NPTEL

NPTEL ONLINE CERTIFICATION COURSE

Course

On

Introductory Course in Real Analysis

By

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Lecture 69: Riemann Steiltjes Integrable Functions

Okay, so this is the one existing result, now let see few functions which are always be Riemann integrable functions like a continuous function, they are always a Riemann integral functions and some other case, so let's take the various type of the Riemann integrable function, okay, various type. So first theorem is if F is continuous on a closed interval A, B, then F is Riemann Steiltjes integral with respect to alpha on A, B, okay.

So let’s see the proof, alpha is given to be a monotonic functions, so once alpha is given to monotonic, alpha is monotonic function either increasing or decreasing defined over the interval A, B, so alpha A, alpha B these are finite values okay, alpha N finite values.
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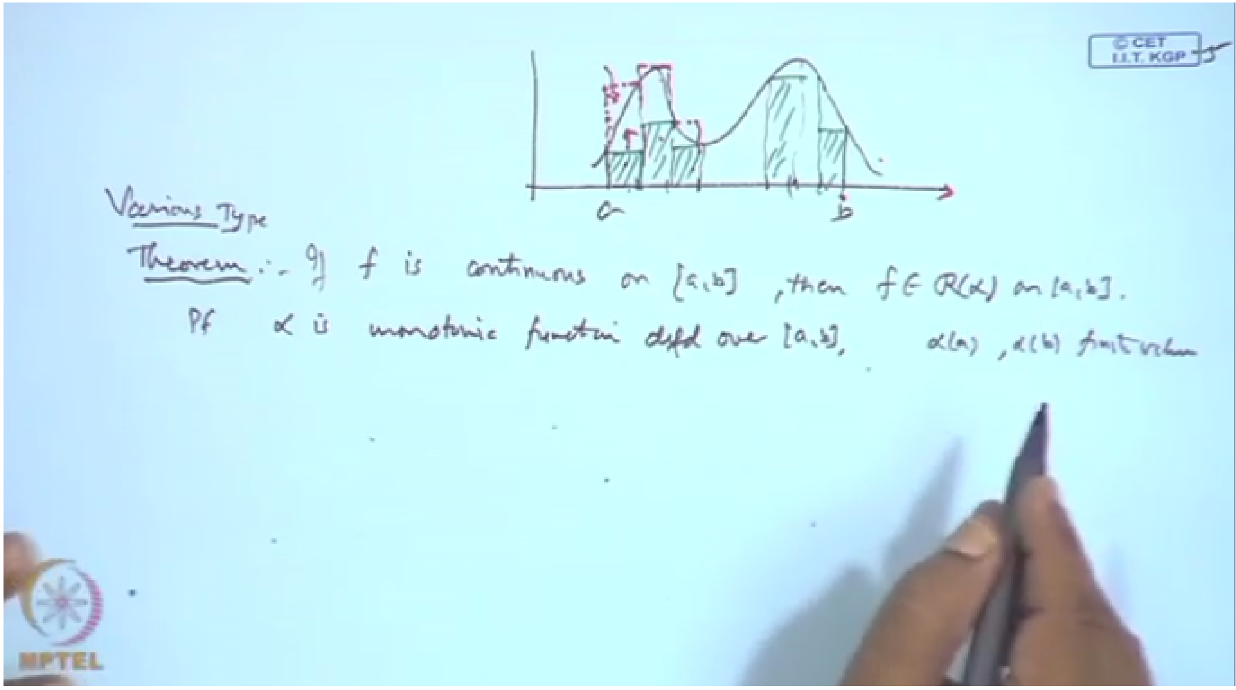

And now alpha B - alpha A is F(x) number, so let epsilon greater than 0 be given, now since alpha B - alpha A is a finite quantity, this is how we can identify some eta such that so choose eta greater than 0 such that alpha B - alpha A multiplied by eta is less than epsilon, this is possible because these are the fixed value, finite values, so we can choose the eta to be epsilon over this number, and since this is the monotonic either increasing or decreasing so it cannot be alpha A = alpha B, okay unless it is a constant function which we are not taking, okay, so we get this one less than epsilon.

