Model 2

Lecture – 7

Some Theorems on Open and Closed Sets

Introductory Course in Real Analysis

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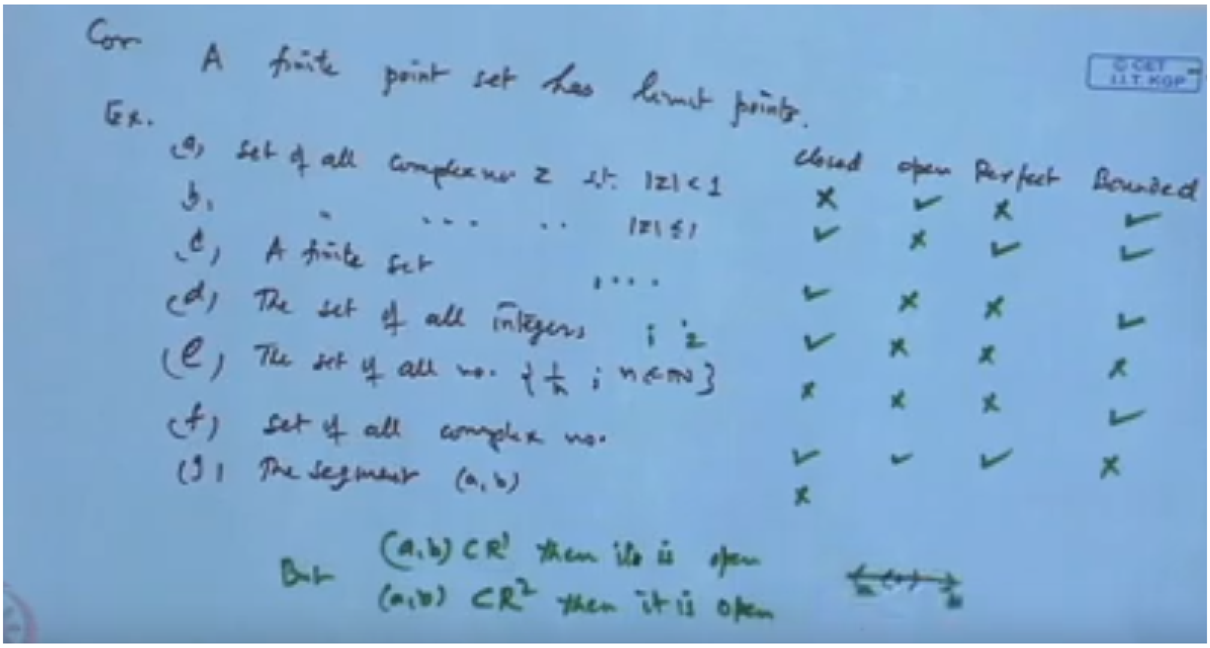

Let us take few examples. We are this suppose we have the sets like set of all complex numbers, all complex numbers complex numbers, Z such that mode Z is strictly less than 1. then set of all complex number Z such that mode Z is less than equal to 1, then C is a finite set, D say set of all Integer, integers is the set consisting of the number is like this the set of all complex numbers, set of all complex numbers, set of all numbers of the form say 1 by n where the n belongs to n. Natural number. Okay all numbers of this and EF set of all complex numbers, complex numbers and then the segment a,b, this segment a, b. Let us see these components see whether these sets are closed, which are open, whether they are closed, whether they are open, whether they are perfect, or whether they are bounded, because these concepts we have already seen. Now you see the first one
obviously it is not closed because closed means the set is closed, when that mode of z is less than equal to one is the point which is the limit point of this so it will not be a closed set. So this will not be a closed set. No, so here is no. Then it is open because more than less than every point will the interior point it will even open. Then perfect set. The set is said to be perfect if it is closed and every point of E is the limit point, say more than, less than one is not then closed set then obviously it will be it will be not be a perfect set and then bound. Yes, it is bounded by 1, means all the point of z is then then equal to one, so it is a bounded set. This set, yes it is closed because all the points which are this limit point all belongs to it the limit point of C is a point of this. limit point is then it is not open because of the closeness is not open perfect though it is closed and then, whatever they every point whether every point is the limit point or not yes it is inside that this mod Z less than equal to one every point is an interior point when mod Z less than one and one is also the limit point in it so it will be a perfect set. then bounded yeah it is bounded because one is there. Finite set, finite set it does not have any limit point, so we can assume all the limit point inside it are there, so we can say it is closed. But it is not open, why it is not open? Because the finite set when you choose these are the sets, these are the sets point, so if we draw the neighbourhood around the point P, then we do not get any point other than this, so this not be an interior point 0 it will not be an open set, since it is close but no limit point so we can say this set is not perfect. Because every point must be either limit point, perfect set it is close and every point of P is a limit

