Module 12

Lecture – 71

Various results of Riemann Stieltjes Integral Using Step Functions

Okay so we will continue our previous lecture, that is the properties of Riemann Stieltjes integrals and in this lecture being particularly deal with the step functions and we will show that whenever alpha becomes a step function, alpha is taken as a step function, then Riemann Stieltjes integrals reduces to a either a finite sum, or maybe infinite some of the terms series, in the form of the series. So one can get the value of the integral, in the form of the series okay, similarly if alpha is a differentiable functions then it can be reduced, Riemann Stieltjes integrals, reduce to ordinary integral, just a definite integrals and one can get the value easily for that, okay.

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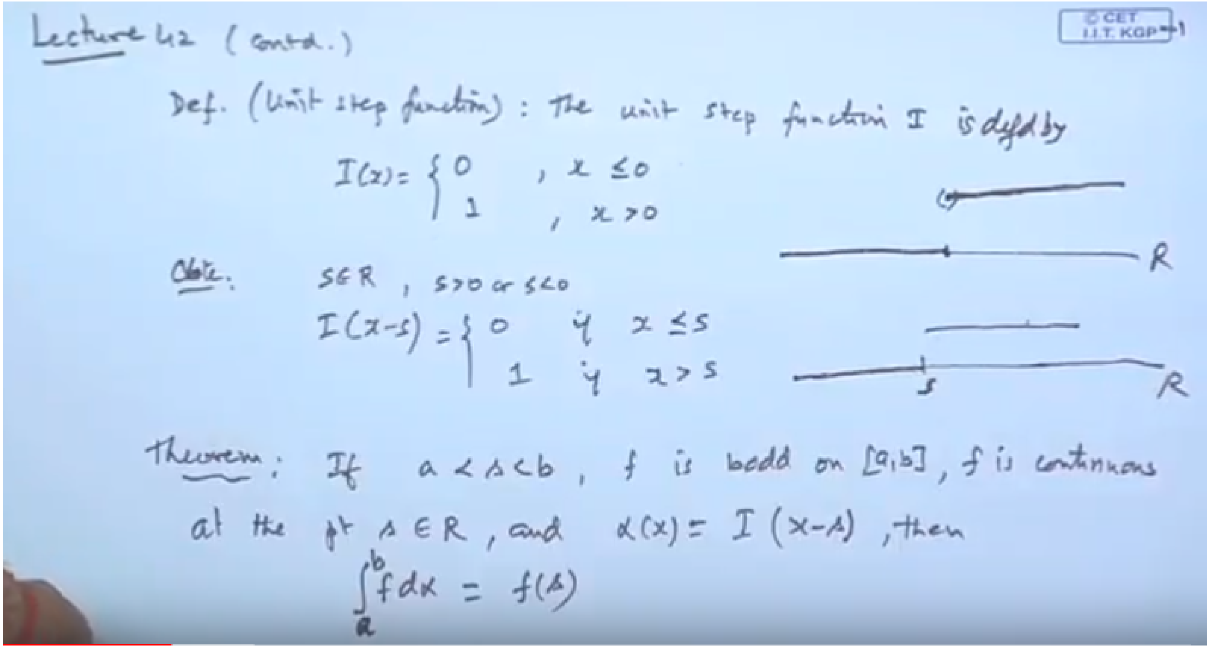

So we prior to this before starting, we will give the definition of any unit step function, we define the unit step function. The unit step function I is defined by, Ix, is zero, when X is less than equal to zero, and 1 when X is greater than zero. So basically this is entirely a line. So ix means, when the x, real number X is less than equal to 0, the corresponding value of IX will be this and when it is greater than 0, the corresponding value of I X will be 1, this point is not included here, it is included here, ok so this is our step function. If we call it as a unity step function, if we as a note, we can say if suppose I take S be any real number, in the some interval say a, b and when we say X minus S, then the meaning of this is 0, if X, is less than equal to S and 1 if X, is greater than S, so this will be starting if S, is say positive or may be negative, or S is negative, suppose I take S here, then all the for all the real number which are less than equal to S, the image of under I will be 0 and while it will be 1, when it is greater than this. So this is the real line so this is all about the step functions. Now we will make use of this step functions and first we will show that if, if, our F is continuous and also alpha, is a step function, then in that case the integral can be computed, just by computing the value of the function at the point where the function is continuous. So the result is this, the result says if suppose S is a number lying between a and b, a, b is a given interval and F is bounded over the closed interval, bounded on the closed and bounded interval a, b. F is also continuous, F is continuous, continuous on and at the point, at the point S, which is in are lying between a and B, a and B and suppose alpha X is the unit step function, X minus s, ok alpha then the result says, the Riemann Stieltjes integrals of the function f, with respect to alpha, from A to B, over the interval a, B,, is nothing but the value of the function at a point is where it is continuous. So that way, we can easily get the value of the function Riemann Stieltjes integrals, if I know this property that, alpha, is a unity step functions with respect to say S, we have alpha X is I X minus H and F is bounded and continuous function or defined over the interval a B, ok. So let's see the proof of this way.