Further since F is continuous on the closed and compact, and closed and bounded interval, this is a closed and bounded, so every continuous function on a closed and bounded set all in a compact set is uniformly continuous, so this shows since F is going to show it is uniformly continuous on the interval A, B, this one is this, so apply the definition of uniform continuity so there exists,
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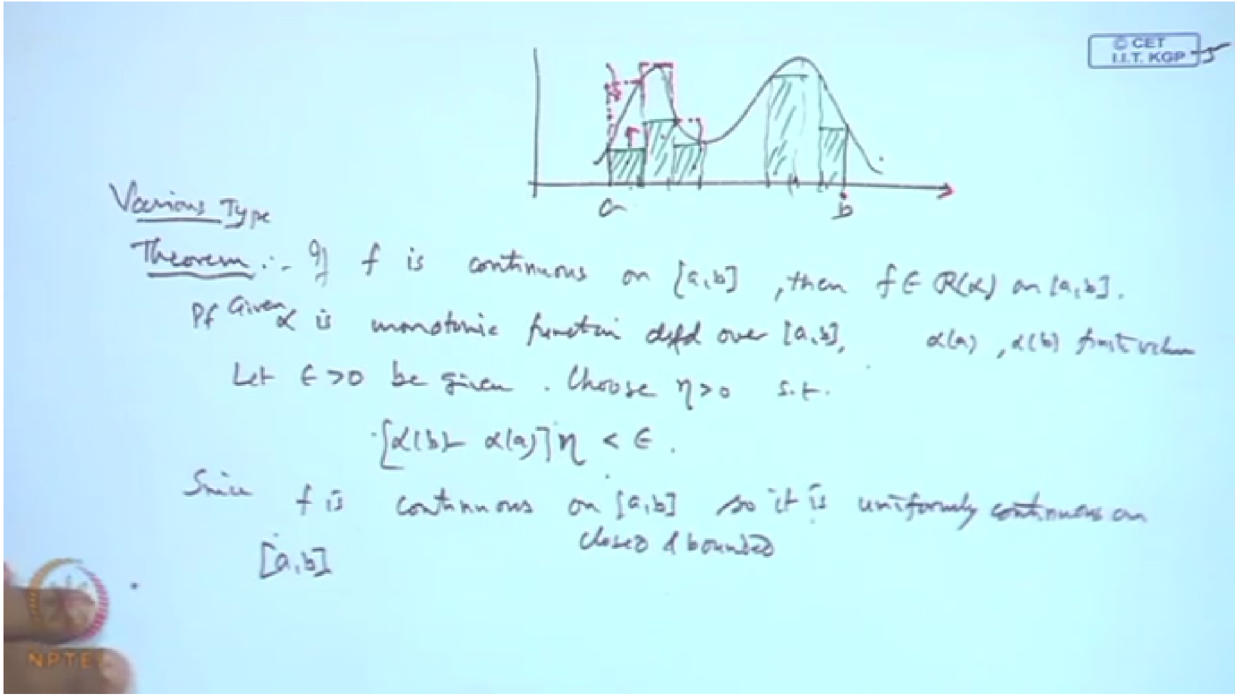

so for given epsilon the same epsilon greater than 0 there exists a delta, there is this delta positive, it does not depend on the point such that the value of this functional value F(x) –F(t) remain less than eta for all X and T in the interval if X belongs to the interval A,

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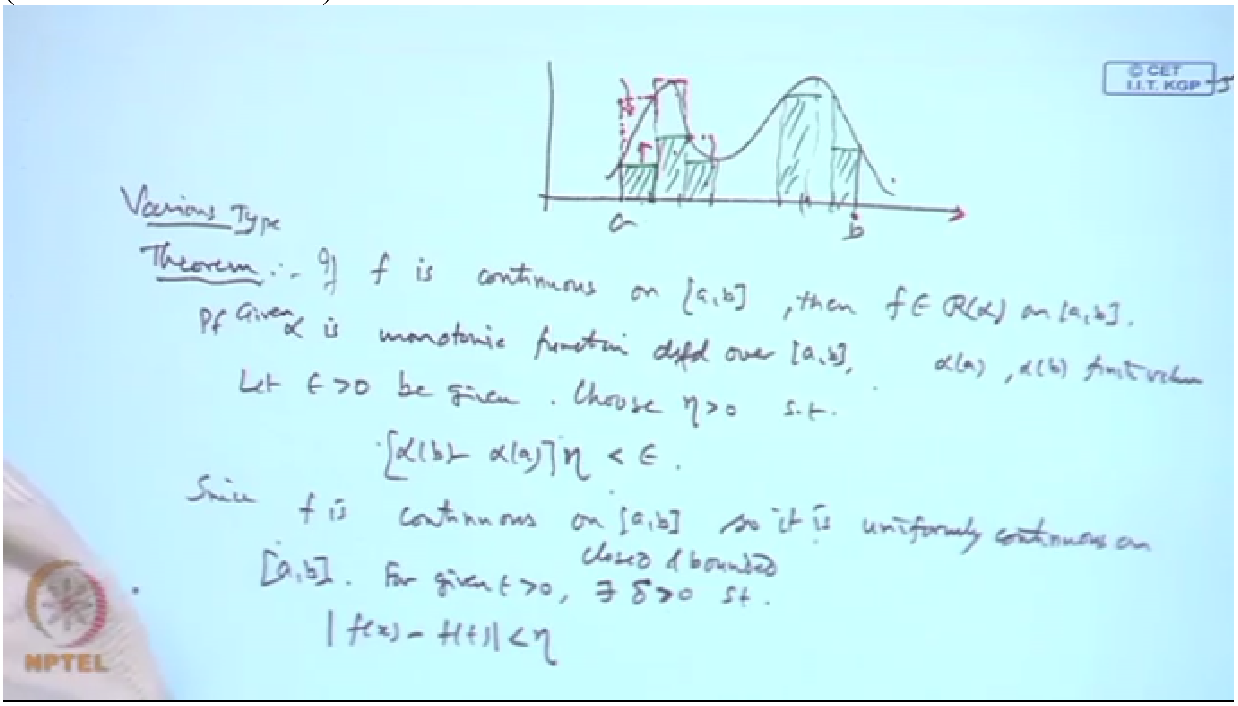