point, but every point is not a limit point, so it is not. And then of course it is bounded because only finite number of that we can identify a point
find out its distance you know from this particular point so you can find a M upper bound so that this boundary. Set of integers. Set of integers, this is closed set, again the same thing, set of integers the point every point is not a limit point because again there is a gap between 1and 2, 1 and 2, there is a gap. So we can draw the neighbourhood around the point 1 which does not include any integer, so one cannot believe it point, similarly all other integers do not have a limit point. So set of integers does not have a limit point, but we can assume all the limits points inside it so we can say it is a closed set. Say D, Okay but it is not open because the points are not interior so it is not open. then it is perfect again every point is not a limit point so it is not a perfect and whether this is bounded or not obviously this way 1 2 3 is the unbounded set, we cannot find it n, so that all the point is less than equal to n. so this is not bounded. Then set of all numbers 1 by n. 0 is the limit point for this except 0 no other point is the limit point for this so this set is not closed, e is not closed because closed means all the limits point belongs to this, because 0 does not appoint in it, so 0 is a limit point, which is not n so it is not closed, it is not an open set why because the point 1 1 by 2, 1 by they are not the limit Points, they again they are not an a limit point, therefore is not open it is not a perfect set. However it is bounded because the bound is a poor bound is 1.
Set of complex number it is the entire set we can say all the limit points are inside, as well as every point is the limit until open because we can draw the ball around the each Point, which is totally contained in the complex plane, so it is open so it is open close, it is perfect also because all the points are limit point, every point is in limit point so we can say it is perfect. And then bounded set is no because it is unbounded plane. Segment, segment is not in closed because it is an open segment it is an open set but when you stay the segment a,b then the openness depends on the topology, on the set. Suppose I take a, b as a subset of R1, then it is open then it is open but if we take a, b as a subset of R2, then it is not open. Why? Because it is not open because every point of this set is not a interior point, the reason is in this case we can choose this is a set a, b whatever the point is you can draw the neighbourhood
around that point which is totally contained in but in this case the a, b interval is this, this is our a, b and the neighbourhood when you take, the neighbourhood will be something like this. So here the points are available here which are not the points belonging to the set? So we can say this point cannot be an integer point, so similarly others so it cannot be an open set. It is yes then it is not open it is not open set in r2. So that is important for this so, that is why we are not am and further perfectness is not there. So we can say across and then bound is yes. Because it is bound is a. whatever the topology take it is bounded, so you can this so this way we can identify. Now here these sources this example shows that openness or closed net our relative concept. say the set which is a subset of a set which is the set of which it is a subset so with respect to that whether it is open not if we consider a, b as a subset of R, then obviously it is open but when you consider a, b as a subset of r2 then it is not open. so it is a relative concept. So we can further in give a definition of the openness or colonial with respect to the set. That is called the relative of open, relatively open or relatively close sets. So we can know. The results and further so, we have another results theorem.

