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Course

On

Introductory Course in Real Analysis

By

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Lecture 70: Properties of Riemann Steiltjes Integral

Now what we do is in the step 3 we remove the open interval, remove the segment UJ, VJ from the closed interval A, B, it means what we are doing is that this is our interval A, B, okay, and here is the point say U1, B1, U2 sorry U1, V1, U2, V2 and like this, suppose these are the points say like UN say VN.

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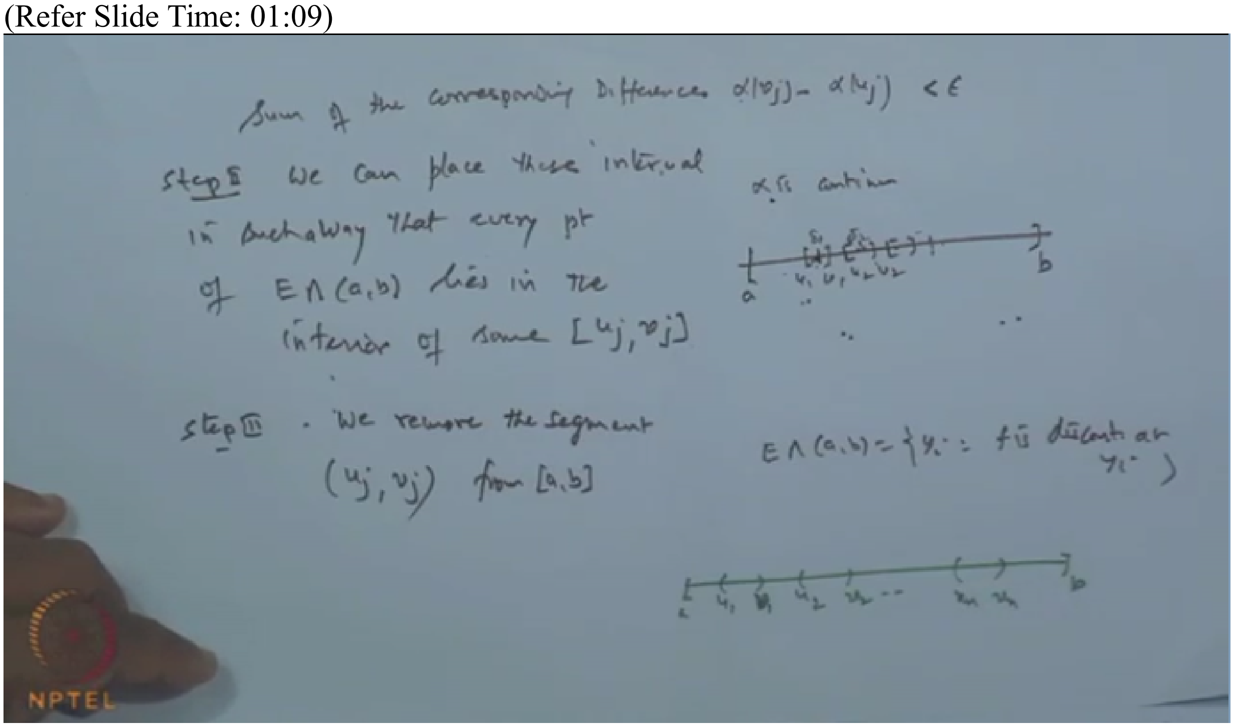

Then what we do is we are removing this portion, this portion we are removing, so we are getting now after remove it A into U1, A U1, this 1, then start with V1, U2, and like this, then start with V2 and continue this way and up to here say B, so this set K which is, in which these things are not available, these things are not available,

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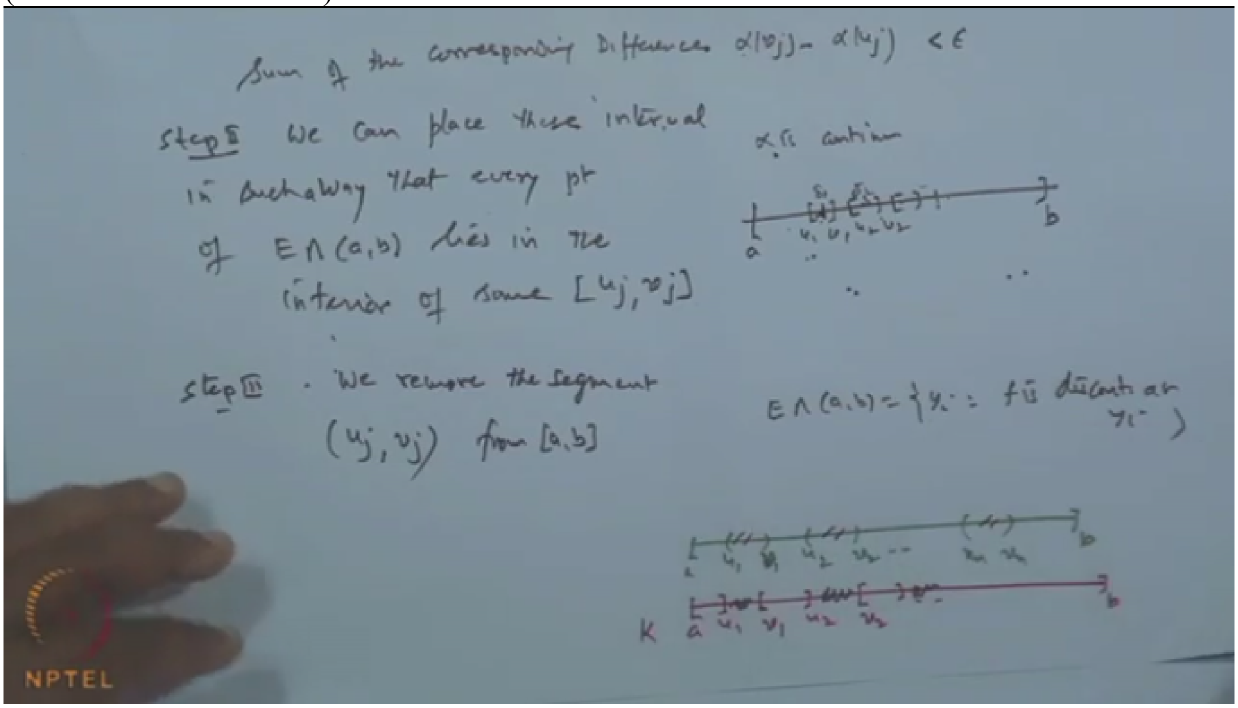

okay, so this set K forms a compact set. So once we remove the segment UJ, VJ from here, then the remaining set K is compact, because it is a finite union of the compact closed and compact set intervals like this, so it is a compact set, is compact or compact set, is okay?

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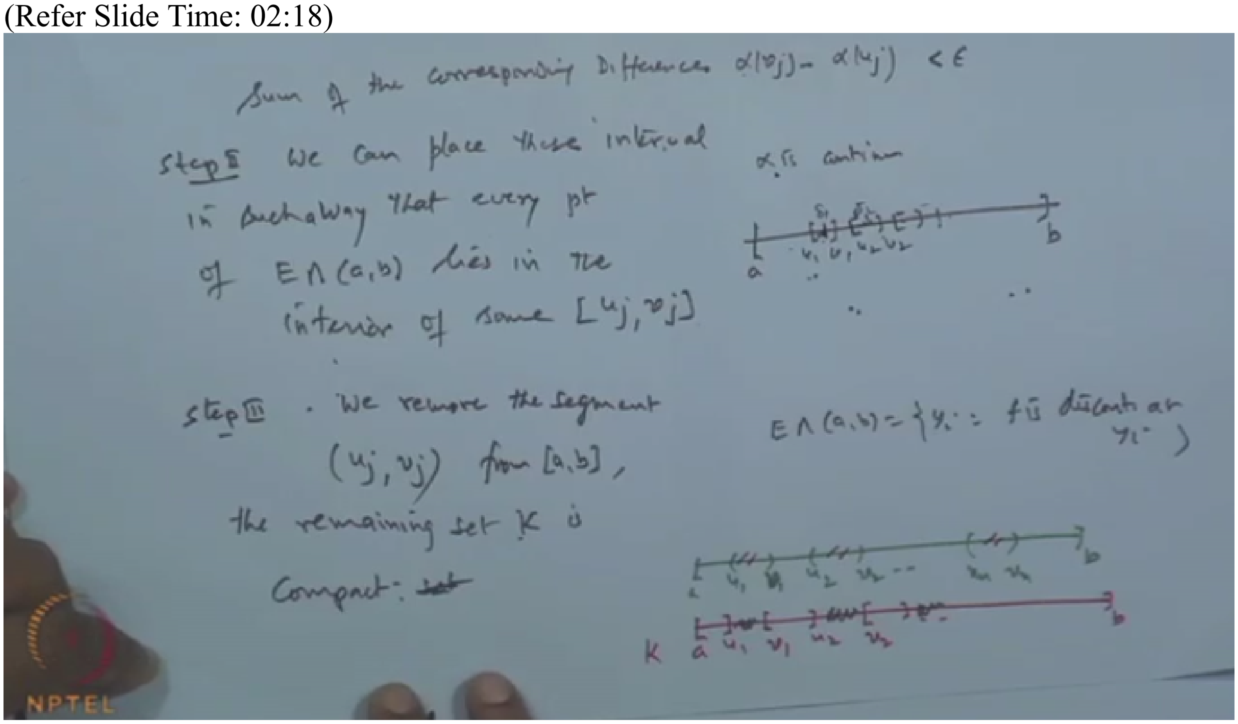

Now function F is given to be continuous, since F is continuous over K because that point of discontinuity we have already removed, so F is continuous over K and K is compact set so every continuous function on a compact set is uniformly continuous,

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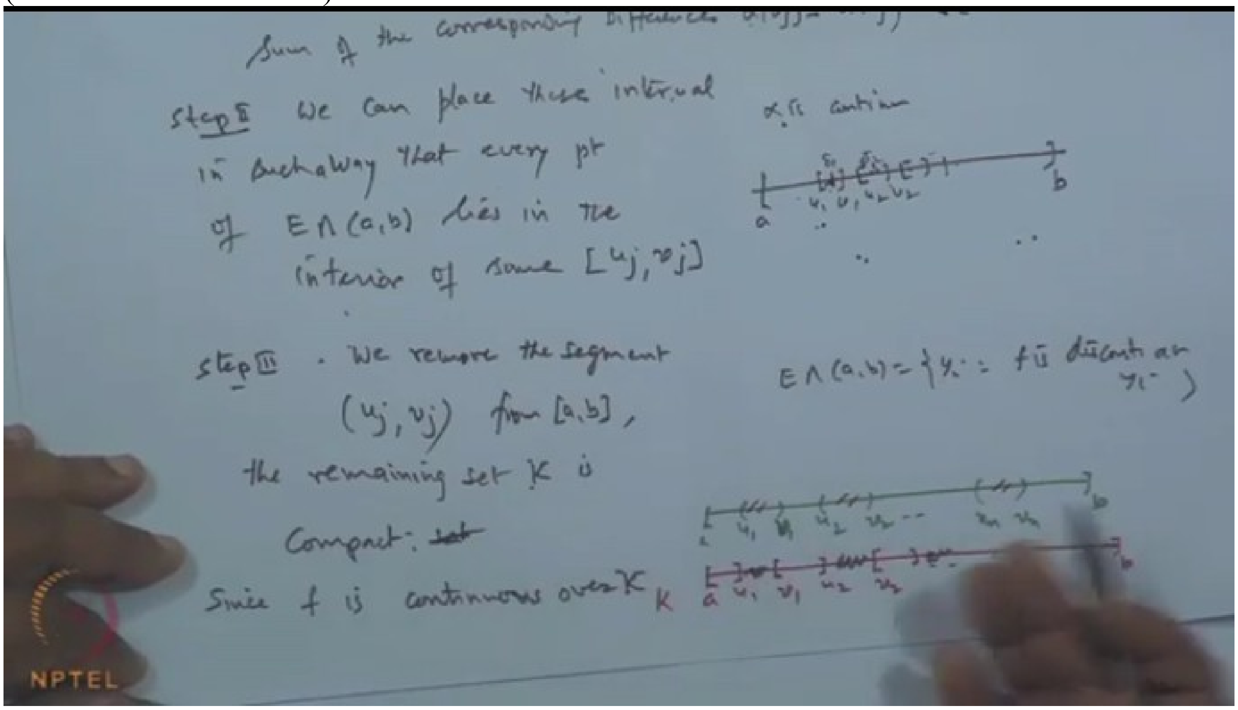

so F will be, so F is uniformly continuous function on K, clear.