B, T belongs to the interval A, B and such that mod of X-T is less than delta, this is by definition of the uniform continuity of the function, so since F is uniformly continuous it means if we pick up any two point in the interval A, B which satisfy this condition that X-T is less than delta that is in the delta neighborhood of the point T or any, then images will fall in the eta neighborhood of F(t), that is this eta this say I think it's less than epsilon, this is less than eta look up okay, fine.

Now choose the partition P, now if P is any partition of the interval A, B such that delta XI is less than delta, delta XI is less than delta for all I,

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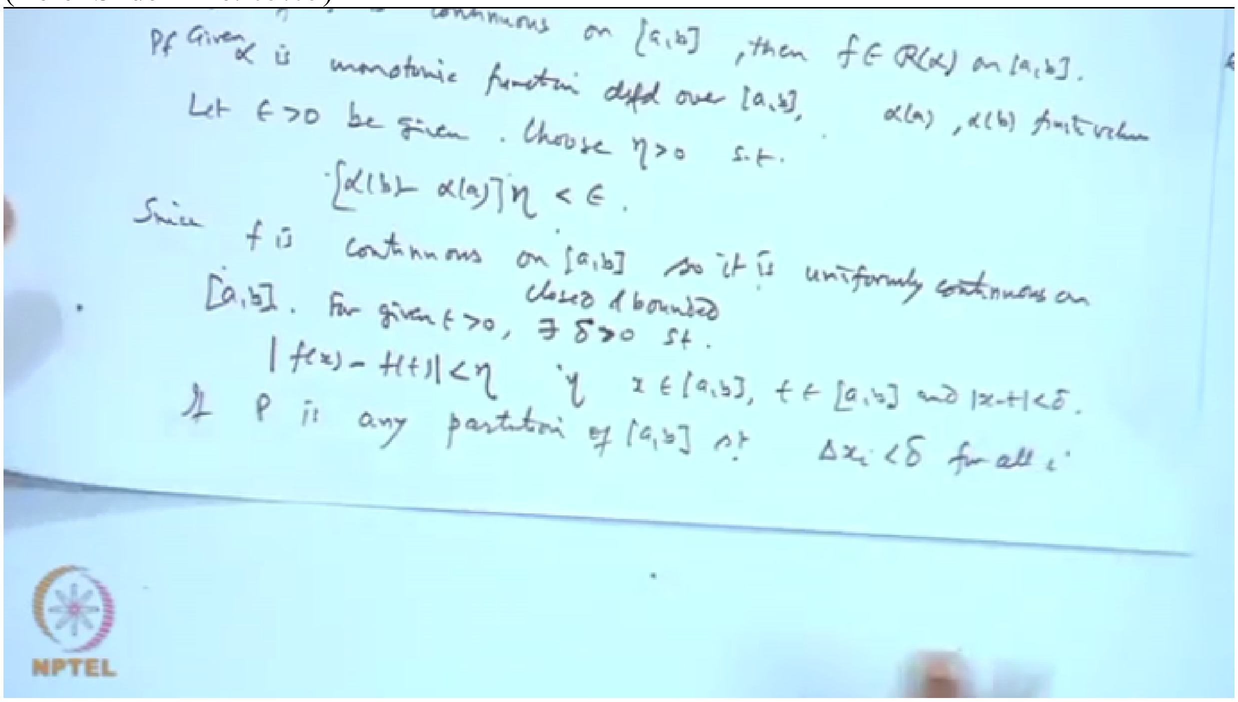

it means we are taking the partition A, B as X naught, less than X1, less than X2, and less than XN = B, so here is the point say XI-1, XI,