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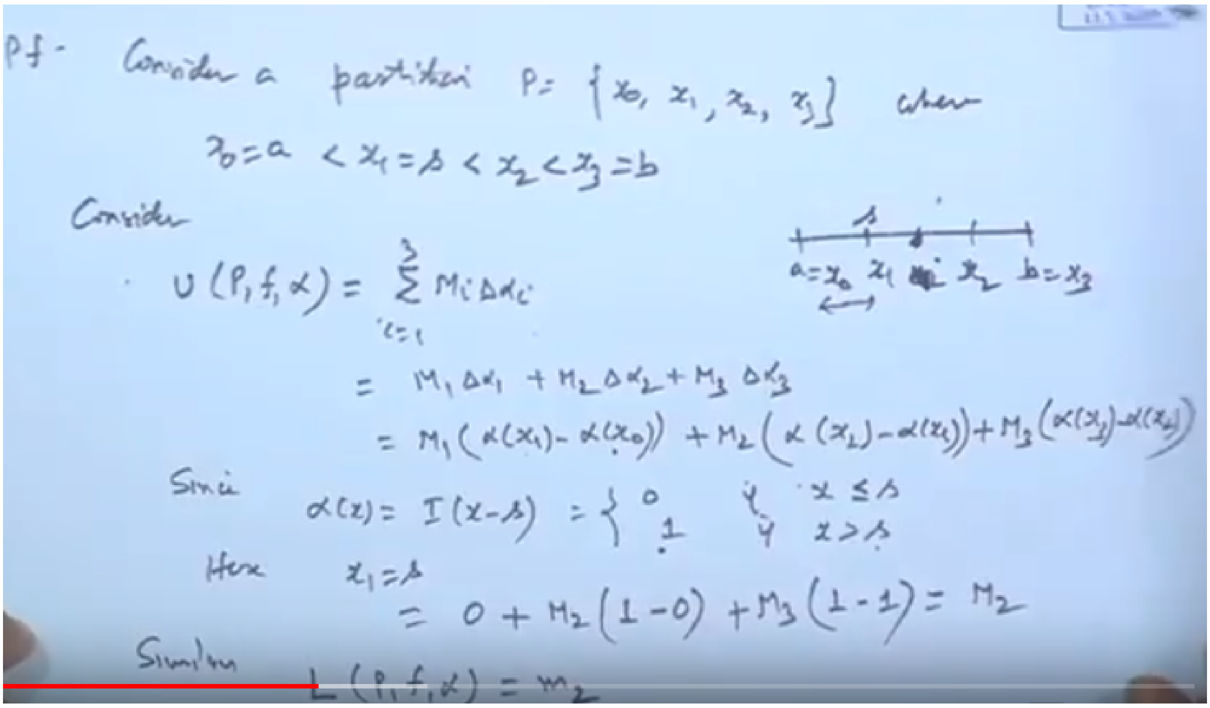

in order to prove this result, let us use that since f is giving to be a, f is continuous and bounded also, so we can function f, over the interval a B, will attend is maximum and minimum value because the interval a B is closed and function is continuous, so it will attain this maximum and minimum value, therefore we can find out the maximum and minimum Riemann sum, for this function f ok. so let us take consider a partition P, having the point suppose X naught, X1, X2, X3, it may be more point but just for simplicity I am taking and the result can be extended, when there are so many other points involved in between a, b, ok.