So by definition what is F is uniform means? So for a given epsilon greater than 0, so for given epsilon greater than 0 there exists a delta which depends only on epsilon not the point greater than 0 such that the mod of F(s) – F(t) this will remain less than epsilon if S belongs to K, T belongs to K, and mod of S-T is less than delta, by definition of the uniform continuity, okay let it be 1, okay.

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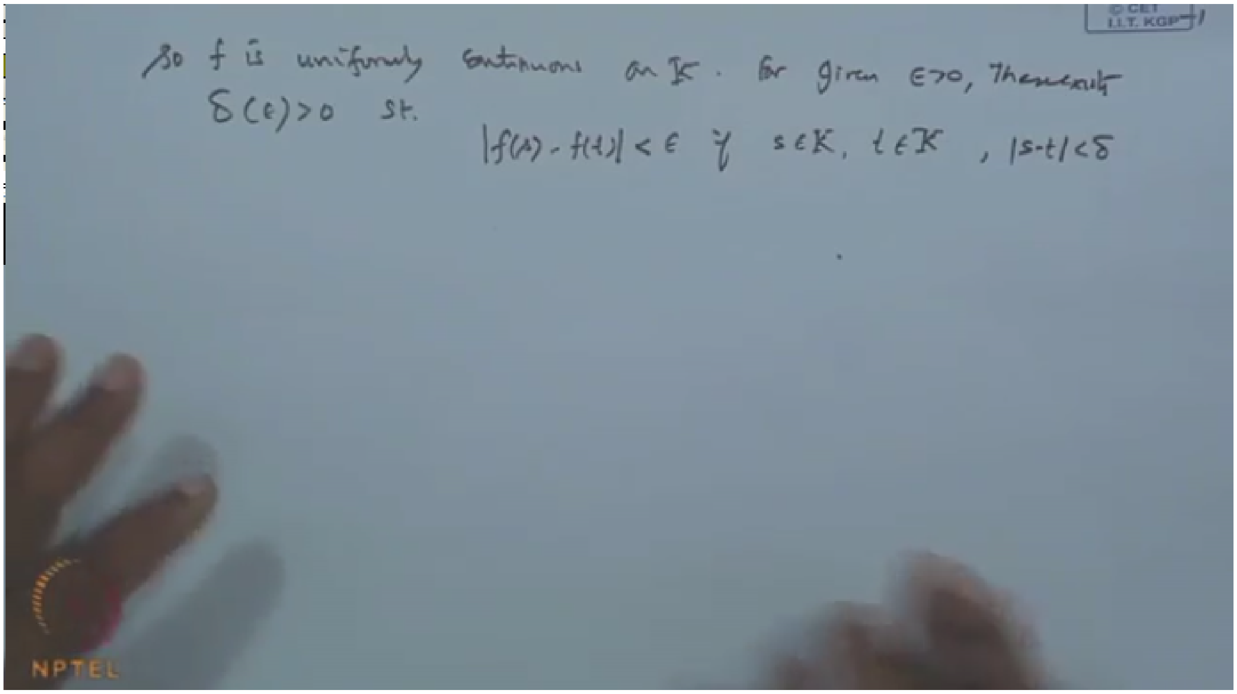

Now let us partition, now partition the interval (A, B) as follows, suppose P is the partition, let P is the partition X naught, X1, X2, say XN of (A, B), then B takes such partition that each U, P is a partition where UJ, each UJ occurs in P, each VJ occurs in P means this U1, E2, E1, V1, V2, VN these are the one of the points of the partitioning point of this interval (A, B), and no point of any segment (UJ, VJ) this open segment occurs in P, this is the restriction we are putting,

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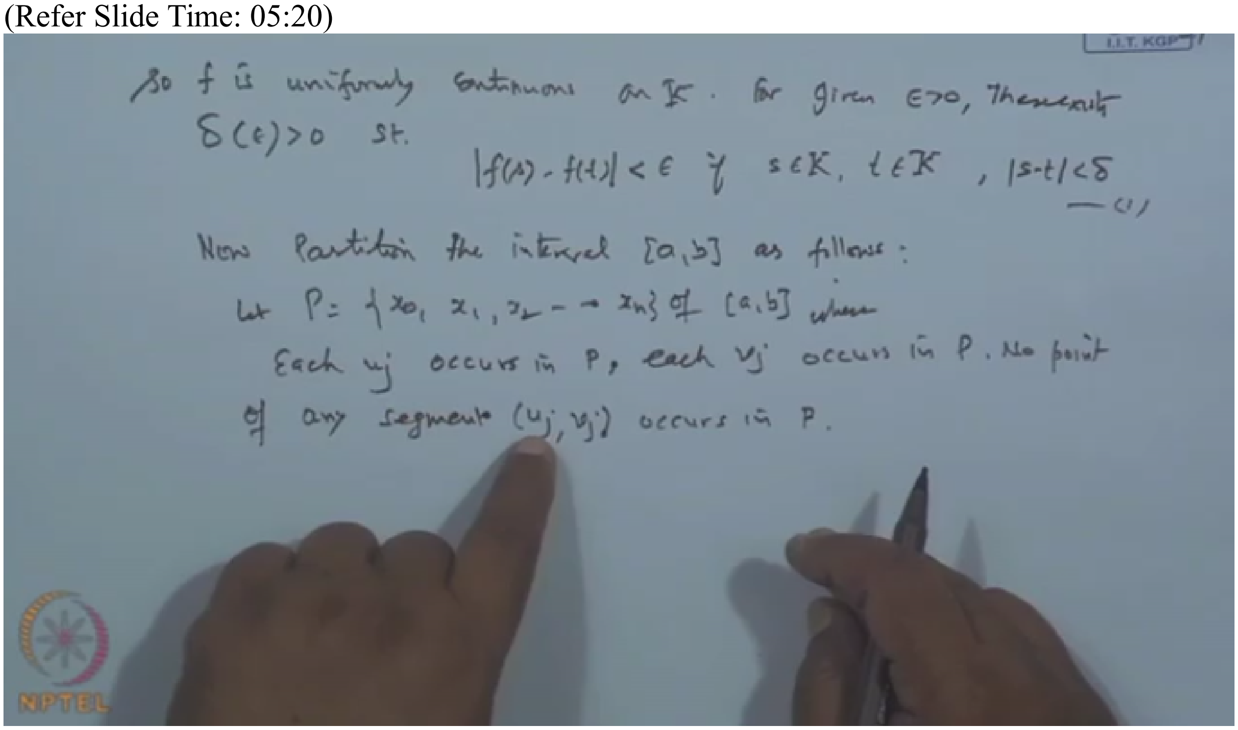

we are choosing the partition in such a way that the corner points of these sub intervals which are covering the point of discontinuities are basically coinciding with the partitioning point, but none of the point in between in this, inside this interval all the partitioning point, are the points of the partition this is one thing.

Second thing in incase if XI-1 suppose is not one of the UJ, because this might be possible, the number of the points are finite, may be the number of points are not as efficient as the partitioning point is there, so some of the XI's will remain untouched, so if XI, MI is not one of the user, then what we do is then we put the restriction here that the XI-1 that is the delta XI this should be less than delta, this is our restriction, okay,
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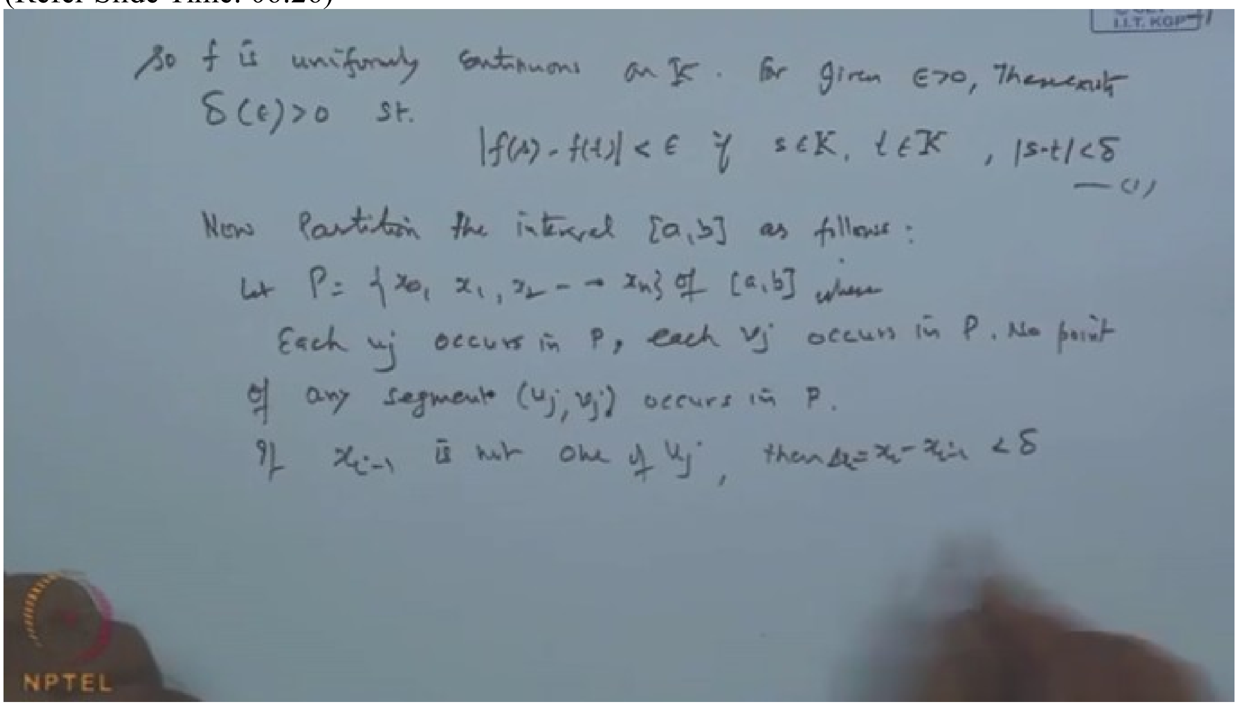

so once you have partitioning this point it means, okay, now we get this, so this is our X naught, X1, X2, X3, XN and so on, say this is A, this is B, okay, and then we are coinciding this as suppose U naught, U1 and so on continue suppose here is UJ, and after this we are not getting anything, and the points in between UJ are not there, one thing.