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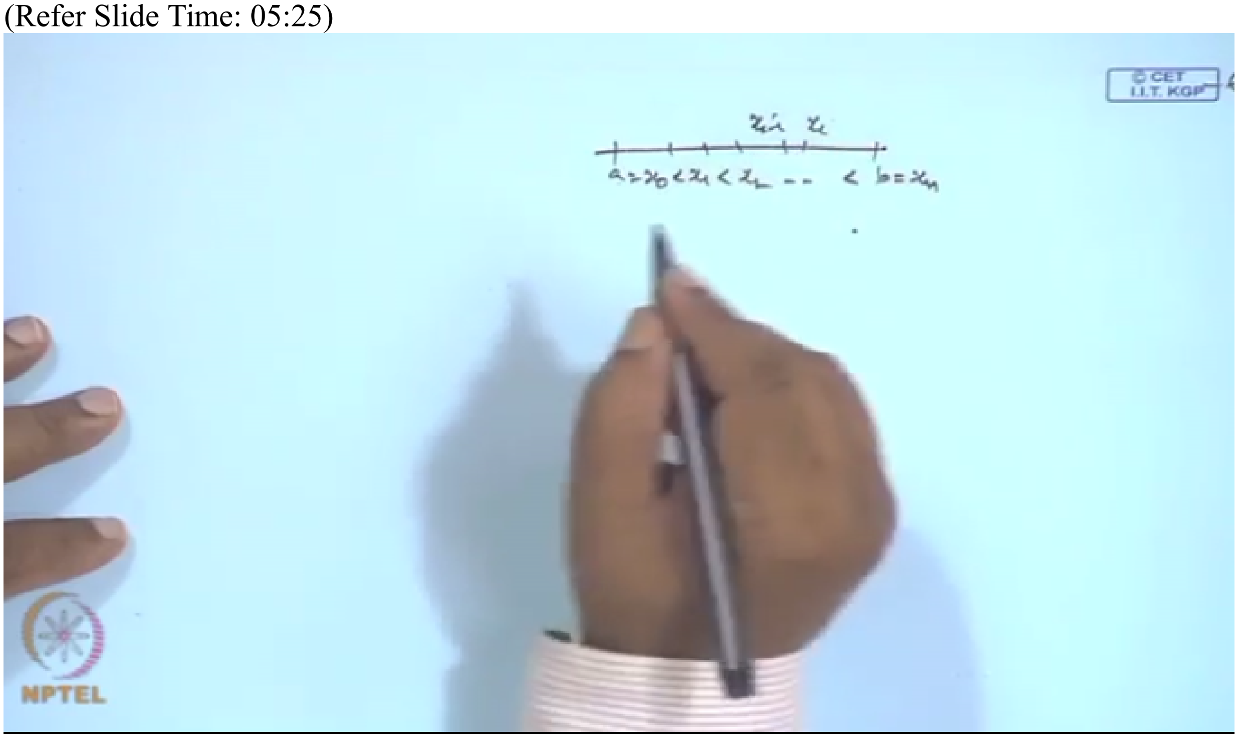

okay so this delta XI is XI-XI-1, now this delta XI is less, because we are choosing the partition less than, so this is positive if the length of this is less than delta for all I really.

Now if we picked up any point inside this delta XI, then what happens? The image is functional value at the point XI - MI this difference will remain less than eta because of this, because any point XY which is less than lying between this interval, lying between this interval XI-1 or TI, if this is the corresponding images will be there, so the maximum value and the minimum value of the function will also satisfy this condition if the point is here in the closed interval XI-1 and XI and function attains the maximum-minimum value,

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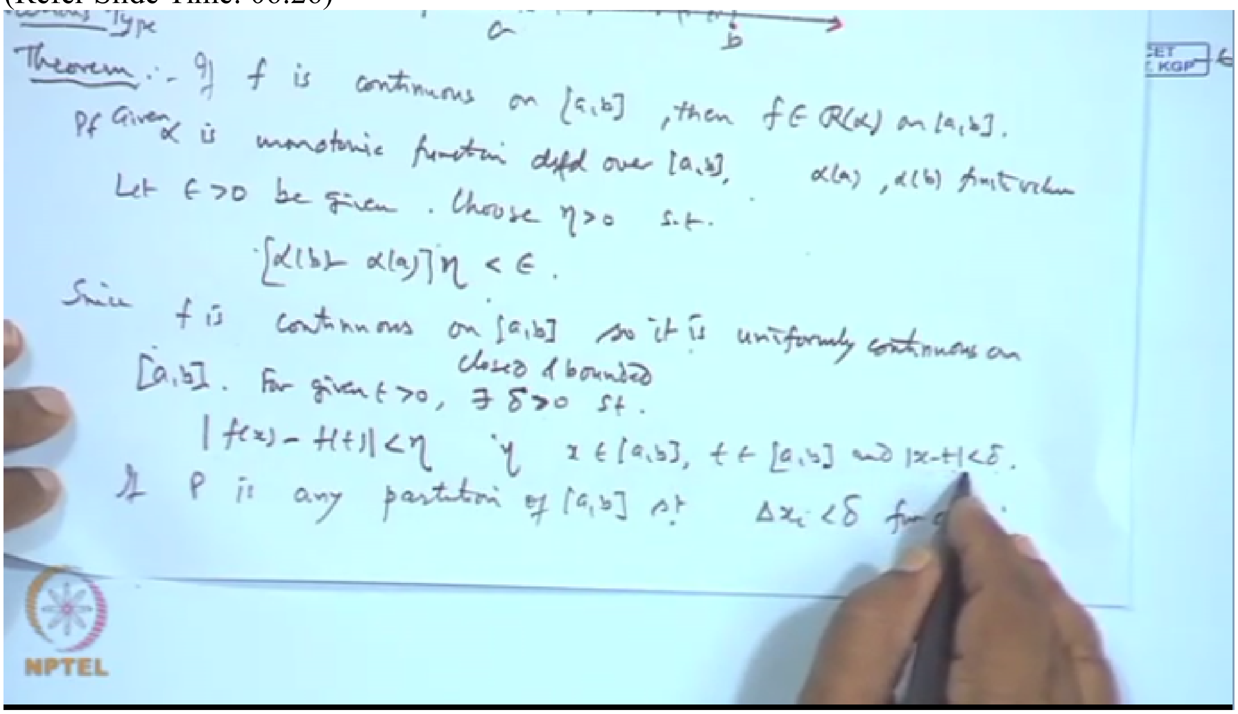