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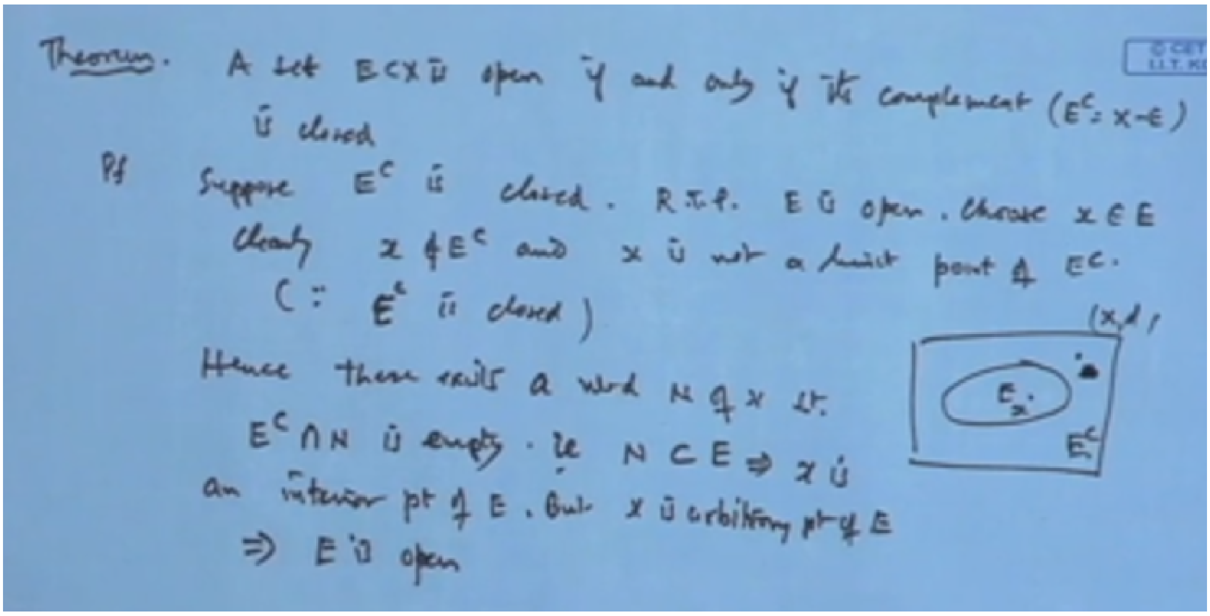

The theorem is a set each open is open, if and only if it is complements complement each closed. Complement means that is if X, E is a subset of X, X be the metric space, then complement of this means denoted by Ec which is X minus E. complement of the set is closed. Okay, so proof is, so let us suppose Ec is closed suppose the complement of this set Ec is closed, then what we wanted to show is E open, required to prove is is open, it means every point of E is an interior point. So choose an element X belonging to E, if I prove that there is a neighbourhood around the point X, totally contained in E then E becomes open. Okay, now since X is in E and E complement is the set of those point which are not in E so clearly X is not an element of a complement okay and so X is not a company in element of this now further X is not a limit point of e complement and X is not a limit point of a compliment he complements okay so is why it is not a limit point of the compliment because the reason is a compliment is closed a compliment is closed so close by definition all the limits point must be the point of the set if set is closed if it includes all of its limit points so if X is not in Eason so X cannot be neutral because if it is a limit point it must be the point of VC so here is this e and this is our space XD and here this is easy so we are taking a point X here which is in X sorry we are taking X here now what we say is we exit neither in EC knowledge the limit point of e see okay so there will be as some neighbourhood around the point we can draw which is totally away from EC or
does not intersect with EC and lies in it so there is a sense there exist there exist because it is not a limit point it is not a limit point so we can find out a neighbourhood around the point X which is totally away from is your intersect of EC is empty so there exists a neighbourhood n of X such that the neighbourhood n of X such that intersection with EC is empty ok because it is not a limit point and if this is a limit point then every neighbourhood of X must contain some point of EC so since it is not a limit point we can identify a neighbourhood whose intersection with this empty we do not have any point inside this neighbourhood that is the neighbourhood is totally contained in E because it contains only the points of e ok so what is so that X is a point who around which we have a neighbourhood and this is totally contained he this source that is open this source X is an interior point of E hence an X is arbitrary but X is arbitrary point of E so every point has in is an integral point therefore E is open.

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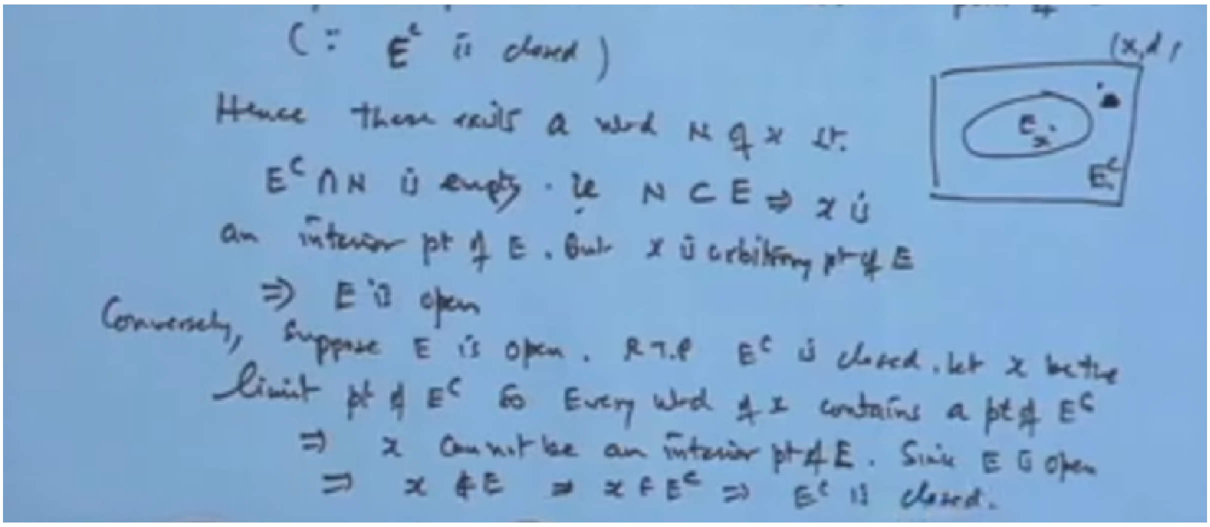