So let us see suppose we take any interval clearly over any sub intervals MI - small mi this will remain less than equal to 2M, why? Because what is our M? M is the supremum value we have chosen,
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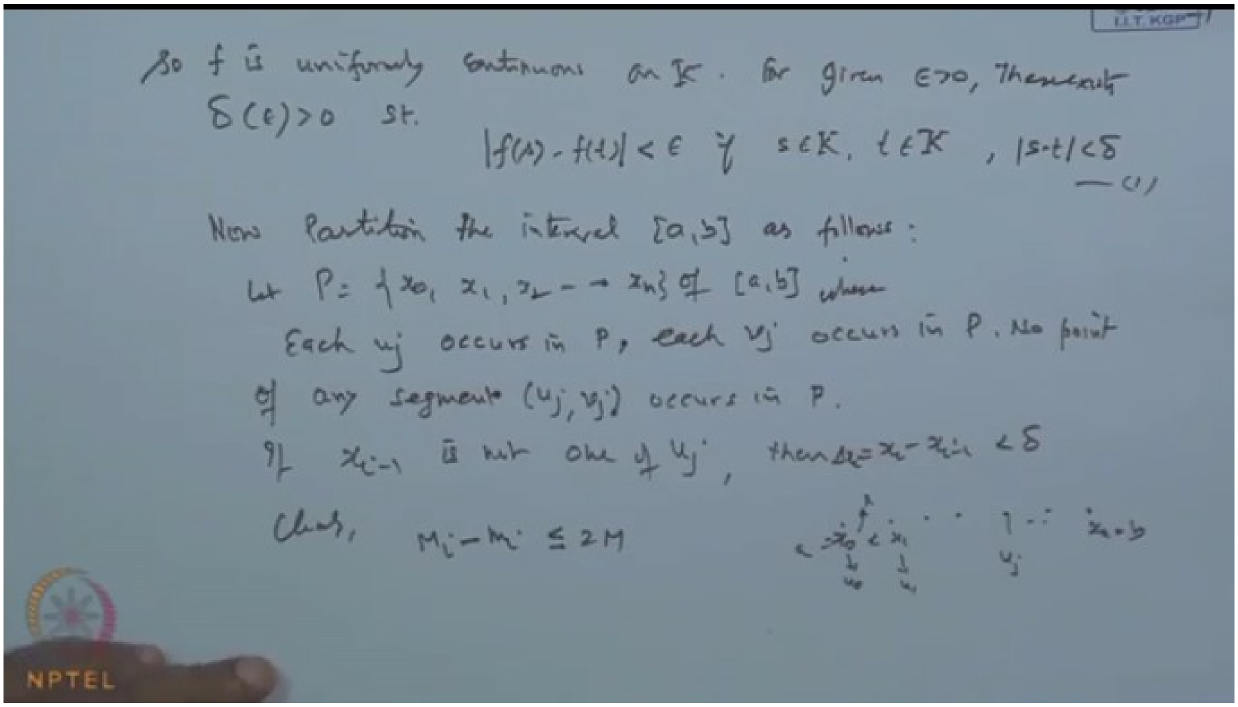

this is M is the supremum value over interval so if I consider the XI-XI-1 clearly because over the interval, over XI-1 to XI the minimum value of the function is MI, supremum value is M, so MI as well as a small mi both are less than equal to M, so MI-mi is less than equal to 2M,

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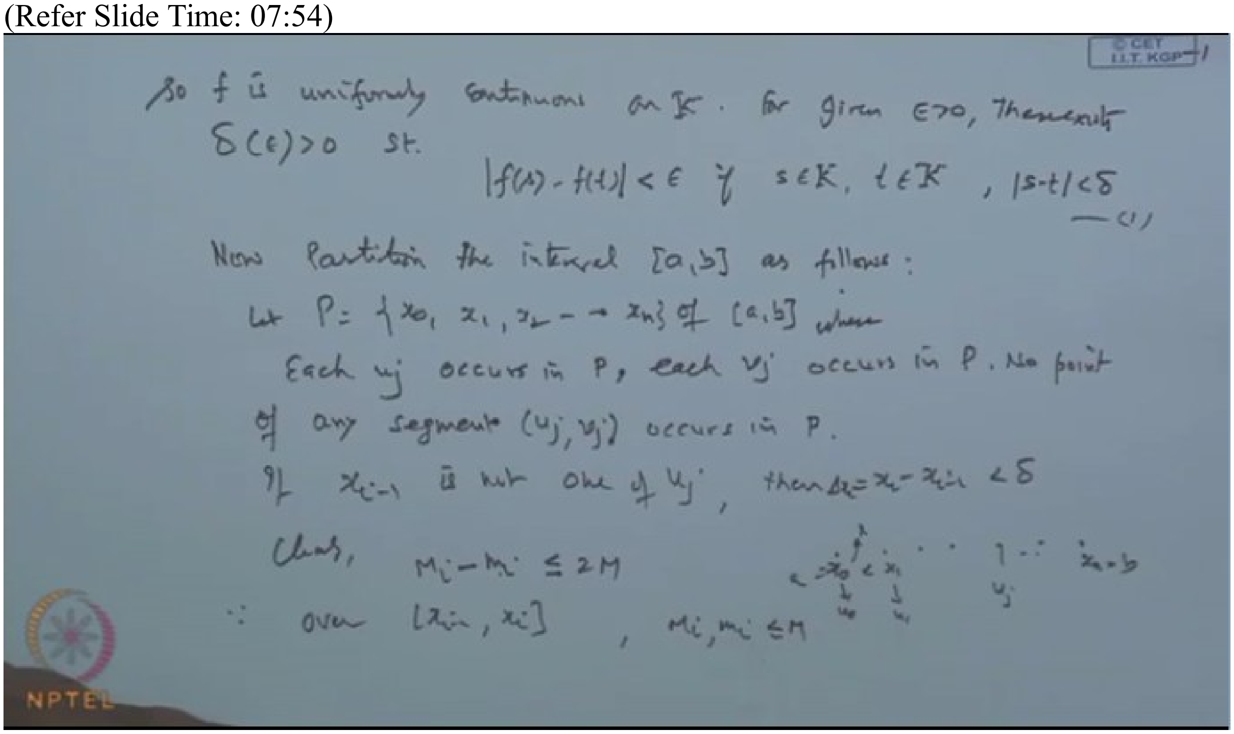

this is true for each I, for every I, this is true.

And further if XI 1 is not one of the point, if XI-1 is not one of the point of UJ,

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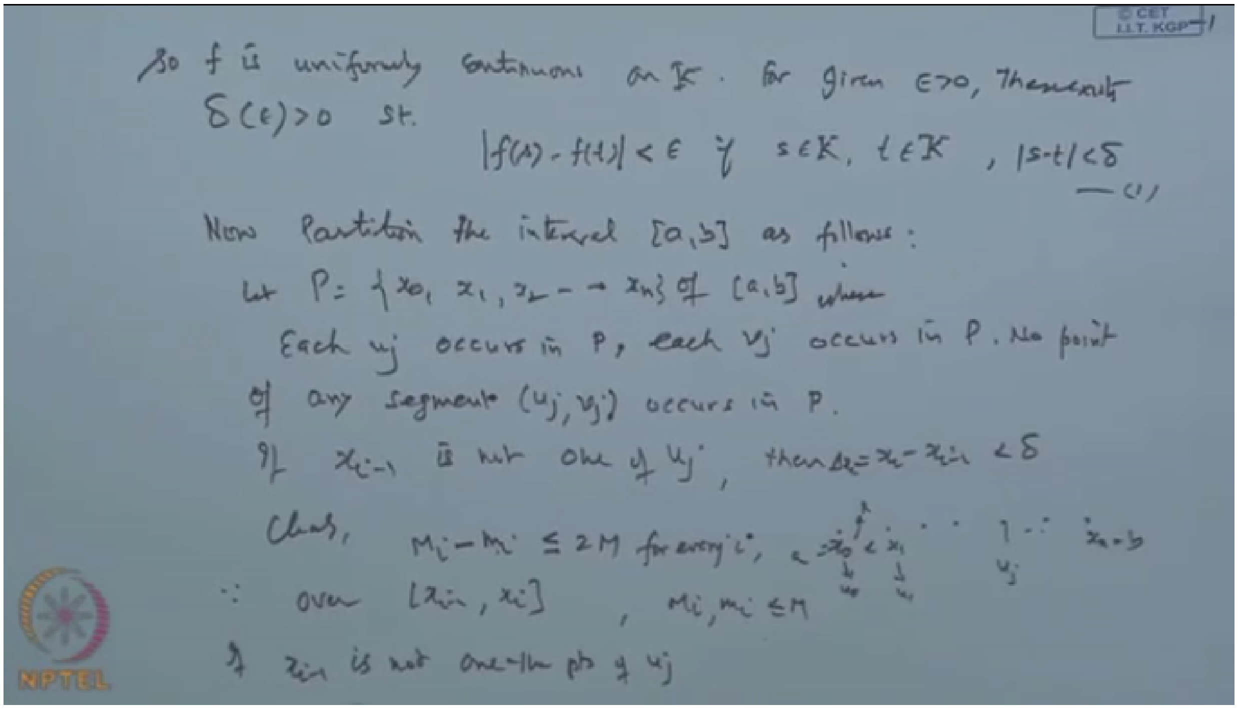

unless this is not one of the point then the difference between MI- small mi, this difference can be made less than equal to epsilon, why?

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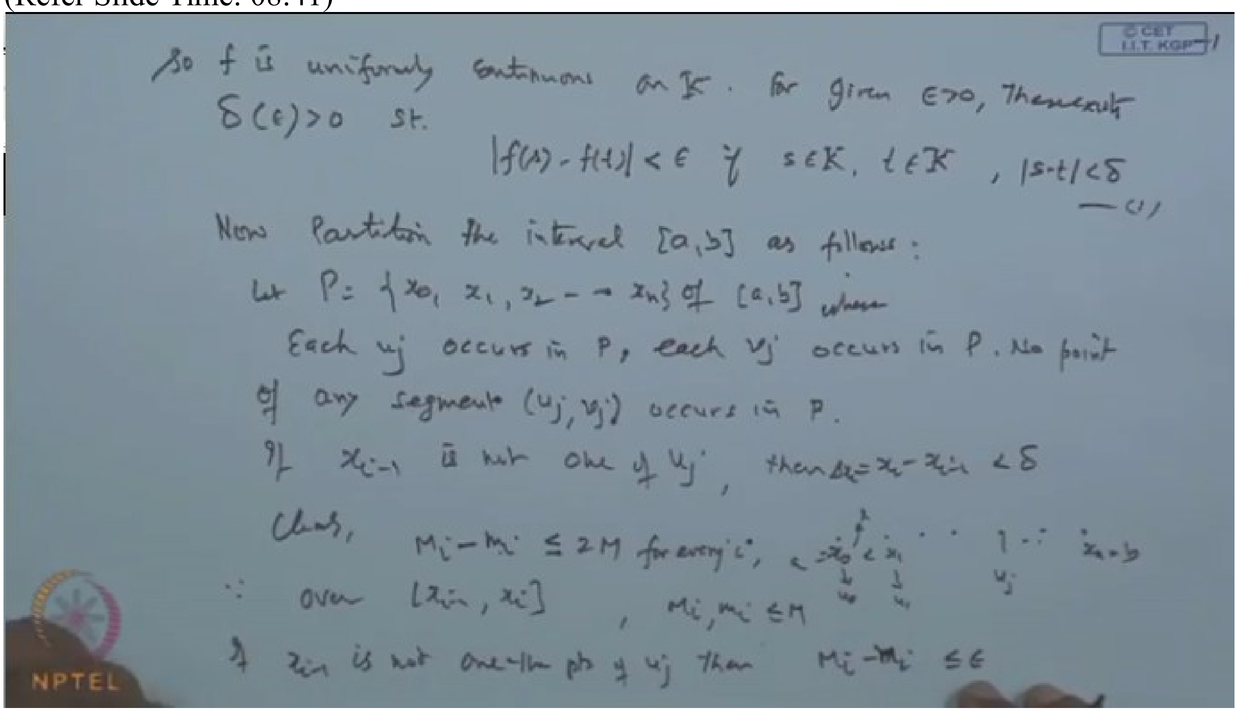

Because of this part, because a function is continuous as well as uniformly continuous over the K, so if the point XI-XI-1 less than delta, if this is because we have put this XI, if XI is not one of the point then this difference is less than delta, so because of the one this difference should be less than epsilon, so MI- small mi will be less than epsilon, okay, so that for each one this will be less than equal to epsilon.

Hence if we constructed the proof upper sum - lower sum, so upper sum of the function F with respect to alpha over the partition that for P - lower sum of the function F with respect to alpha over the partition P, this is basically what sigma, I = 1 to N, say MI - small mi into delta alpha I, which can be break up as sigma over K + sigma over (A,B) – K, we got that drop till double, okay, this which are dropped, so in this case is over K this is less than epsilon, so we get epsilon into sigma of this part, but sigma of this delta alpha I, I is 1 to N is nothing but the alpha B -alpha A, so this will, and this part over this here, already we have, function is bounded, so +MI is less than equal to 2M, and since sigma of alpha I, alpha B - this total is less than epsilon, this is because of the continuity this sum of these corresponding is less than epsilon, so we get from here is that this is less than epsilon, okay, and this part will be epsilon alpha B - alpha A + 2M epsilon, but epsilon is arbitrarily small number, so this shows that the right hand side will go to 0 the left hand side, so alpha F will be the element of F alpha, this is proved that, okay, so this is what proof.