Where X naught, I am taking as a, which a, which is less than X1, is suppose s and less than X2, which is less than X3, is suppose B. so this is the interval a, B, we have partitioned this interval by choosing this partition X naught, X1, X2, X, and X3, is B. So this is our say X 3, ok. So let it not be this here, this will be X2, so X1 is s, this is our point s, we have taken this one. Now consider this a person, a person of the function f, with respect to alpha, over this partition P, so this sum is Sigma, mi Delta alpha i, i is 1, 2, 3 and that is the meaning of this is M1 Delta alpha 1, M2 Delta alpha 2, M3 Delta alpha 3 and if I further expand it Delta alpha 1, means it is defined over this interval, so the choosing as, Alpha X1, minus Alpha X naught, this is our M then M2 alpha X2, minus Alpha X1, then plus M3, alpha x3 minus, Alpha X2, this is my definition now function since our Alpha X, is given to be the iX minus s, is it not? so this is the value is zero if X is less than equal to s and 1, if X is strictly greater than s, now here X1 is s, so it means when the value of alpha at the point X1 will be 0, value of alpha at the point X naught is 0. So first term will be 0, the second term will be m2 alpha of X2, since X2 is greater than s, so alpha of X2 will be 1 and then alpha of X1, because it by definition it will be 0 and then m3, alpha X3 and alpha X2 will all be 1, so basically you are getting M2, so upper sum will always be m2, similarly we can say the lower sum of the function f, with this proto alpha will be the small m2 like this ok.

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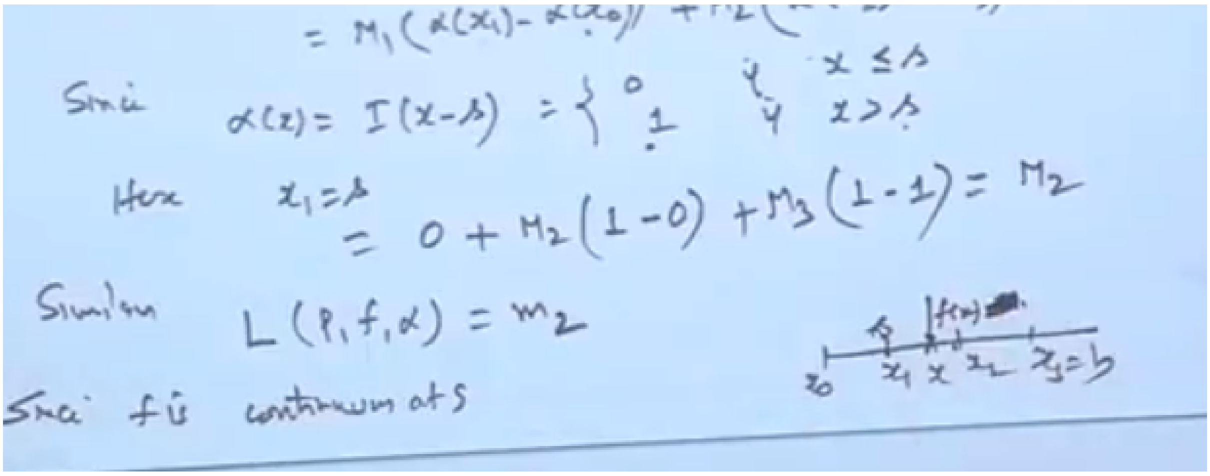

Now since F, is continuous since F is continuous, at s, so it means this is our X naught, X1 which is s, X2, X3 which is B, so if I take any arbitrary point here, say X, the value of fx, in this value is nothing but what? this is always the maximum value of this, will be because it m2, m2 is the maximum value over this interval, so it will be the maximum value of this FX, will be m2, okay so it will be close to this.

