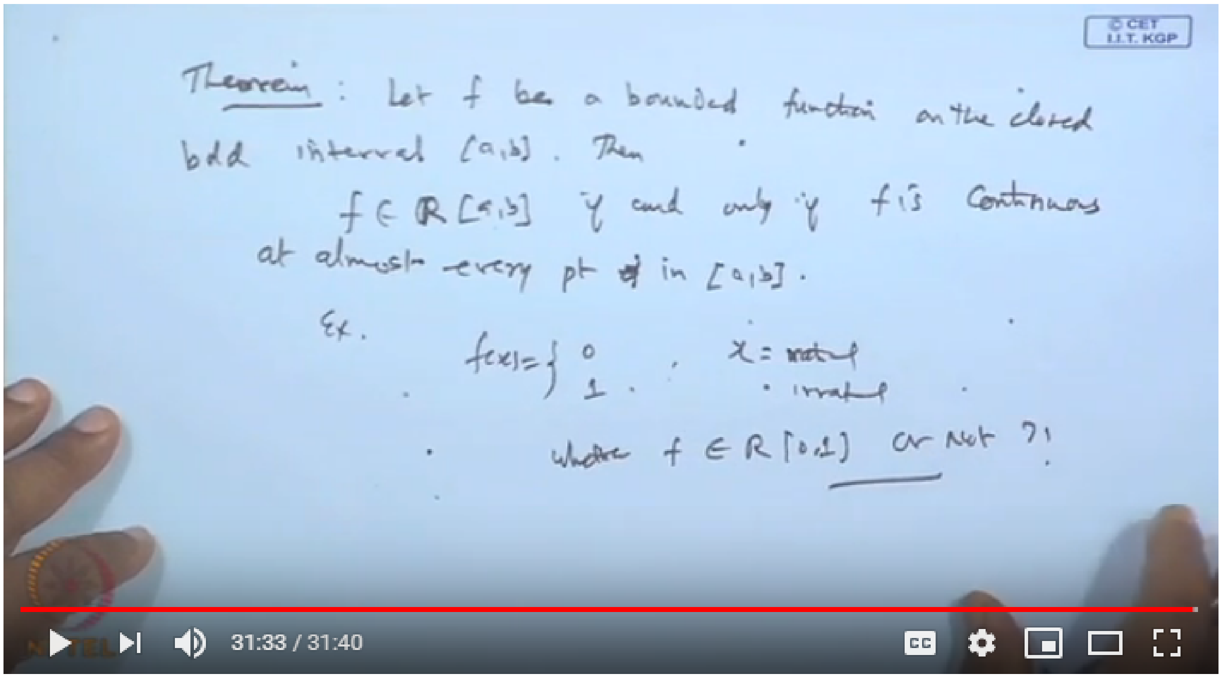

Now, based on this we have a very in result important theorem, and the proof is ,proof is of course we are just keep neglecting because, this proof based on already we have proved this earlier, the theorem says, let F be a bounded function, bounded function on the closed interval, closed bounded interval say, a B then, f is Riemann integral over interval a b, f is Riemann integral functions over the interval a B if and only if, if and only if F is continuous, F is continuous at almost every point, of point in the interval a B, almost every point in the interval a B, then we say, the function is continuous for this, okay. For examples let us take this function suppose I define FX, 0 and 1, when X is say, X is irrational and X is rational and this is irrational it is no point, okay ,this is bounded function now, let us see whatever we will discuss whether question is whether f is Riemann integral functions over 0 1 or not. So, this question we will discuss next when you go for the tutorial, okay, similarly other.

Thanks you very much.