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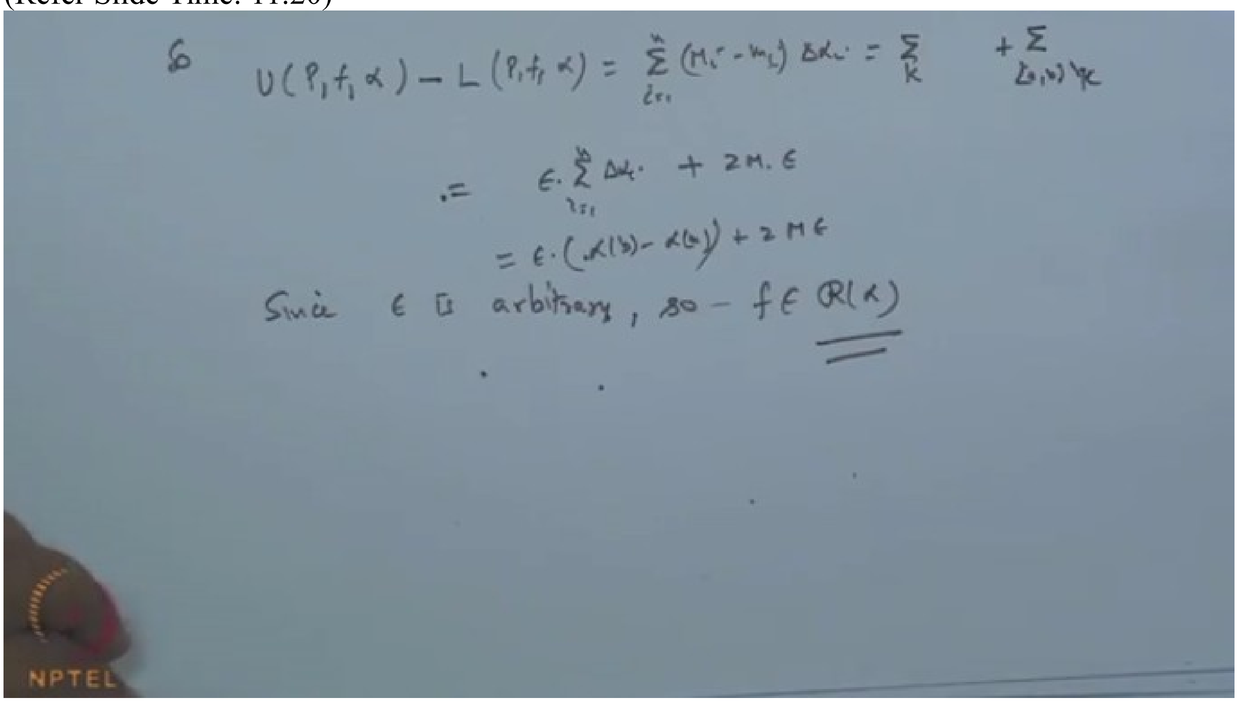

Now another results also which is interesting, the result says suppose F is in R(alpha) on the closed interval (A,B), Riemann Steiltjes Integral with respect to alpha over that interval (A, B), and F is bounded function, bounded by say m in capital M, phi is continuous on the closed interval m and capital M, this is phi. Then an H, H(x) is phi of F(x) on the closed interval (A, B), then this result says the H belongs to R alpha that Riemann Steiltjes integral on (A, B),

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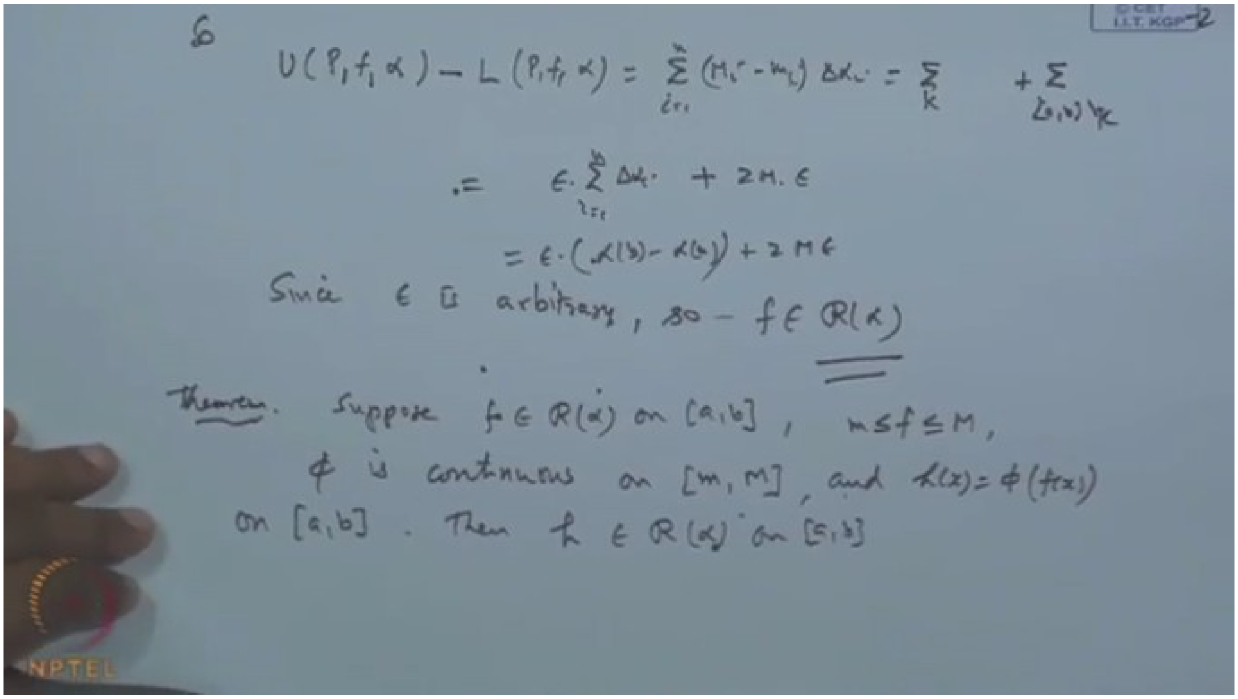

so if function F is Riemann integral function and phi is a continuous function on the (m, M) that is where the function have a range set, range of the set F is (m, M) interval, then the composition of this function continuous, image of a Riemann Steiltjes integral function will be Riemann Steiltjes integral, that is what it says, okay.

So this is just like that here we are taking this function this is our interval (A, B), function F is given, so here is our m and capital M where the function F is attains the minimum and maximum value, and over this is the function H is defined, okay, so what is F? When we combined the phi composition F(x) then it will transfer directly from here to here which is H composition H(x) = phi composition F(x).

And if this is in RL, this function phi, oh this is phi sorry, this is phi, and if this phi is continuous and this is continuous, this is in R(alpha),

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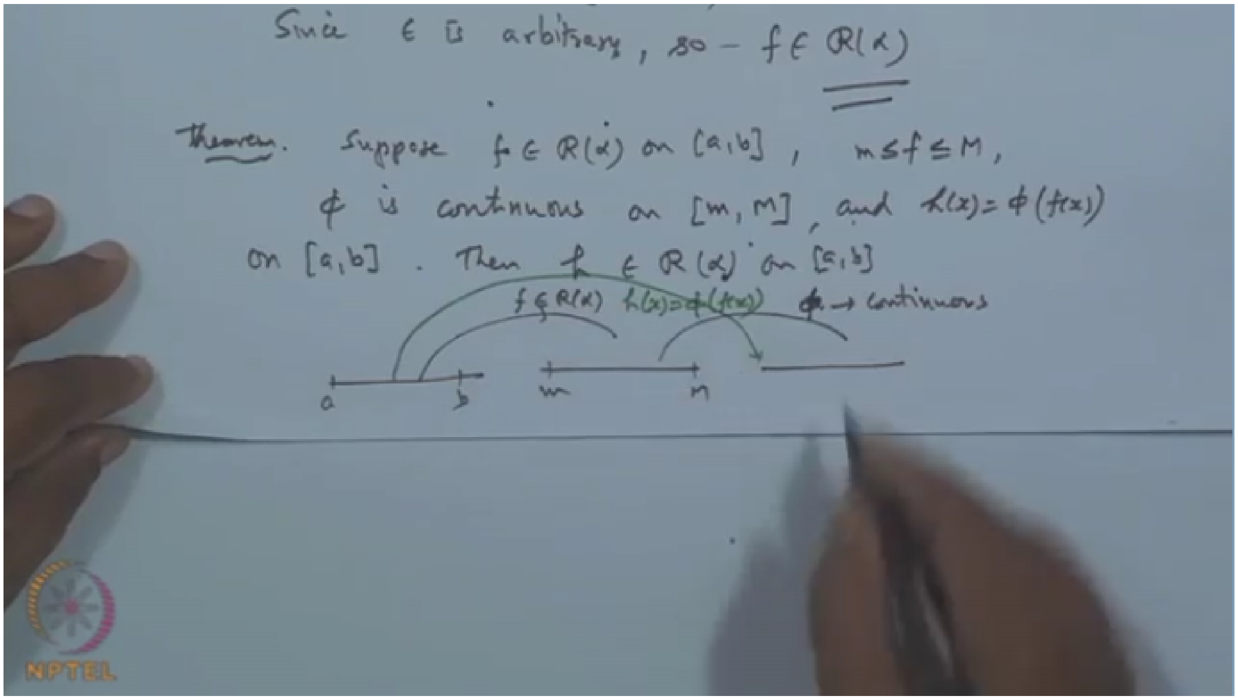

then our H will be in R(alpha) that is what it says, okay, so let's see the proof of it, choose epsilon greater than 0, okay.

Now given that phi, since phi is continuous on the closed interval m and capital M, so it will be uniformly continuous over this interval m and capital M, because every continuous function over a compact set is uniformly continuous, so by definition epsilon we have already chosen, so let us choose the delta, so for this epsilon greater than 0 which is chosen earlier there exists a delta greater than 0, such that now I'm putting the restriction on this delta is smaller than epsilon, because this is any delta is true then all the delta which are less putting delta to be less than epsilon this image will go there, okay, so let us put and take the delta to be less than epsilon, and phi(s) – phi(t) is less than epsilon, if mod of S-T is less than equal to delta, what you say is because phi is uniformly continuous, so for any epsilon greater than 0 there exists a delta such that image of this any point in which satisfy this condition must fall within this range.

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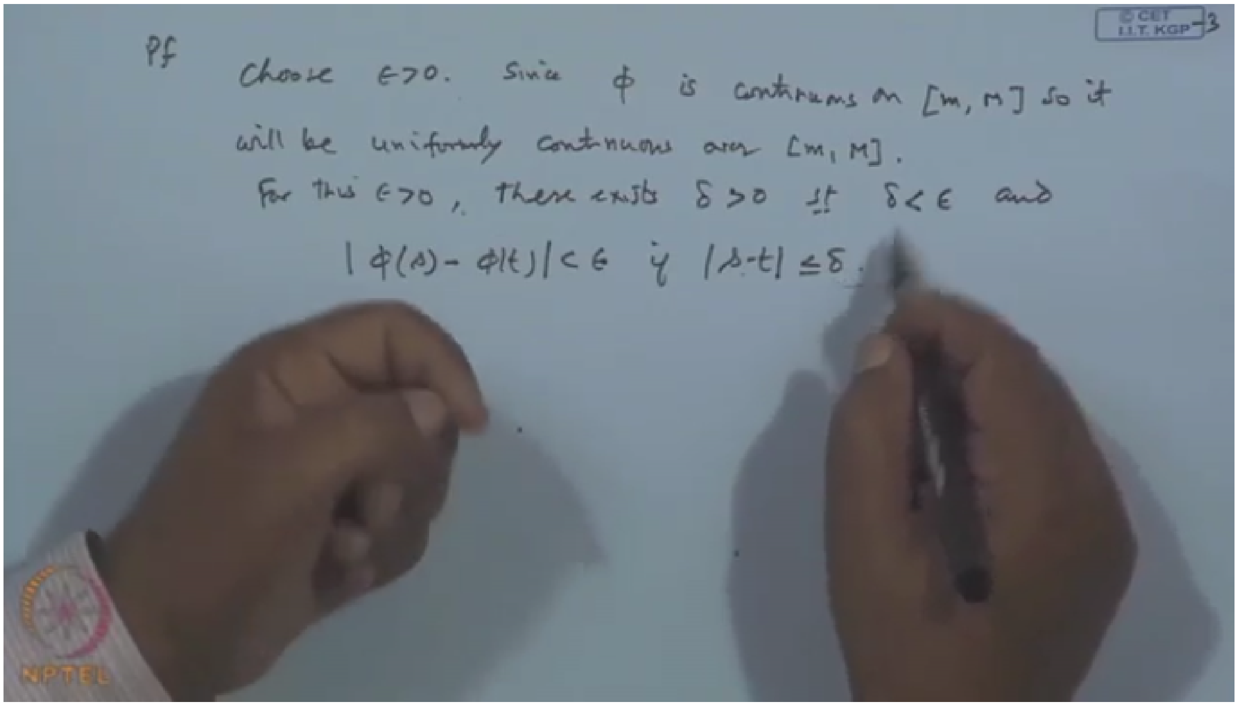

Now suppose I take delta which is greater than epsilon then I can picked up another delta which is smaller than this less than epsilon, so again this image will fall here, so nothing, we are not losing anything, but it will be advantageous while proving the whole theorem, okay.