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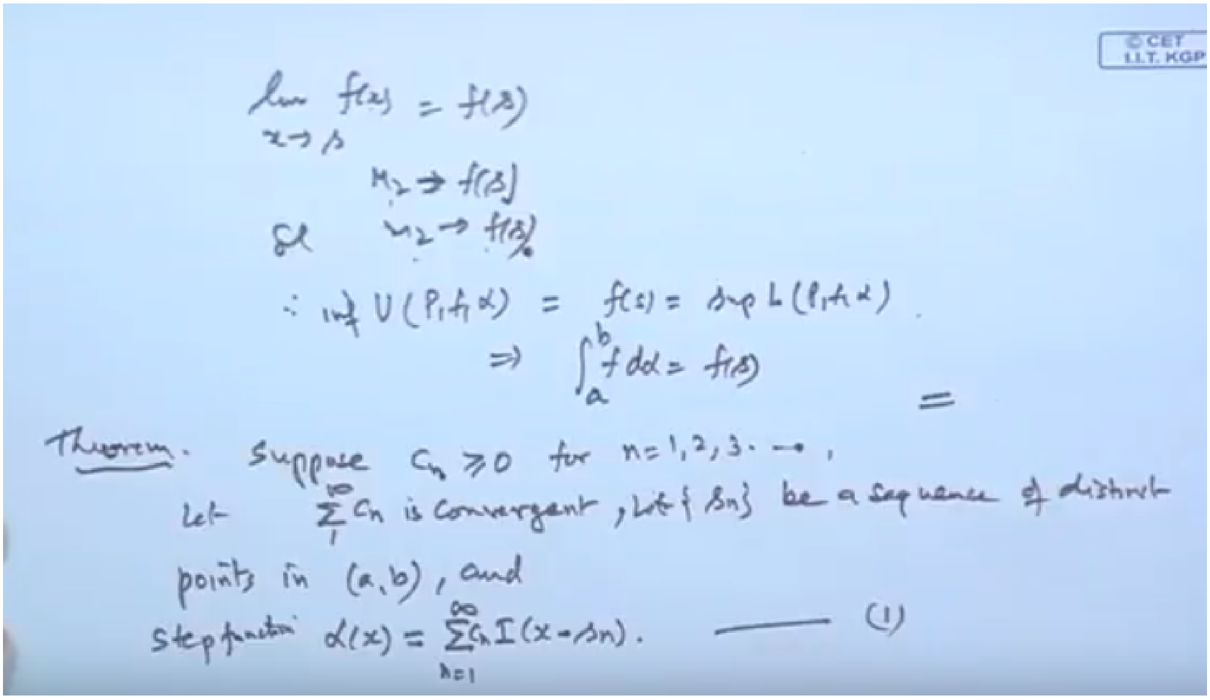

So function f is continuous s, therefore limit of the function FX, when X tends, to s, will be FS. Okay, but the function f, is such F itself which is defined over this interval X1 to X2, the function Fx, each of our alpha X, into this, so when you take the value the value will be m2 and this will come out to be F of s. so both m2 sorry, m2 will go to F of s and M one similarly small m2 will also approach to F of s, because of the continuity, so both m2 and kept is small M two will go to FS and small FS, sorry small m2 will also go to a small FS and but this is the largest value in the smallest value, but as X approaches to s this will go to FS, so m2 will approach m similarly M, small M two will approach to FS so this shows, that both upper sum of the function when the limiting value, limiting value, of the is limit of this X tends to s, is our and FS and that is the infimum, of this and which is equal to supremum of the lower sum of f with respect to alpha, so this source integral A to B, F D alpha, will be FS. because always this limiting value is coming to be FS, so upper sum when you take the infimum value, it is FS when you take the supremum value, of the lower sum, it is also FS, so it is integral and integral comes out to be this, so this proves the result.

The second results also in connection with the step functions, which will give the result that if alpha is a step function, then Riemann integral, integral, stage integral, will be reduced to the infinite series or a finite series depending on this. So suppose Cn is greater than equal to 0, for n is equal to 1, 2, 3 and so on and let Sigma Cn one to infinity is convergent, let this converges, okay, is convergent, that this is convergent. The sequence SN, is a sequence of distinct point, let sequence SN be a sequence of distinct point, distinct points, in the interval a, b, in the open interval a, b and alpha X, is the step function n equal to 1 to infinity, this we call a step function alpha, X is this Cn, i Cn i of X minus Sn. Let it be number one.

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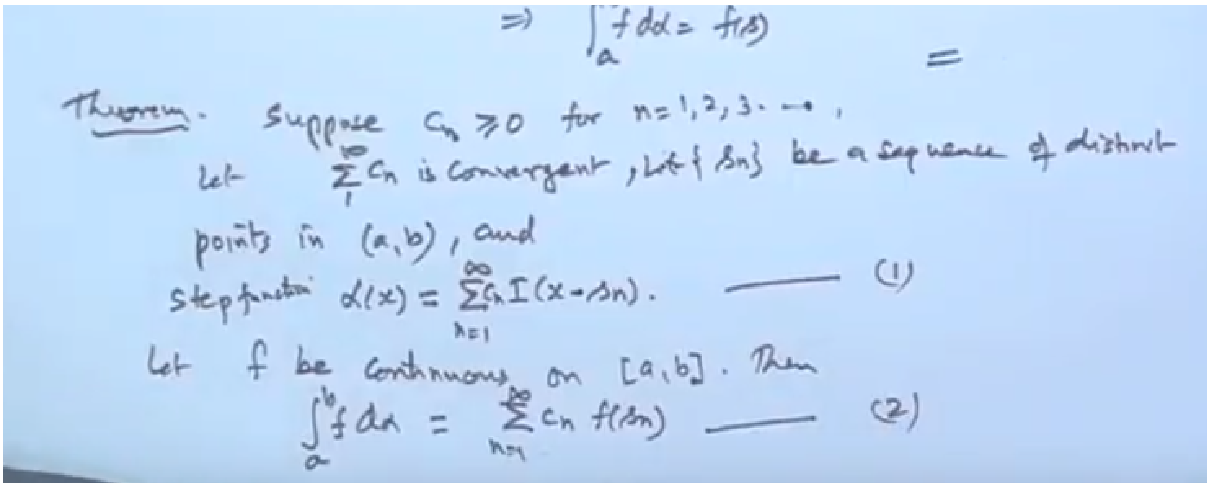