then they have to follow this result so because of F uniform continuity of F we get, we have Mi-small mi is less than or equal to eta for I = 1 to up to N, is it okay?

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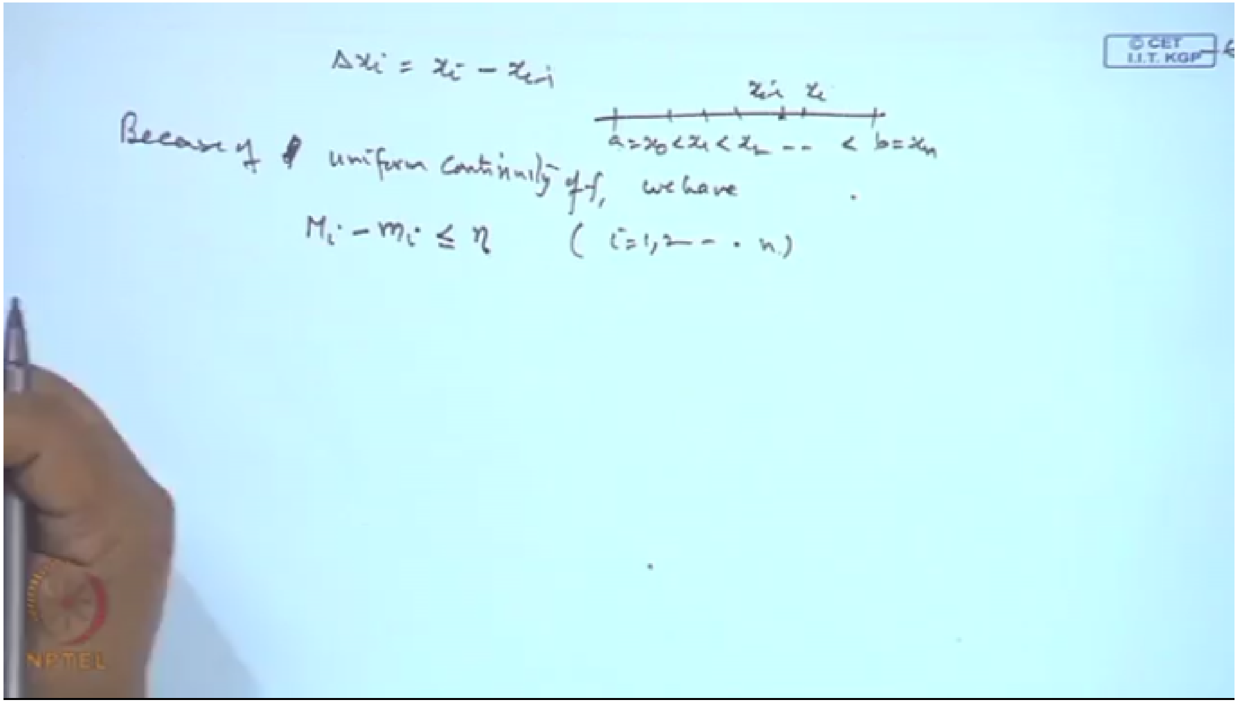

Now consider this, we want this, we want the function is the Riemann Steiltjes integral, it means we wanted to use that result that if the upper sum – lower sum is less than epsilon for some partition for a given epsilon there exists some partition such that upper sum - lower sum is less than epsilon, then the function F must be a Riemann Steiltjes integral, so let us consider, choose the partition P first and then consider the upper sum of the function F with respect to alpha over this partition P - lower sum of the function F with respect to this partition, now this is by definition is nothing but I = 1 to N, Mi capital Mi- small mi delta alpha I, is it okay?