Now further since F is giving to be an element in R(alpha), so by definition the result a necessary sufficient condition for a function to be a Riemann Steiltjes integral or reviewing the class of this if there exists a some partition, for a given epsilon there exists some partition such that upper sum - lower sum is less than this, so let F belongs to this then there is a partition, there is a partition say P, X naught, X1, X2 and XN of (A,B) such that the upper sum of the function F with respect to alpha - lower sum of the function F with respect to alpha over the same partition is less than say delta square, clear? So this we are getting, so let it be quiz 2.
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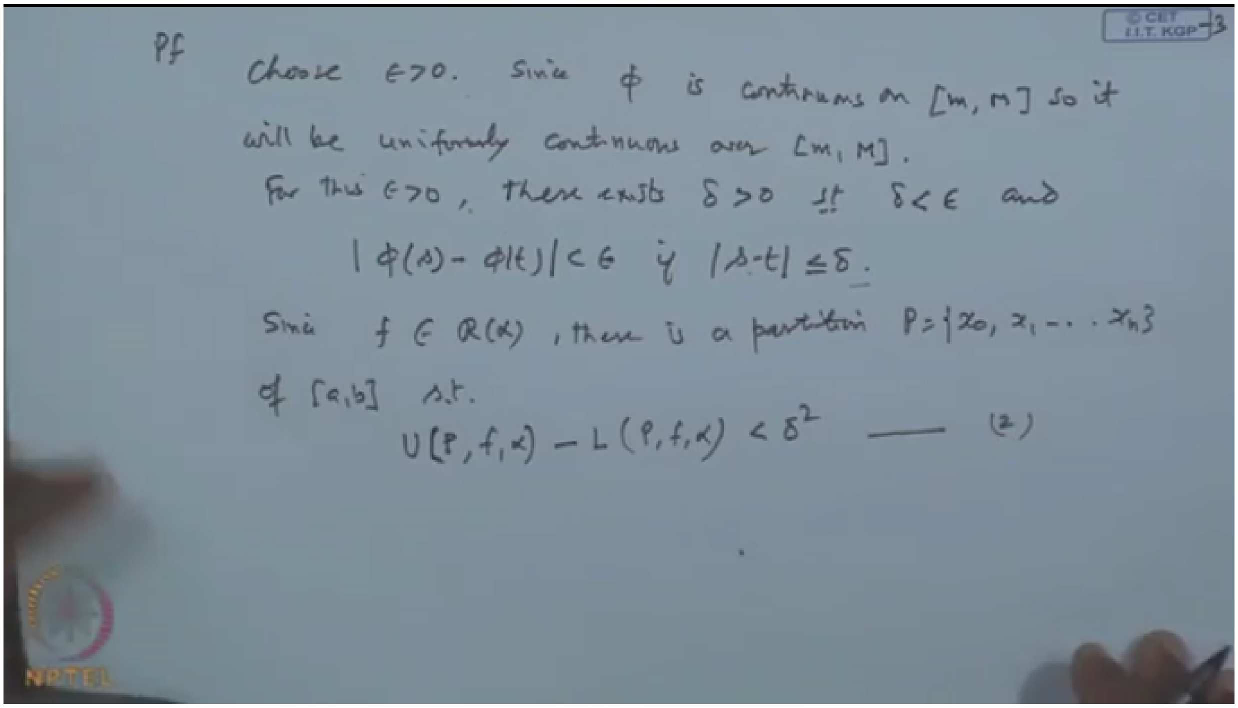

Now over the sub intervals we have let XI-1, XI+, this is the sub interval of this partition (A, B), this is the partition (A, B), this is how, now here the function attains this minimum value say mi and the maximum value is suppose capital MI, so let mi and capital MI, MI is the supremum of the function F(x) when X lies between XI-1 to XI, while the mi, small mi is the infimum value of the function F(x) when the X lies between XI-1 to XI, clear, let be this one.

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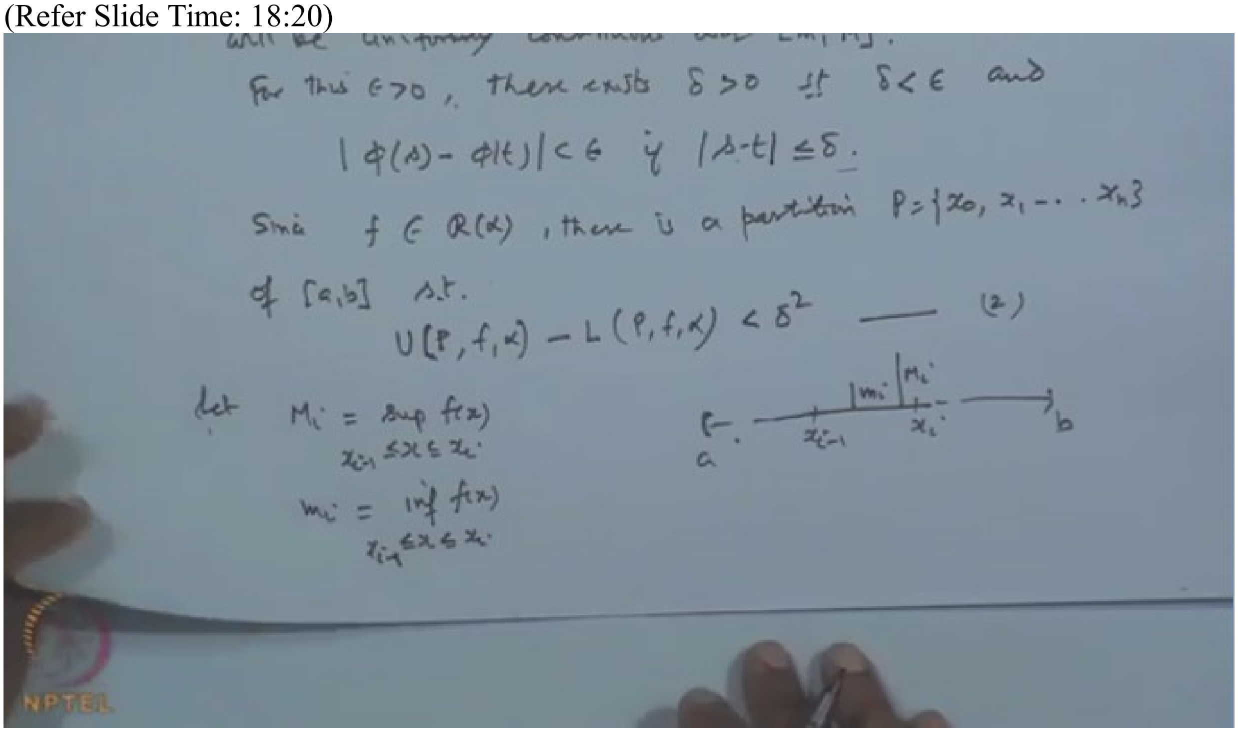

Now function H is defined, since H is defined because what was the result is, that H is yeah, F is Riemann integrable, F is this and phi is continuous function, phi is continuous on this and H is a function which is basically phi of H F(x), so what are the range of F, H is defined on it, so basically the H is defined on m and capital M let's say, so in this case when you partitioning this (m, M) say m and capital M and suppose I partition it, say M naught less than M1 and so on, say here is MN = N, so if we picked up the say MI-1 and MI, now in this case when you consider the upper and lower bound for the function H, then we say we denote this as let us denote, let MI star is the supremum value of H(t), I am using the T where T lies between MI-1 to MI, and small mi star is the infimum value of H(t) when T lies between MI-1 to MI, is it not? Just I'm taking this one.

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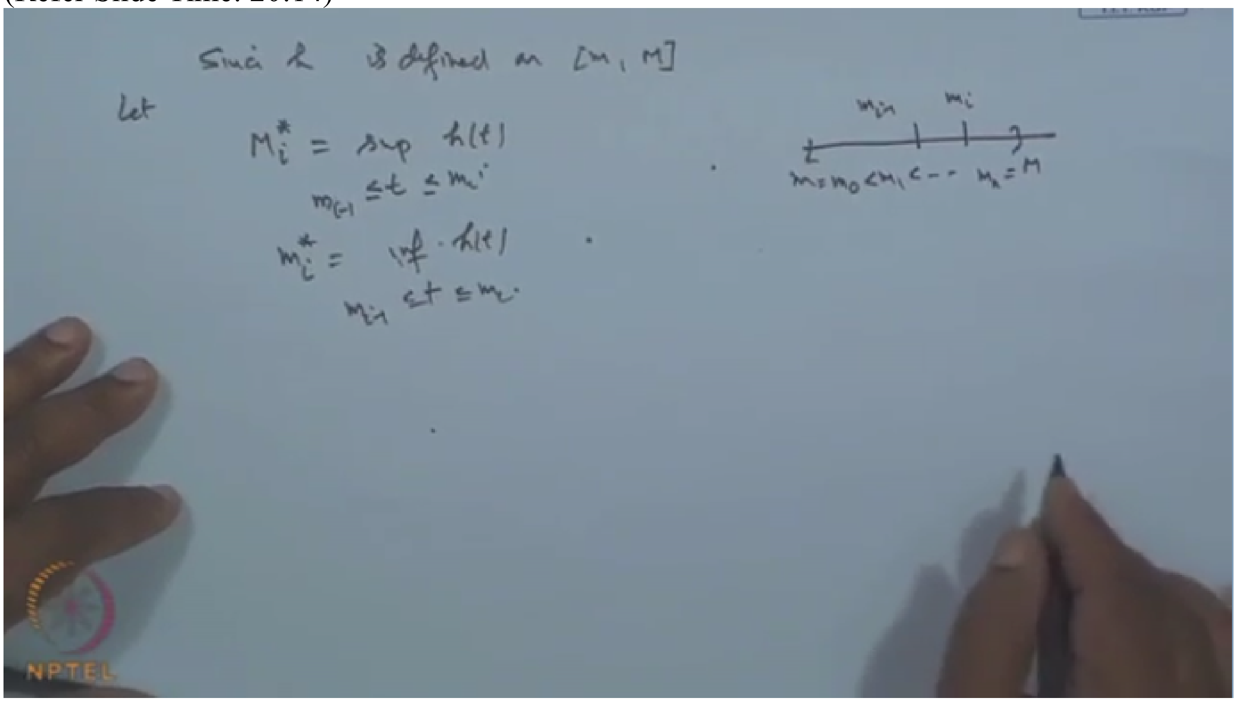

Now, so this is our interval (A, B) here I am taking one say sub interval XI, XI, one over one, and here is say suppose MI's, so this is our say m and here is capital M say this is abort, so basically this will be that, okay, that all the functions will be somewhere here, clear, in fact this is wrong, this one, like this, so here is now we are dividing here and then H is defined over this, H is defined on this side, okay.