So this was the concept conversely suppose each given to be open conversely let us resume is open we wanted to show the closed EC is closed okay so - so EC is closed so we wanted to prove that all the limits point of the EC are the points in E so let X be the limit point of e see I see okay if we than once it is the Nabal limit points in every neighbourhood of X will include the point of EC so every neighbourhood of X every neighbourhood contains a point of EC so when's the every neighbourhood contains the point of EC it means X cannot be an interior point of EC because if it is an interior point then there will be a neighbourhood around the point X BC totally contained in E but every neighbourhood contains the point of EC also so this shows that this never this X cannot be an interior point of E. cannot be an interior point of E. Okay, the further what is given is E is open, so what do you mean it means if X is a point in E then because E is open so X must be an interior point, but here we have shown X is not interior point. Therefore this implies that X cannot be in E. so this implies that X will be in E Compliment. so it will be an in compliment. So this follows that E compliment is closed. That is what is proved.

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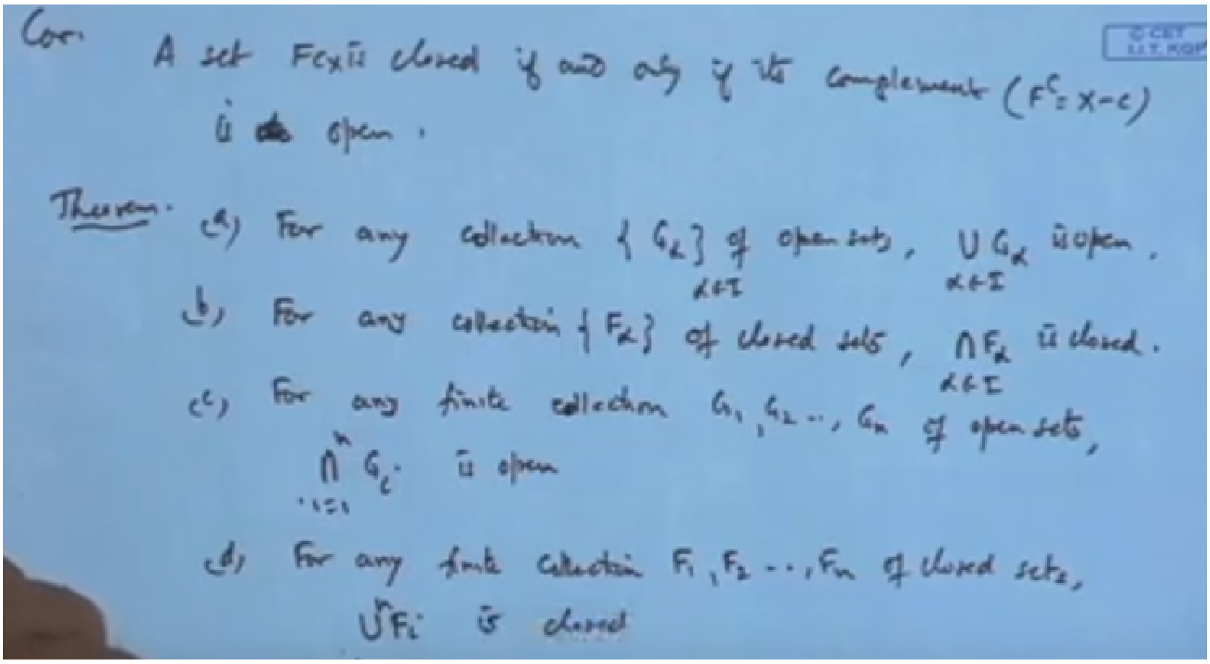