Now what is this? This is already given to be less than eta, so eta and sigma I = 1 to N delta alpha I is it nothing but what? This is the value of alpha at the point means I is 1 so delta alpha 1 means alpha X1 - alpha X naught + alpha X2 - alpha X so all gets cancelled and finally you are getting to be eta alpha B - alpha A, but we have chosen the, for epsilon we have taken the eta in such a way that this is less than epsilon, so this shows this implies F belongs to the Riemann Steiltjes integral with respect to alpha and that’s prove the result,

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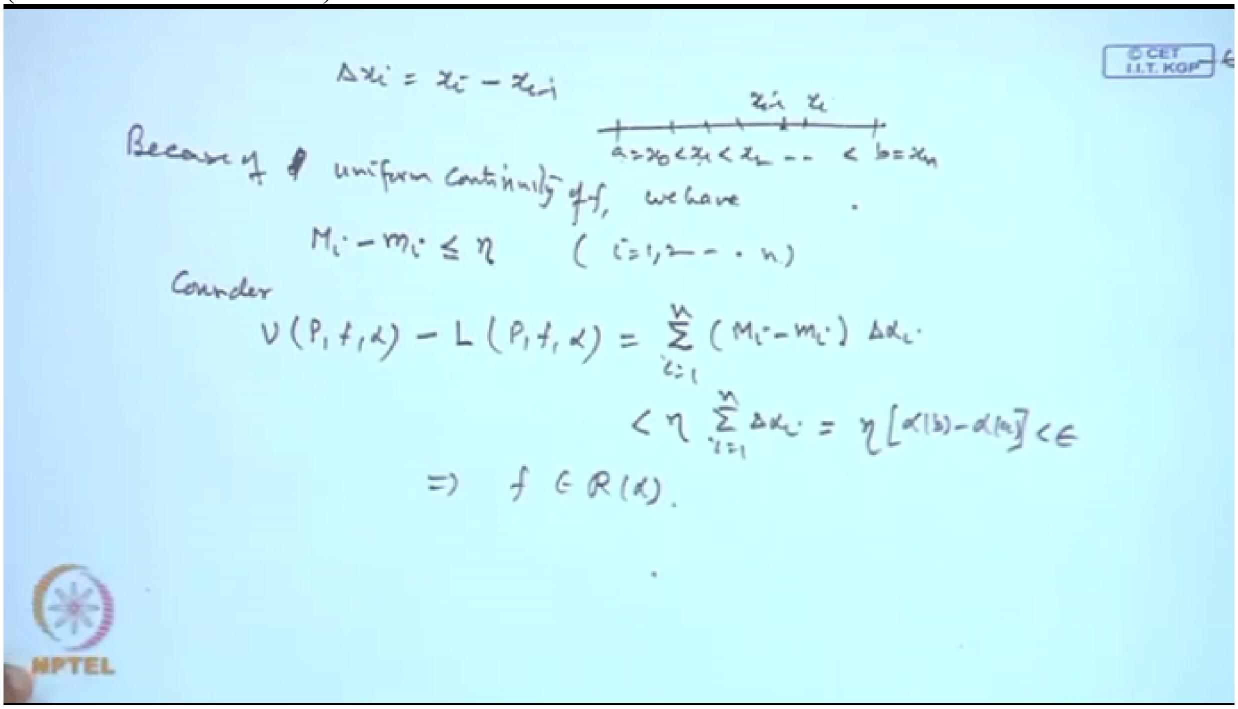

so every continuous function is a Riemann Steiltjes integral and in particular every continuous function is a Riemann integrable function.

Next shows that's also if F is a monotonic function on the closed interval A, B and if alpha is a continuous function, is continuous, now apart from this alpha is also monotonic, remember this I need not to write because we are assuming this is already monotone function, but maybe monotone function need not be continuous throughout, so here we are assuming exclusively alpha to be a continuous apart from its monotonicity on the closed interval say A, B, then F is Riemann Steiltjes integral or F belongs to the class ring, okay, let's see the proof of this again,
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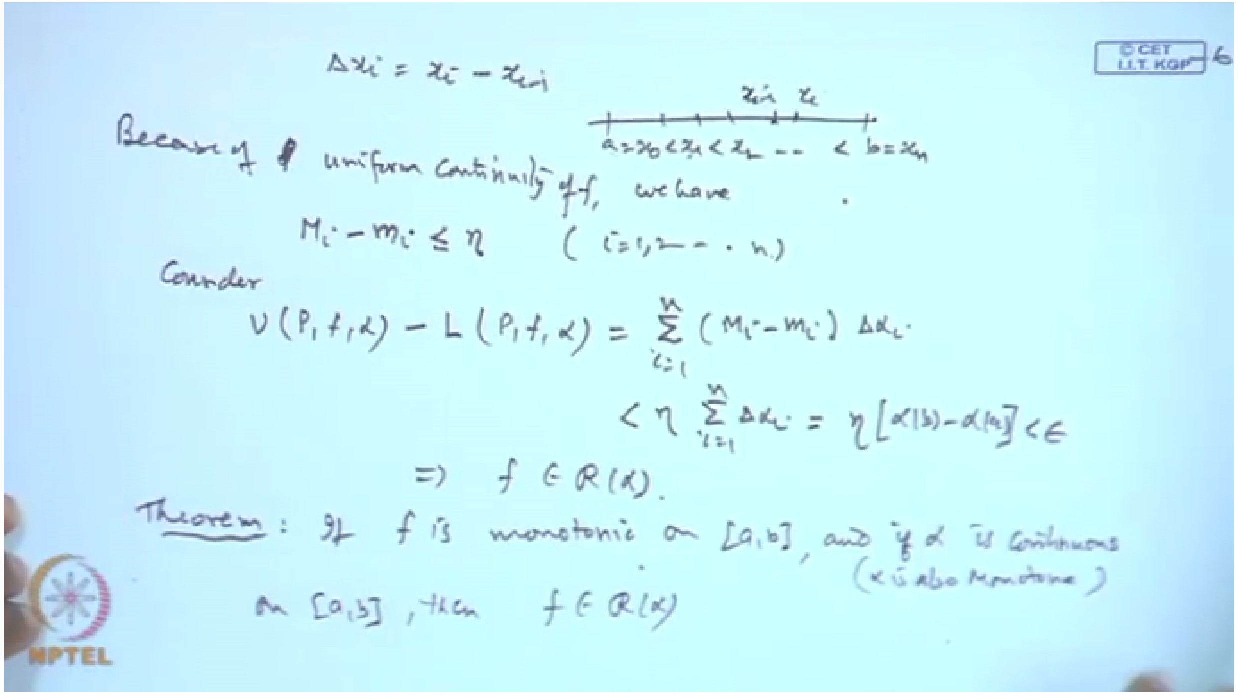