Okay and let F be, continuous on the closed interval a, b, continuous on the close interval a, b, then the Riemann status is integral, of the function f, with respect to alpha over a, B, is equal to the summation of the series 1 to infinity, CN, f of sn. So Riemann status integral, can be calculated in terms of their series if alpha is given to be a step function. So that is very important result we get.

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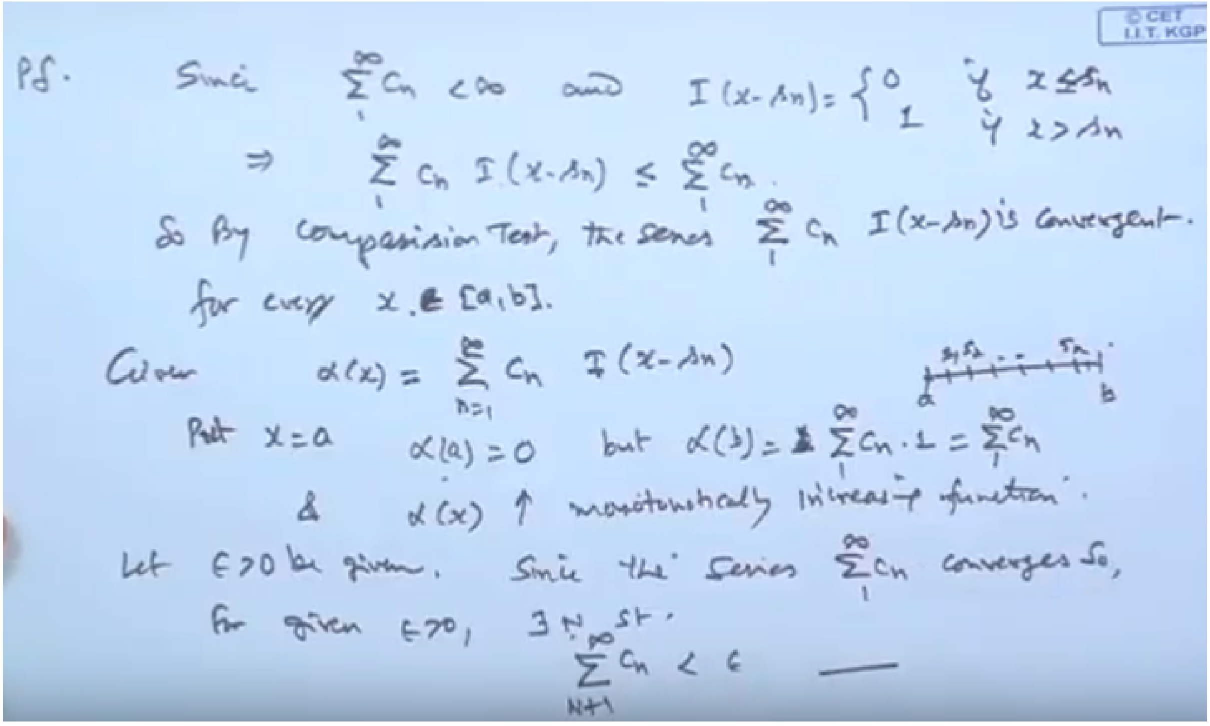

so proof of this ok, first thing is that since Sigma of Cn 1 to infinity is convergent and i of X minus SN, is 0, if X is less than equal to Sn and 1, if X is greater than Sn. So this implies that the series 1 to infinity, Cn, Cn of say i of X, minus Sn, will always be dominated by, the series Sigma Cn, 1 to infinity because this term this will help this were reduced, when all the terms, for all X, which are less than SN, the terms which i of this will be 0, so the series will have a lesser term and all the positive terms, so it is less than equal to Cn. But this is given to be convergent, so this is, so by comparison test, comparison test, the series Sigma 1, to infinity, Cn, i of X minus Sn, is convergent, is convergent, that's the first thing which you get it and convergent for every X, for every X, belongs to the interval say a, b, here of course we can choose the real line also their support for every X. Okay now let's take that alpha. What is alpha?