Okay as a corollary of this we can corollary a set F is closed, if and only if its complement is open. Complement means if it is a subset of X, complement means Fc, which is X minus c, each closed is open. Okay, now these proofs course just a previous from the previous column we can drive this corollary easily there's no point of wasting. now following results are for any collection for any collection G alpha of open sets real alpha belongs to the index set sets the union of G alpha when alpha belongs to III index set alpha is in I I is the index set okay then G alpha is open this is one second result says for any collection for any collection F alpha of closed sets. The intersection are between intersections this arbitrary Union, arbitrary intersection F alpha, alpha belongs to I is closed, for any finite collection for any finite collection G1, G2... Gn of open sets. The finite intersection of these open sets is open. D is for any finite collection for any finite collection for any finite collection F1, F2... FN of closed set of closed sets. The finite union of these Fi is closed. So here this C and D is not valid for a arbitrary or infinite collection or countable intersection of Gi and countable union of Fi it is not valid, it may not be true. Okay, so only for finite cases it is true. The proofs again follows like this it is given that each of G alpha for each alpha is a open set, we want the arbitrary Union is open.

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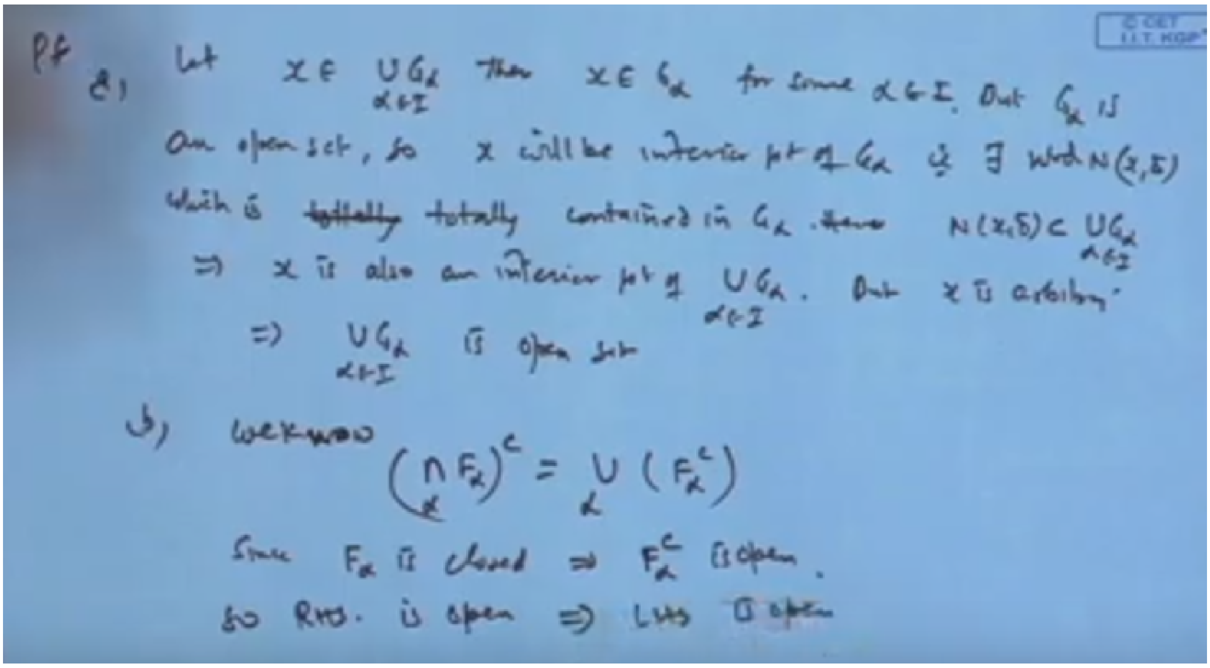