so it means the monotonic functions if F is monotonic and alpha is continuous, then this will be a Riemann Steiltjes integral.

Now in case of the Riemann integral function alpha x we are choosing already X so over the closed interval A, B if you check alpha X = X it is automatically a continuous function and it is a monotonic function, so basically it satisfy this condition so you need not to take this condition when you are dealing with the Riemann integral, what simply say every monotonic functions on the closed interval A, B is Riemann integrable function, but in case of this we have to take it condition on alpha as it, see the proof of it.

Let epsilon greater than 0 be given then again because alpha is continuous so it will assume all the values from alpha A to alpha B, it will assume all values therefore we can partition it delta alpha I we can choose the equal partition alpha B - alpha A/N as a N partition for this means sub partition of the interval A, B like this, so we can choose that so for any positive integer N choose a partition P in such a way such that the alpha B - alpha A divided by N is our delta alpha I, when I is 1 to N means equal part, delta alpha 1 is the same as delta alpha 2 is the same as this and the value is alpha B -, and this is possible since alpha is continuous. If alpha is not continuous it means that there's some point in between A and B the function alpha may not be defined at that point, so we cannot talk about all these things, so since it is continuous therefore all the values in between alpha A and alpha B is possible therefore we can divide it and get the equal values of delta alpha I, okay.

Now suppose F is monotone, this is given F as monotone so let F is monotonically increasing function, the similar case when monotonic decreasing function can be proved in a similar way, so once it is monotonically increasing function it means when you choose A to B partition as X naught, X1, X2, XI-1, XI and XN is B then value of the function at a point XI is greater than the value of the function at a point XI,

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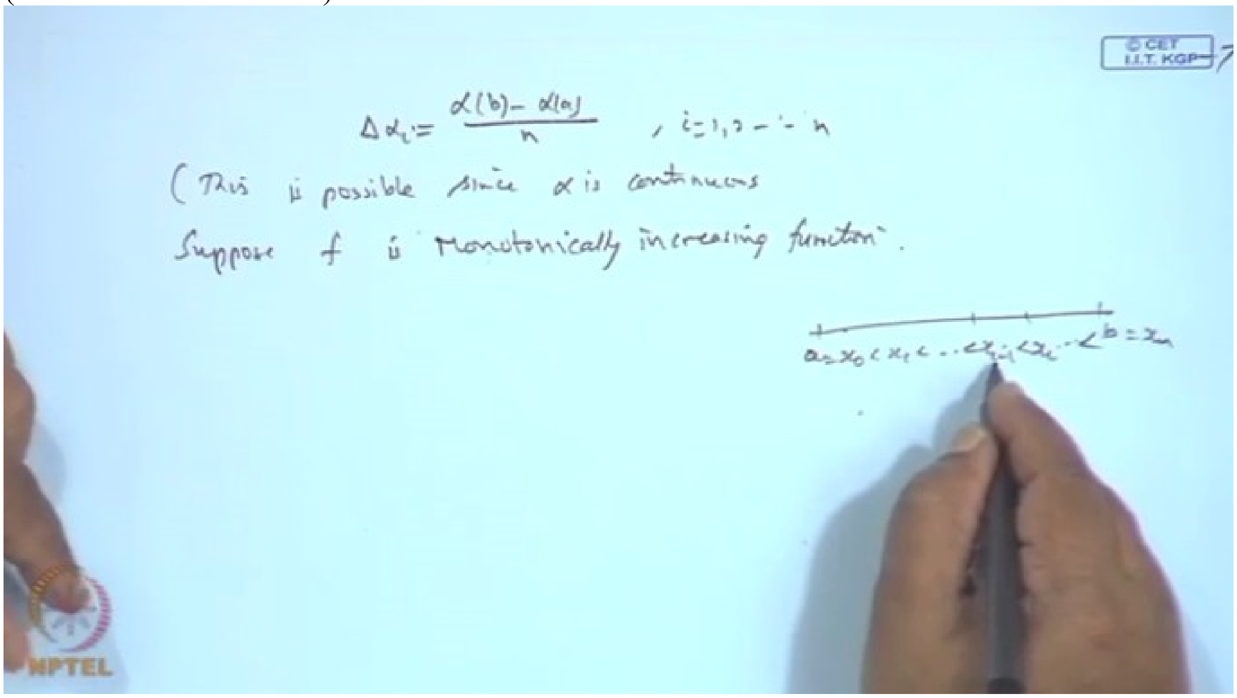