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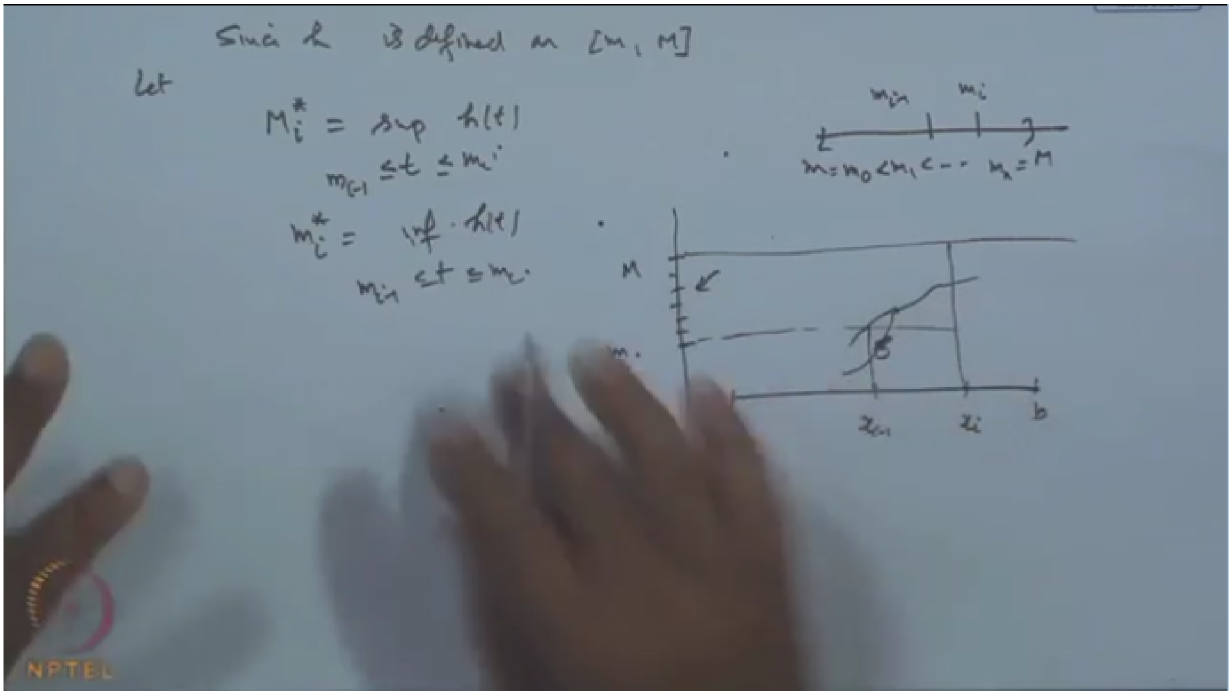

Now let us choose the part, suppose for I divide the number 1, 2, 3, N into two classes, first class is A, two classes A and B as follows, when I belongs to A, for I belongs to A, our choice of delta shows, I belongs to A means that is, if I belongs to A for I belongs to A, if just I belongs to A, if MI - small mi is less than delta, and I belongs to B if MI-small mi is greater than or equal to that, so this will be like this.

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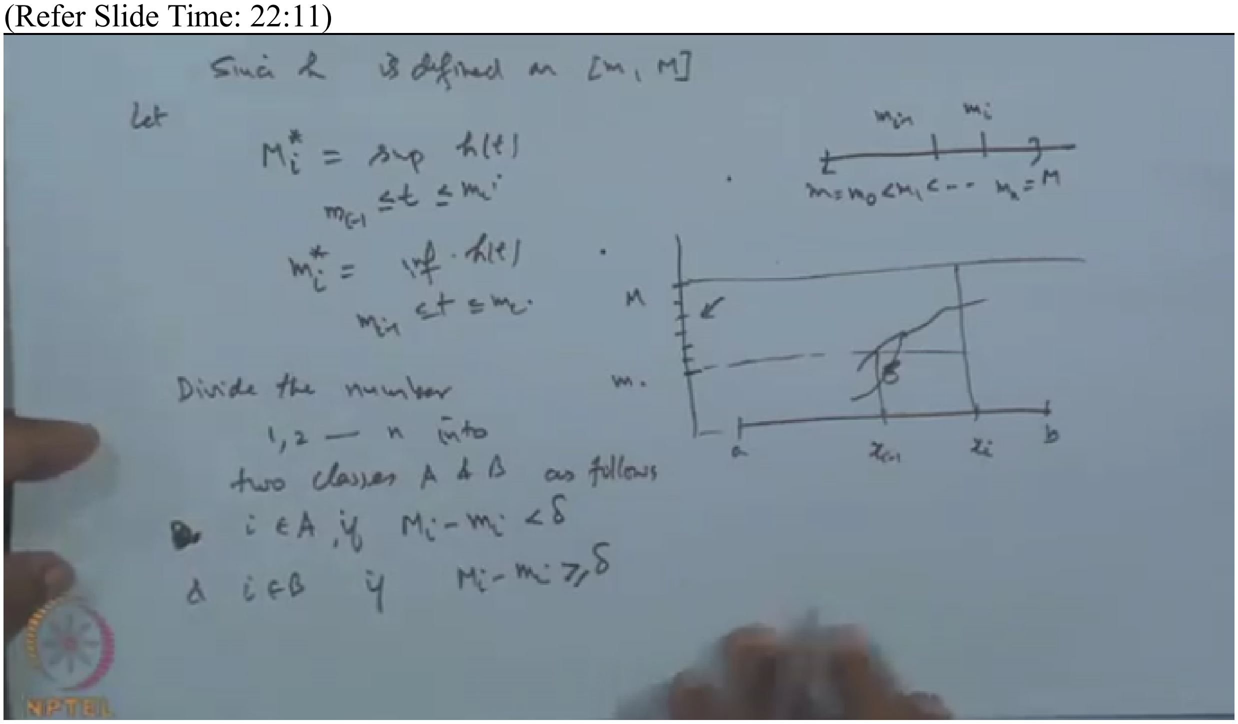

Now picked up now class one, so for I belongs to A, what is our choice? When I is in A the MI - small mi is delta, okay, so in this case what is the MI star - mi star, small mi star? This will be because phi is what? Phi is uniformly continuous, so if I belongs to A it means it satisfy this condition, once it satisfies this condition then because the H or phi is defined on what? On this interval, so is satisfied this condition therefore the difference of this MI - small mi star must be less than epsilon, so since our I belongs to this, so our choice of delta shows that the MI star – small mi star is less than, and this follows from this equation that is the relation, I will say the relation is say X term, okay,

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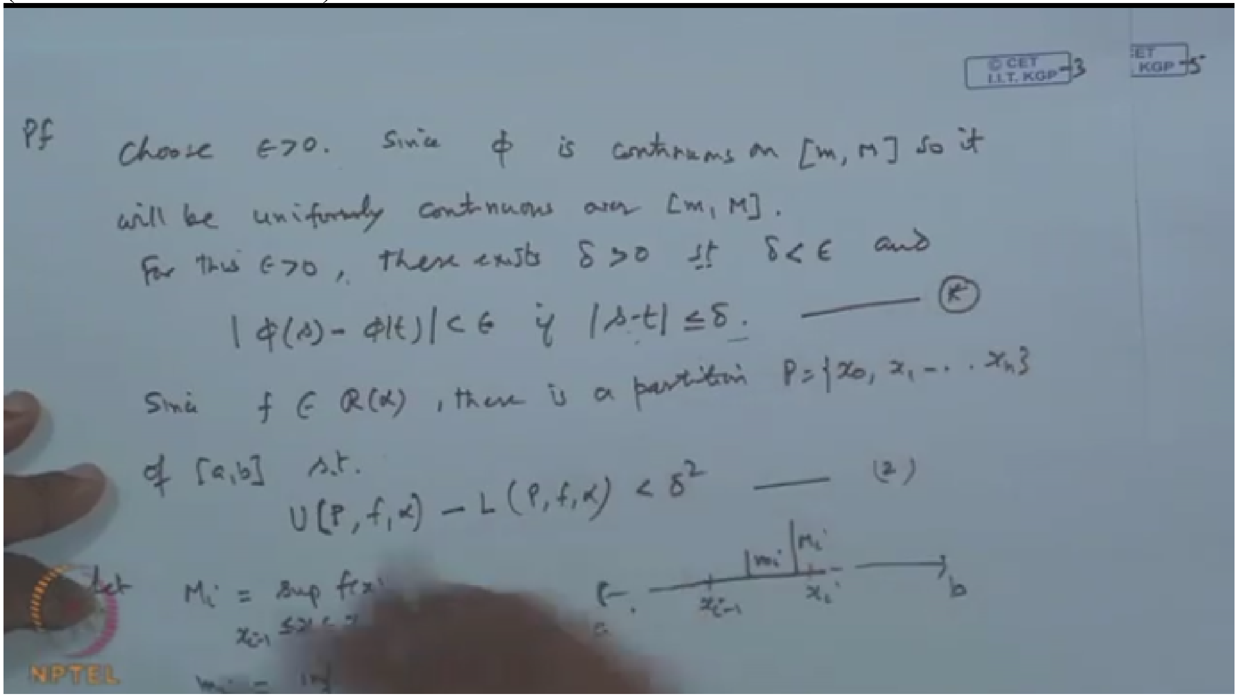

so it follows from the relation star, okay, so nothing.

So first thing and if for I belongs to B where this is greater than equal to delta, we have in this interval the MI star - small mi star is less than equal to 2K, where K stands for the supremum value of the phi when T lies between m and capital M, so just supremum value, and since it is greater than, so use the condition 2, where condition 2 is this condition we have taken that is MI - delta square, where is that delta square condition? So this is yeah condition 2, this is our condition 2, this MI, sigma of MI - small mi delta alpha I is less than delta square, so use the condition 2, so use 2, then in that case the U(P, F, alpha) - L (P, F, alpha), this is what? This is sigma MI- small mi delta alpha I, I is 1 to N, but if we take and this is given to be less than delta square, this is given to be delta given.

Now if I take the only for sigma MI-small mi delta alpha I over the set B only, then obviously it will remain less than delta square, because this sum is smaller than this, so it is less than delta square, but this will be greater than equal to what?

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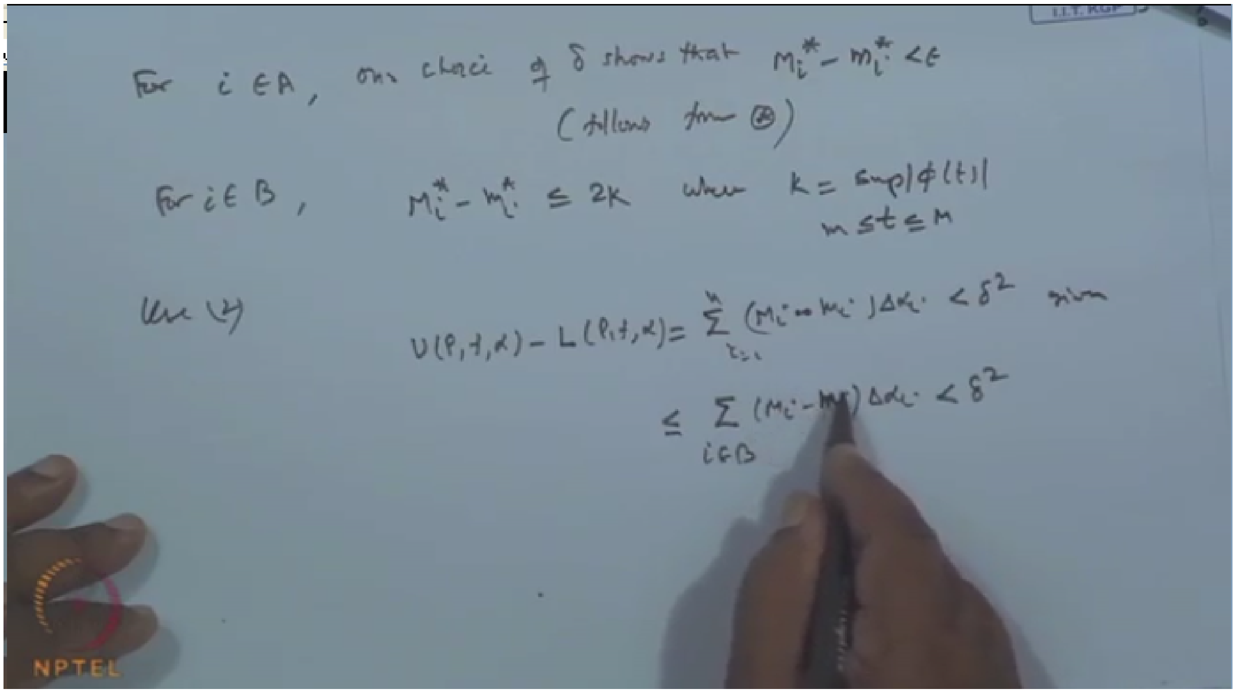

Over B this is greater than equal to delta, so it is delta sigma I belongs to B delta alpha I, okay, so what does imply? That's implies that sigma delta alpha I, when I belongs to B is basically less than delta, let it be 3, okay.