so let us take a point let X belongs to the R between you know G alpha, alpha belongs to this it means and that then X belongs to G alpha for some alpha belongs to I, but G alpha is open, is an open set. so there exist a neighbourhood around the point G which is totally contained inside it. so so X will be an interior point, of G alpha, that is there exist a neighbourhood and with set suitable radius Delta, such which is totally contained totally contained in Gl1. hence this neighbourhood Nx Delta will also contained in the countable union of G alpha. So this implies X is also an interior point of Union G alpha. Alpha belongs to Y, but X is in arbitrary point. But X is an arbitrary point.
So what we say that countable union of Gl alpha, is of it because every point becomes the integer point. so this implies that countable union G alpha, alpha belongs to Y is an open set. Similarly for the Part B but we follows from this result we know this is a simple set theoretical result. arbitrary intersection of this if I come find the complement of this, this becomes the arbitrary Union of the complement of this F alpha C. now what is given is F alpha is closed, so F alpha C will be open since F alpha is closed, so this implies that complement of this F alpha C is open and right hand side is the arbitrary union of the compensator right side, side is open therefore this implies the left hand side will be open. So this implies the left hand side is open. So complement of complement is closed so this implies that arbitrary intersection alpha belongs to I, is closed that is proofs.

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an open set, so x will be interior pt of G_α i.e. $\exists$ nbd $N(x,\delta)$ which is ~~totally~~ totally contained in G_α. Hence $N(x,\delta) \subset \bigcup_{\alpha \in I} G_\alpha$

$\Rightarrow$ x is also an interior pt of $\bigcup_{\alpha \in I} G_\alpha$. But x is arbitrary

$\Rightarrow$ $\bigcup_{\alpha \in I} G_\alpha$ is open set

b) we know

$$\left(\bigcap_{\alpha} F_\alpha\right)^c = \bigcup_{\alpha} (F_\alpha^c)$$

Since F_α is closed $\Rightarrow$ F_α^c is open.

so RHS is open $\Rightarrow$ LHS is open $\Rightarrow$ $\bigcap_{\alpha \in I} F_\alpha$ is closed.

c) let $H = \bigcap_{i=1}^{n} G_i$ where G_i are open sets

let $x \in H$ $\Rightarrow$ $x \in G_i$ for each $i = 1, 2, \ldots, n$.

Then second the C part we wanted to show that finite intersection of the open set is open. So let us take H is the finite intersection of gi, i is 1 to n, where gi's are open sets. so let us take the point x belongs to H. let x belongs to H so this implies that x belongs to Gi for each i, 1 to n, now each GI is open so there will be a neighbourhood around the Hg around x with a suitable radius R

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So $\exists$ nbd's N_i of x with radii r_i s.t. $N_i \subset G_i$ $(i = 1, 2, \ldots, n)$

Choose $r = \min(r_1, r_2, \ldots, r_n)$

Then $N(x, r) \subset G_i$ $\forall i = 1, 2, \ldots, n$

$\Rightarrow$ $N(x, r) \subset \bigcap_{i=1}^{n} G_i = H$ $\Rightarrow$ H is open.

d) follows as earlier.

Remark: (c) & (d) do not hold for arbitrary $\cap$ & $\cup$ resp.

eg $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ open sets but $\bigcap_{n=1}^{\infty} G_n = \{0\}$ which is closed

Sol $F_n = \left[\frac{1}{n}, 1\right]$ closed set $\bigcup_{n=1}^{\infty} F_n = (0, 1]$ which is not closed.

Right so we can say so there exist neighbourhoods ni of x with radii with radii Ri we are i is 1 to n such that the ni this neighbour is, contained in GI when i is 1 2 then means each n X X is in all J so for each I there will be a neighbourhood a night which is totally contained in GI not choose the radius R, choose R as the minimum of these R1, R2,... Rn. so if we draw the neighbourhood then the neighbourhood with the centroid X and radius R obviously will content in GI for each I 1 to n therefore it will contain this intersection contained in the intersection which is H so this source that H is open H is open okay the second part follows in the same D that is a finite collection like so ABCD D follows as earlier so Nina now let's take the case when they are infinite intersection so C and D does not do not hold for arbitrary, arbitrary intersection and union respectively the examples are suppose I take the GN is minus 1 by n 1 by n then these are all open sets but the intersection of GN n is 1 to infinity is the singleton set 0 which is closed. similarly if you take the say our FN age say 1 by n + 1 these are all closed X the RVT union of these offense n is 1 to infinity is the semi closed interval 0 to 1 which is not closed so this shows the arbitrary intersection they are not true.