Alpha is given, given alpha X, as Sigma, n is 1 to infinity, Cn, i of X minus Sn, so what is the value of us? So if we put X equal to a, X line will be a, b, so let it be in the interval a, b, okay. So X is suppose a, then what will be the alpha a? this is our interval a, b and here, these are the s1, s2,.. sn, these are the points where s1, s2,.. SN, they belongs to the open interval, remember because these are the points of listing so say s1, s2,... sN and so on, these are the points, so a is a strictly less than s1 is less than s2 and so on, so when a is strictly less than 2 sn for each n, therefore i of X minus sn will be 0, so it means 0, but what is alpha B? When you take X of B, all the Sn‘s are less than equal to B and basically less than B, is it not? So it will be completely the value of this will be 1. so what we get from here that alpha a, is 0, alpha B is 1, and alpha X, is an increasing function, is a monotonically increasing function, that alpha s1, when you take X equal to x1, then it is less than X equal to x2 and sn. So it is a monotonically increasing Function. So alpha is a monotonic Function, first thing and this is. Okay and alpha B, okay so what is sorry this i, fX and 1, so basically 1 into Sigma, here I am sorry, this will be i of you can write this thing as Sigma Cn, 1 to infinity and this will be one, so you are getting Sigma Cn, Sigma Cn. So this is 1, ok we are getting this one, so one thing is clear that this will be a monotonic increasing function. Now let epsilon greater than 0, be given. Okay the series since the series Sigma Cn, 1 to infinity converges, so the remainder term of the series, is less than, so we can choose, so for given epsilon, greater than 0, there exist an N, such that the sum of this series, n plus 1, to infinity CN, will remain less than Epsilon, let it be, because by definition of the converse the remainder term of the series is convergent so this is less than Epsilon.

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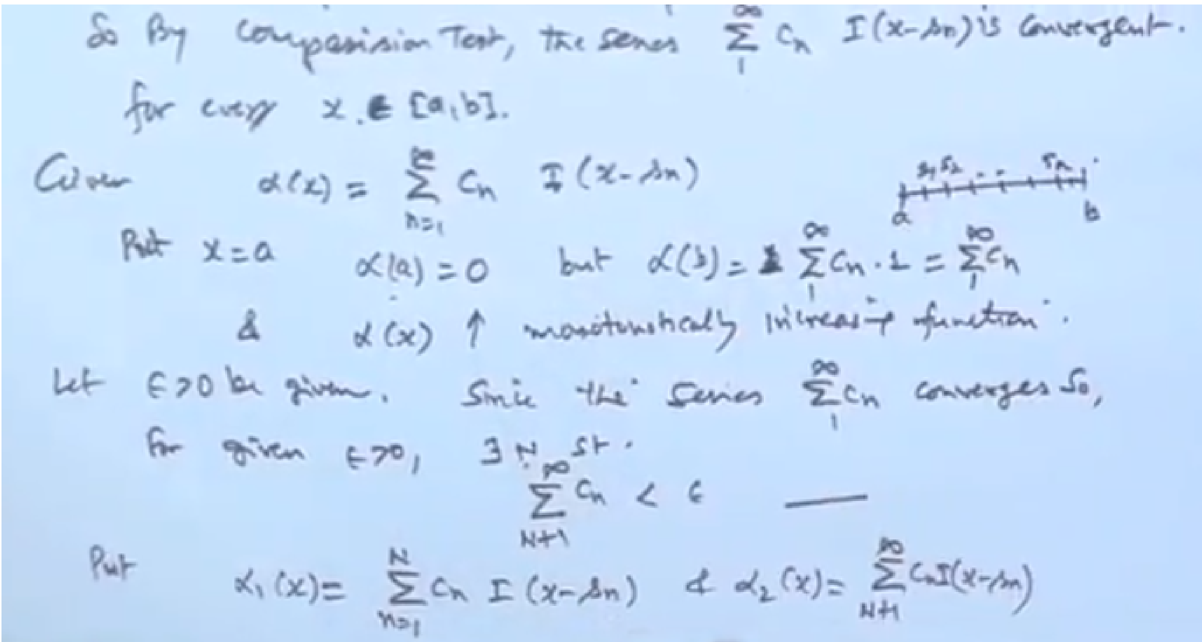