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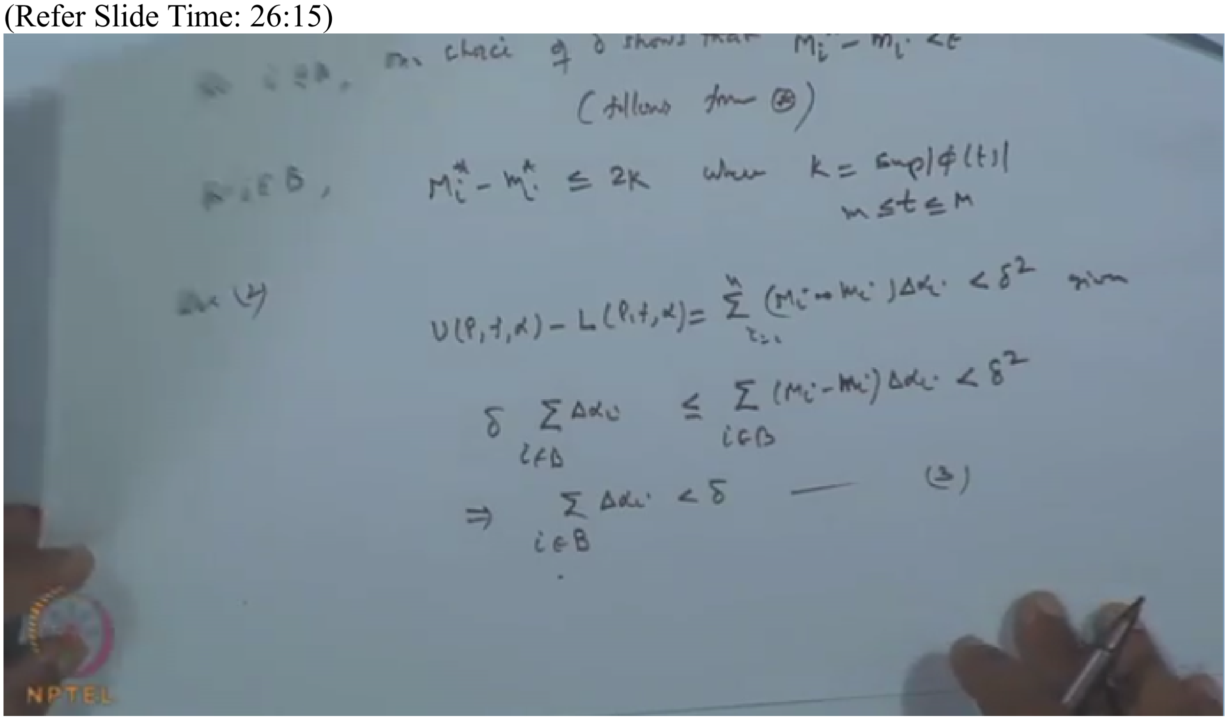

Now consider the partition, now consider the upper sum of the function H with respect to alpha - the lower sum of H with respect to alpha, and this we can divide in two parts when I belongs to A, MI star – small mi star delta alpha I + when I belongs to B MI star - small mi star delta alpha I, okay.

Now see this will be MI star -, when I belongs to A we have already justified this is less than epsilon, so this is basically less than epsilon and then delta alpha I will be less than, sigma of delta alpha I is nothing but less than alpha B - alpha A, so we get this part, okay.

The second portion is this entire thing is less than 2K, because this is 2K so it is less than 2K and then over B the sigma is less than epsilon, sigma is less than this delta alpha I when you choose what is, sorry delta, sigma I is less than delta so it is delta,

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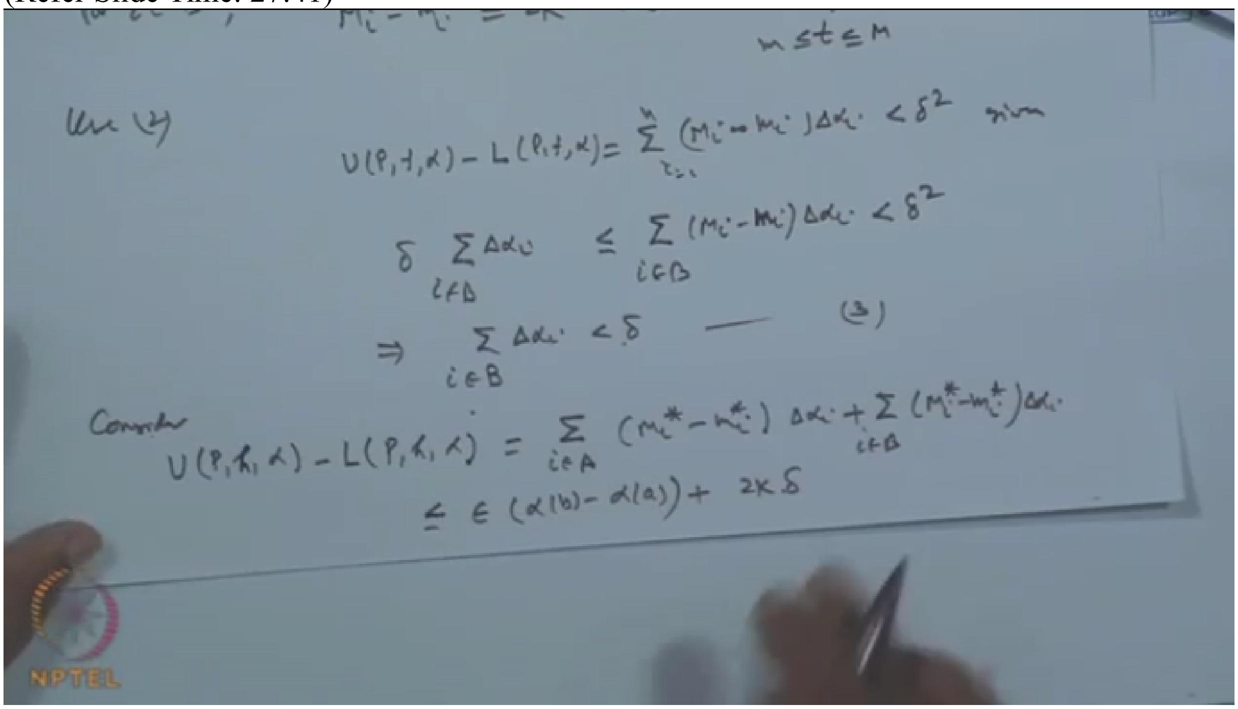

so once it's delta but delta is less than epsilon this is our choice in the very beginning, beginning of this we have taken delta to be less than epsilon, so we can choose the delta outside so we get the P, H, alpha –L(P, H, alpha) is less than equal to epsilon outside, what you are getting is alpha B - alpha A + 2K, but epsilon is arbitrary, since epsilon is arbitrary, so this shows that satisfies the necessary sufficient condition of that theorem, therefore H must be in R(alpha), some few properties of the Riemann Steiltjes integrals, we will list the proof, property and proof, one or two will be good enough.

Now if F1 is a Riemann Steiltjes integral with respect to alpha, F2 is a Riemann Steiltjes integral with respect to alpha on the closed interval (A, B), then the sum F1 + F2 will also be a Riemann Steiltjes integral with respect to alpha, and CF will also be a Riemann Steiltjes integral with respect to alpha for every constant C, and the integral A to B, F1+F2 D alpha the Riemann Steiltjes integral, the sum of it will be some of the integrals F1 D alpha + A to B, F2 D alpha and C times FD alpha is C multiplied by A B to be FD alpha, so this is the first result.

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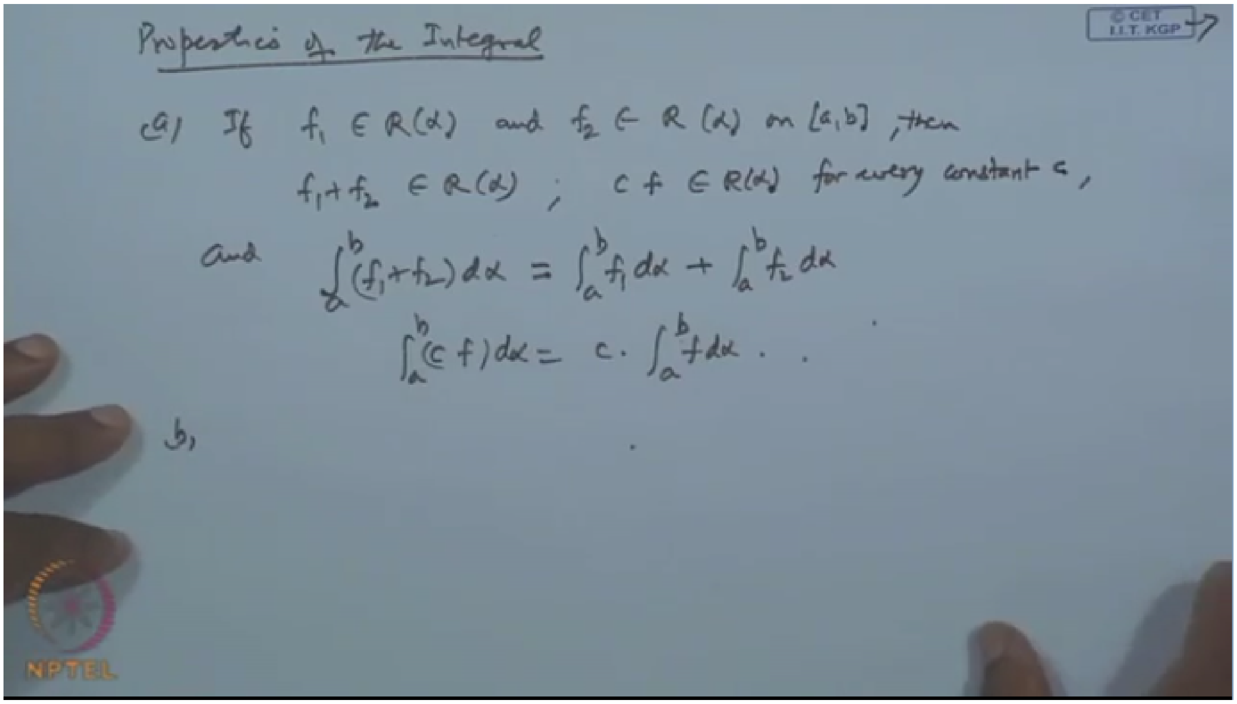

Second result says if suppose F1 and F2 are the two Riemann Steiltjes integral, if F1 and F2 are the two functions such that F1(x) is less than equal to F2(x) on the closed interval (A, B) and they are also Riemann Steiltjes integral and then integral A to B F1 D alpha is less than equal to integral A to B F2 D alpha, of course where F1 and F2 both are in alpha, R(alpha) okay.