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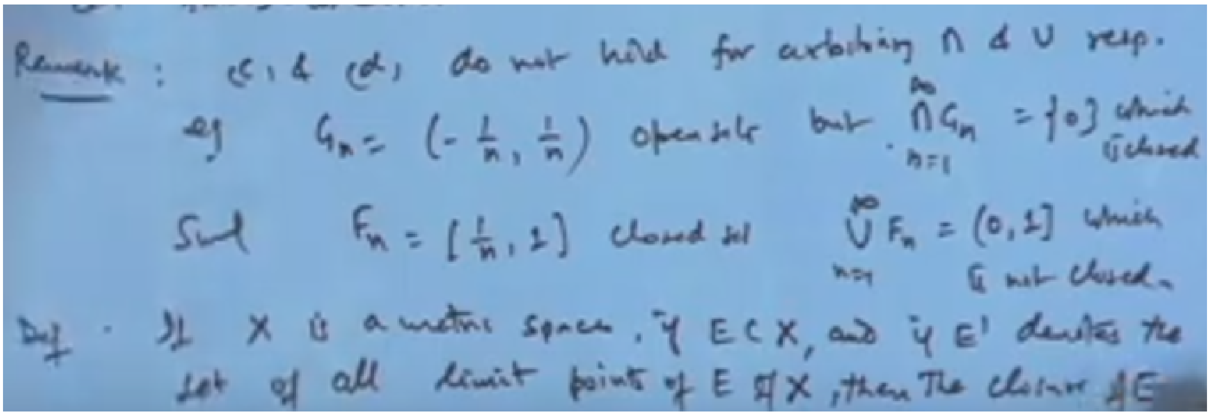

Ok now we define the concept of the sets if X is a metric space X is a metric space and if E is a subset of X and if E dash denote the set of all limit points set of all limits points of E of X. Then the closure of closure of E is the set is the set denoted by E bar which is E Union E dash, so closer of this. So what is this is closure means suppose a set is given then set of all limits points if I include it in E, then the collection will be known as closure. So closure of the set is the set which includes all of its limit point as well as the point of E, of course, then there is a result.

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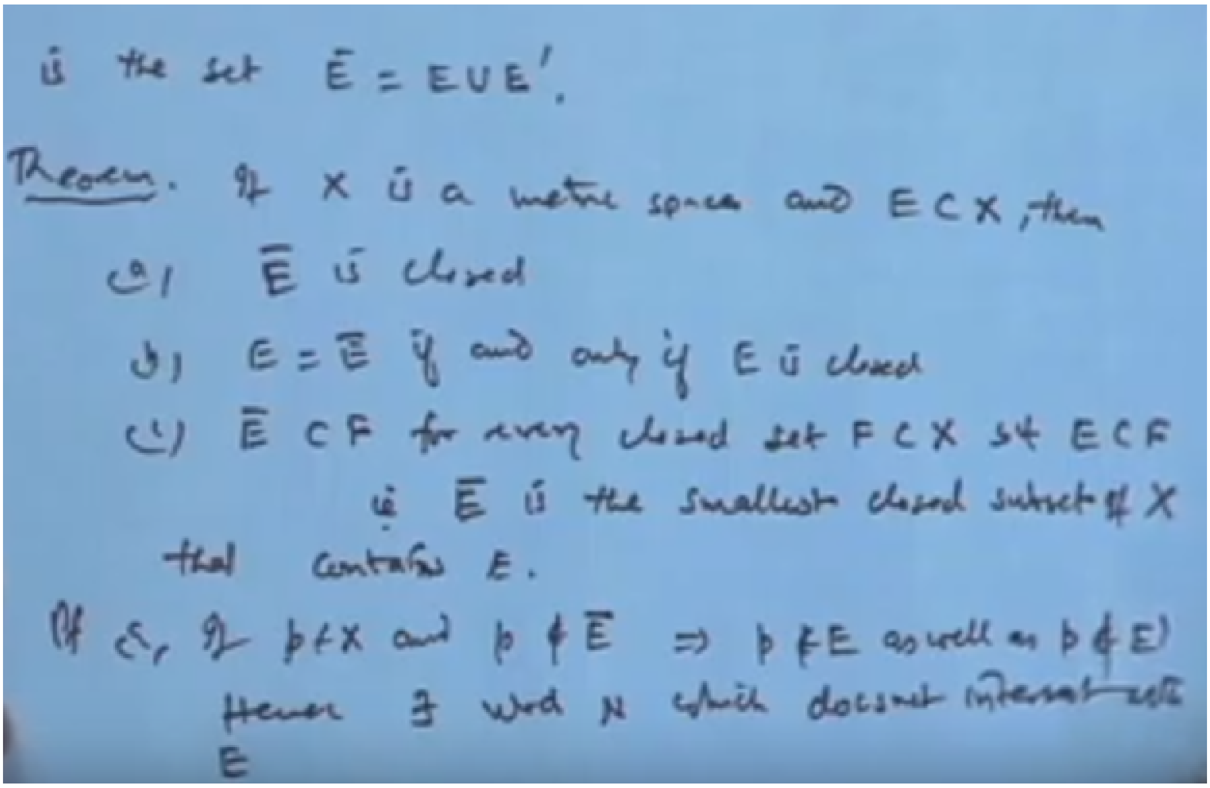