okay now this will be, let us take, put alpha 1X, be the sum of this series, of first n terms, Cn, i of X minus SN and alpha 2X, I am taking the sum of this series, where the terms are taking from n plus 1, onward of the series CN i of X minus SN, let it be this 2. Now since i of X minus n, because of this, this becomes a step function, so some of these step function, in case of this step function previous theorem we say it is a Riemann Stieltjes integral functions, ok. So the sum of this Riemann Stieltjes integral functions, we can write the sum of the this, therefore we can say F is alpha D alpha 1, its sorry, step function, so integral FD alpha, becomes the, by the if alpha is a step function, then integral F alpha a, is nothing but the FS. Is it not? That is what we have seen.

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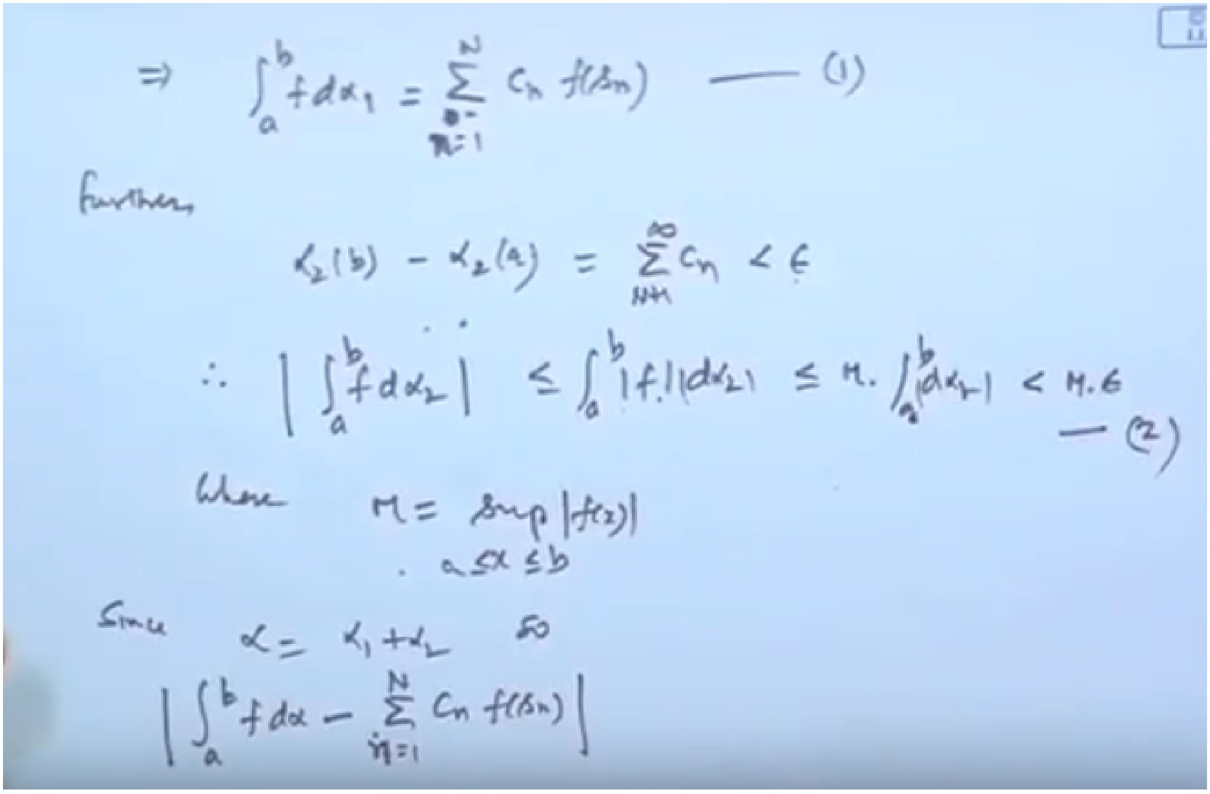

So here, so we get from Here is, so this implies that integral A to B, F, D alpha 1, is nothing but the Sigma, i is equal to 1 to N, 1 to N, i is equal to 1 to N, n is equal to 1 to N n equal to 1 to N CN, F of sn. Ok so this would be alpha 1 and this is number one. Because it will follow from the previous result, this result is there which, we have proved earlier the result is this. is it not, according to this. alpha is giving to be continuous f is continuous and this one so already F is continuous given and bounded and alpha X, is this, so value will be this FS. So here if you take this sum, if you take the integral of this F of, F of alpha 1 X and then we get immediately this can be written as the Sigma into this form. So there is nothing in to explain. Now further, what is our alpha 2b, minus alpha 2a? What is alpha 2?