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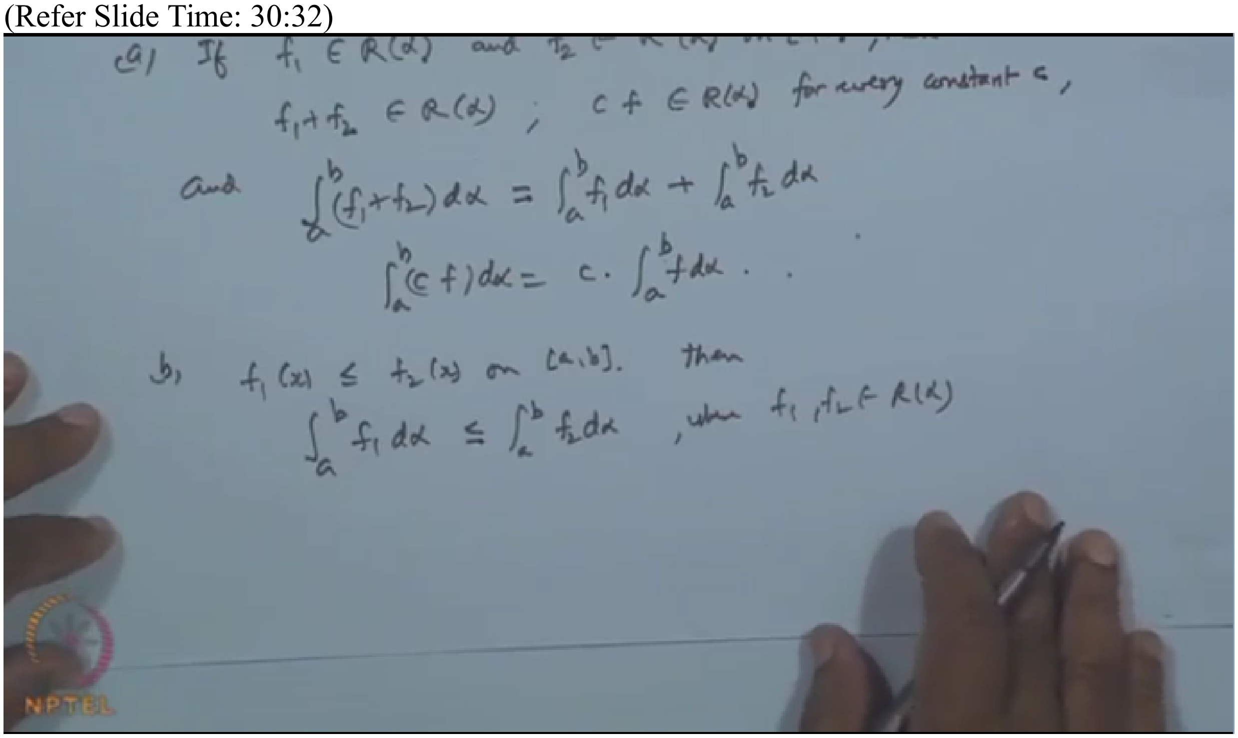

Third is, third property is if F is, if F belongs to R(alpha) that a Riemann Steiltjes integral on the closed interval (A, B) and if A is less than C, less than B, then F is Riemann Steiltjes integral on the closed interval (A, C) as well as on (C, B) means if we divide that (A, B) into

two parts AC to CB then over each subinterval function F remain Riemann Steiltjes integral with respect to same alpha and the value of the integral will be the sum of this two,
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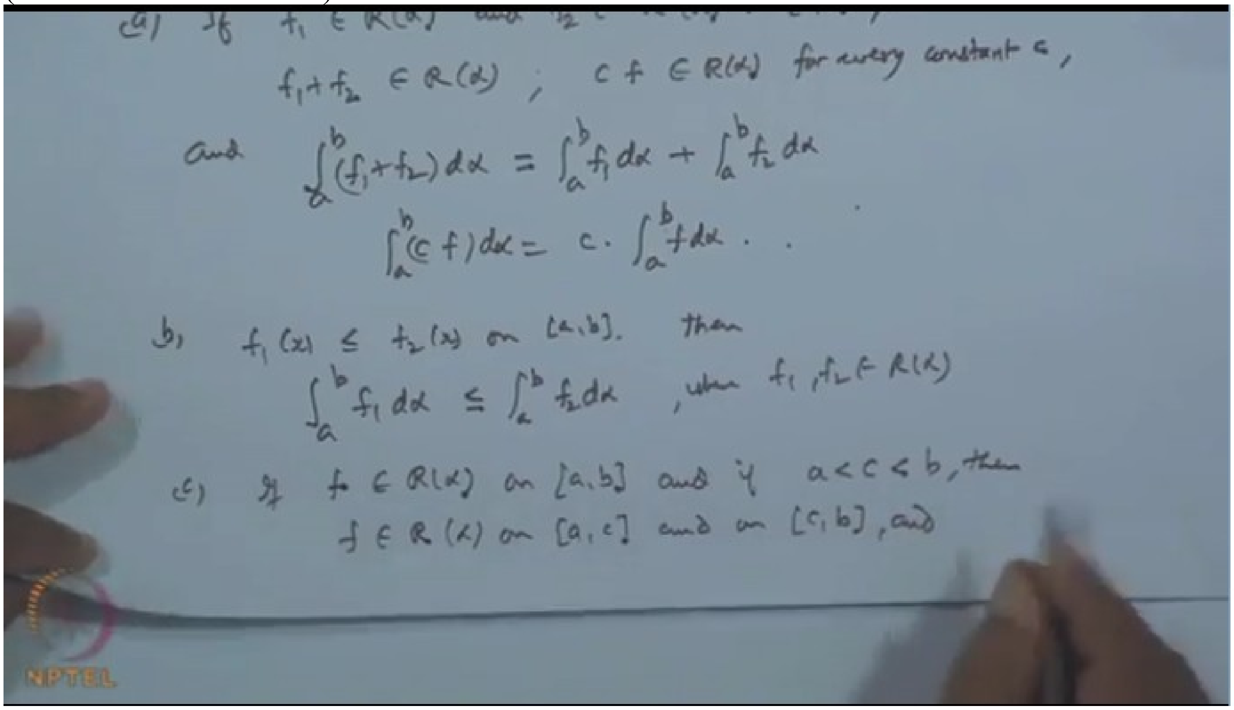

and integral A to B FD alpha is the same as A to C FD alpha + C to B FD alpha.

Fourth property is if F is a Riemann Steiltjes integral with respect to alpha on the closed interval (A, B), and if the function F(x) is bounded by M on the closed interval (A, B) then integral A to B FDX, FD alpha into FD alpha modulus of this, modulus of this is less than equal to M times alpha B - alpha A, okay, modulus of this D alpha. Then E property if F is Riemann Steiltjes integral with respect to the monotonic function alpha 1, and F is a Riemann Steiltjes integral with respect to monotonic function of alpha means F is Riemann Steiltjes integral with respect to the two monotonic functions alpha 1 and alpha 2 respectively,
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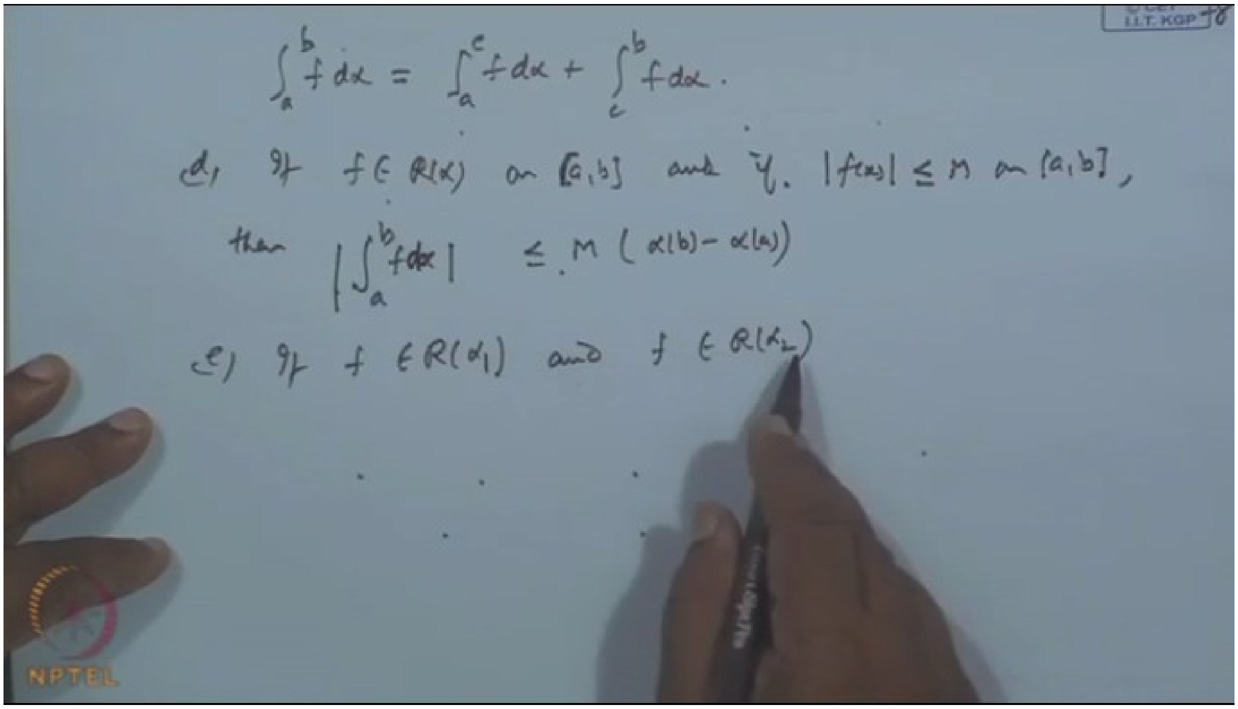

than F will remain Riemann Steiltjes integral with respect to the sum of these alpha 1 + alpha 2, because if that alpha 1, alpha 2 are monotonic increasing the summation will also monitoring increasing, if they are monitoring decreasing they will also be monitoring decreasing, and hence and integral of F alpha 1, D alpha 1 integral FD alpha 1 + alpha 2 over A to B is the same as A to B FD alpha 1 + A to B FD alpha 2.

And if F is Riemann Steiltjes integral with respect to alpha, and C is a positive constant, then F is Riemann Steiltjes integral with respect to C alpha, and integral of this A to B FDC alpha is C times A to B FD alpha,
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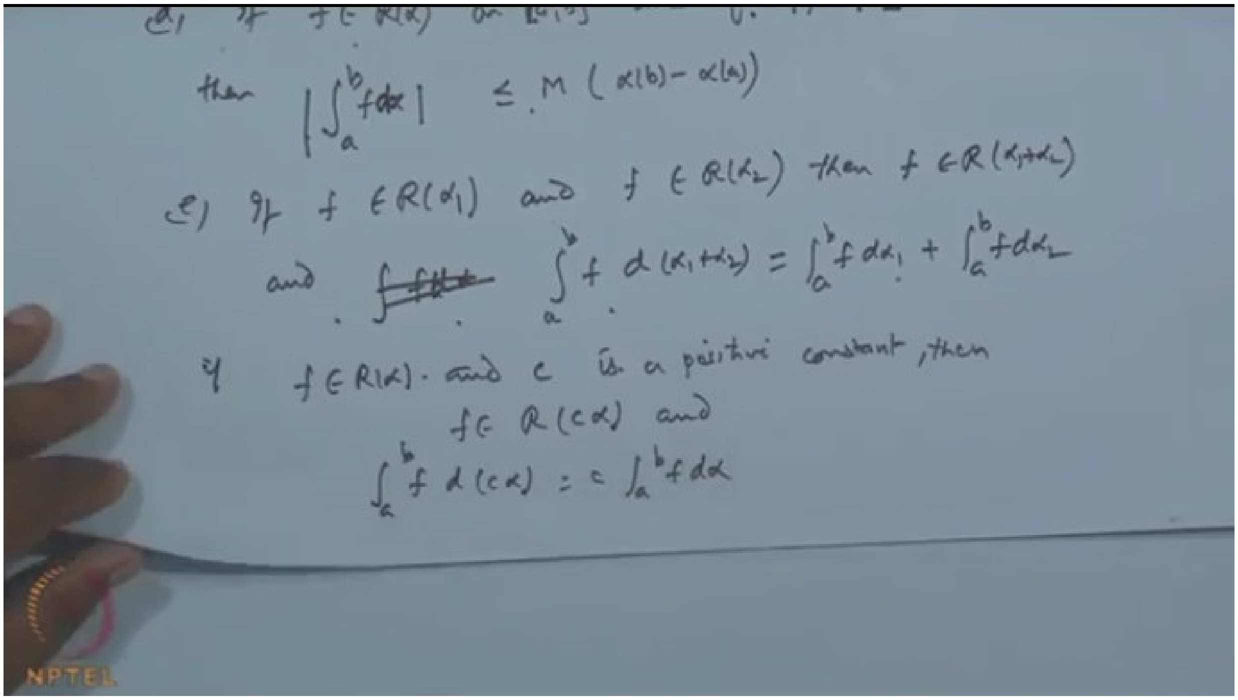

the proof of these results property follows by using the difficult property is also which we just mentioned, if F is in R(alpha) and G is also in R(alpha) on the interval (A, B), then the product of this is in R(alpha), then second part is if mod F belongs to R(alpha) and modulus of integral A to B FD alpha is less than equal to integral A to B mod FD alpha and less than equal to B, the proof is which.

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