if X is a metric space, and E is a subset of X, then the closure of this set E bar is closed set. V if E is equal to E bar that is then, if and only if, if and only if E is closed. and third is if E bar is contained in F for every closed every closed set F, which is contained in X, such that E is contained in F, it means that is E bar closure of this is the smallest, is the smallest closed subset of X, that contains that contains E. the proof is simple just I will give the first proof a E bar is closed is given to be a this then closure of this is closed. So let us say if P belongs to X and P is suppose not in the closure, Okay? Then P is neither a point P does not belong to yes as well as well as P is not a limit point of this, as well as P is not a limit point of this. So once it not so there exists, a neighbourhood, there exists a neighbourhood and which does not intersect, does not intersect with E with E.

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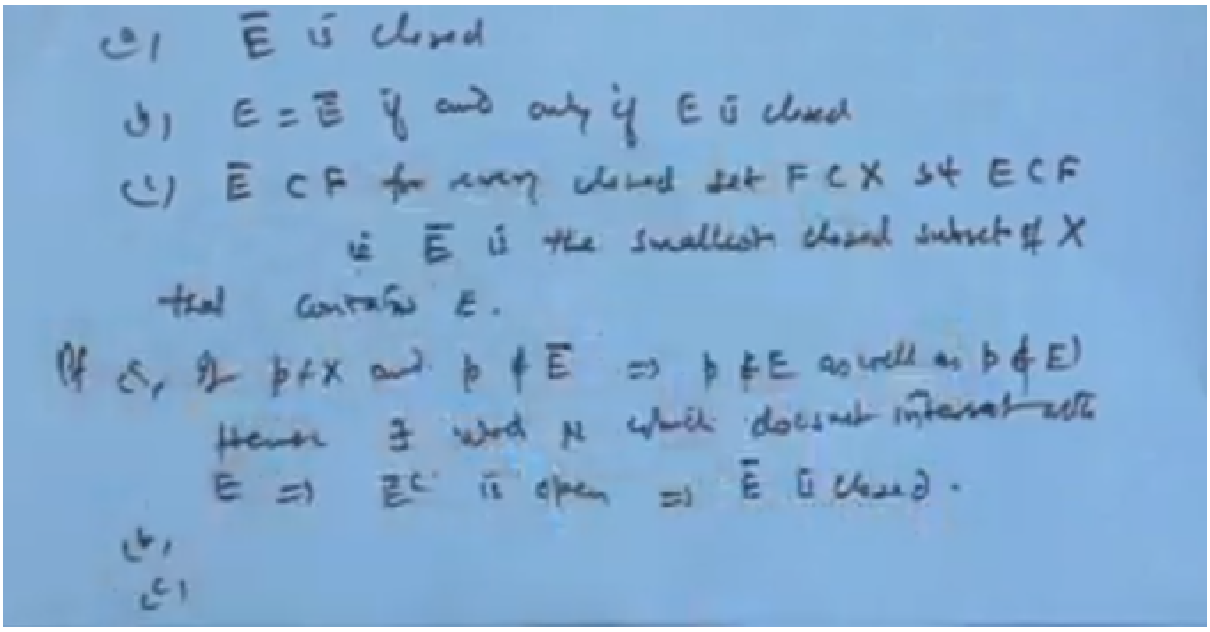

So if does not intersect with E, and then compliment of E bar is therefore open. Therefore E bar is open the complement of this is open Ec is open, then the complement of E bar C is open, complement of E, is open and this shows E closed, sorry this one E bar is closed. So this second and third part follows very easily, so we are just dropping and get.

Thank you very much.

Okay.