The alpha 2, is defined as this, this is alpha 2X, so if I clip as X by a, then alpha 2X will be 0 and alpha 2B, will be Sigma of CF, so basically this is equal to the Sigma n plus 1, to infinity CN, but which is given to be less than Epsilon, because of the series is convergent, so remainder will be less than Epsilon, hence therefore, therefore the integral a to b FD alpha 2, with respect to alpha 2, the modulus of this is less than equal to a to b, mod of F, D alpha 2, is it not? D alpha 2, or mod of D alpha 2 and then D alpha 2, again this is bounded, so it is less than equal to, M times, and M times, integral A to B, D alpha 2, mod of this, and mod of this and this value is nothing but the alpha 2B, minus alpha a, which is less than Epsilon. So this is less than M into Epsilon, okay. So let it be the equation 2. So over alpha 1, we are getting this thing, over alpha 2 we are getting this thing. So where M, is the supremum value of the Function, FX, over the interval a, okay, that's what. Now since our alpha, is nothing but the alpha 1 plus, alpha 2, because when you take the Alpha X alpha X is the sum of this area 1 to infinity I break up into two parts, 1 to N and 2 and plus 1, so alpha is alpha 1, plus alpha 2 X, so we get from here is then, so integral A to B, FD alpha, minus Sigma i is equal to 1 to N, n is equal to 1 to N, and then CN, F of sn, mod of this. Now this will be,

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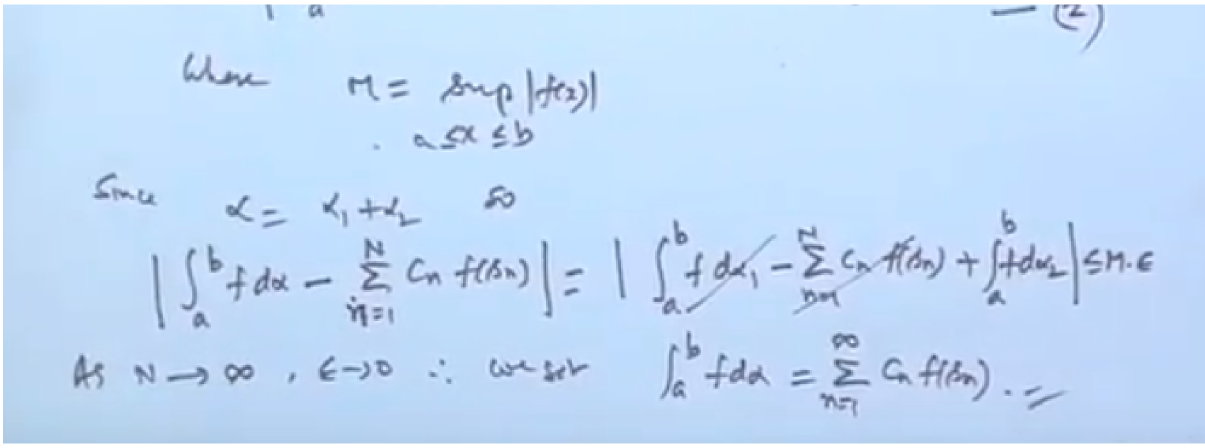

this can be written as what it is nothing but the mod integral A to B, F D alpha 1, minus Sigma, n is 1 to N, CN f of SN, plus integral A to B, F be alpha 2, because this D alpha, is alpha 1, plus alpha 2, we can write this now this part is already giving to be less than Epsilon this part is less than if basically this entire thing alpha 1 is the same as this so this part is 0 we get this sentential only this part in there and this part we have shown this is less than M into F signer so it is less than equal to M but epsilon is arbitrary so when as n tends to infinity epsilon all will go to zero so epsilon will go to zero therefore we get integral A to B, FD alpha, is nothing but the Sigma n is 1 to infinity, CN, F of sn and that's complete the results, okay, so this one, clear? So what this result says is that in case of the function alpha, is a monotonically is a step function, then our Riemann Stieltjes integral functions reduces to infinite series or finite.