so over the interval XI to XI-1 the Mi, the maximum value of the F(x) will attain at the point XI, while the minimum value will attain at the point XI-1 because it is increasing function, monotonically increasing function and this is true for every I, I1 to N, okay.

Now consider the upper sum of the function F over this partition - lower sum of the function F with respect to alpha over this partition, now this will be equal to what? If you write this thing as sigma I = 1 to N Mi - small mi into delta alpha I, but Mi - small mi is this one, so we are taking delta alpha I we are choosing same, it is independent of I so we can take it outside by N and this sum I = 1 to N, what you are getting is F(xi) – F(xi-1), this is the value, now this when you substitute I = 1 to N that terms get cancelled then only you get the FB – FA, so finally you are getting alpha B - alpha A/N, F(b) multiplied by F(b) – F(a) this you are getting.

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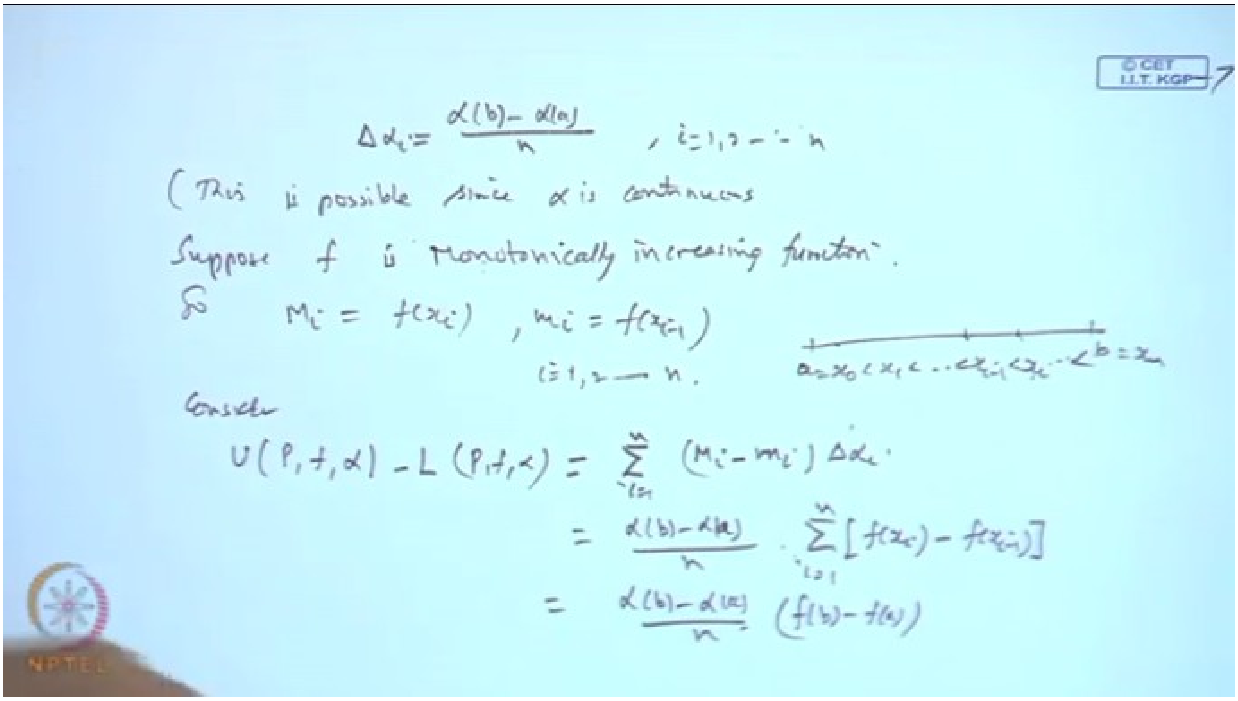

Now this one is less than epsilon this follows from F(b) condition is okay, this is a small quantity alpha B - alpha A is less than this, now this part when you're taking is less than epsilon given, why? So if you are taking this alpha B – alpha A/N F(b) – F(a) and this divided by N, so N is sufficiently large, okay, so we can take this is less than epsilon as N is taken sufficiently large. If N is taking sufficiently large, okay, so when N is sufficiently large the total thing can be made less than epsilon, therefore is satisfied that condition one which is necessary and sufficient condition for a function F to be in the class alpha, so this shows F belongs to R alpha, okay, so that's what.

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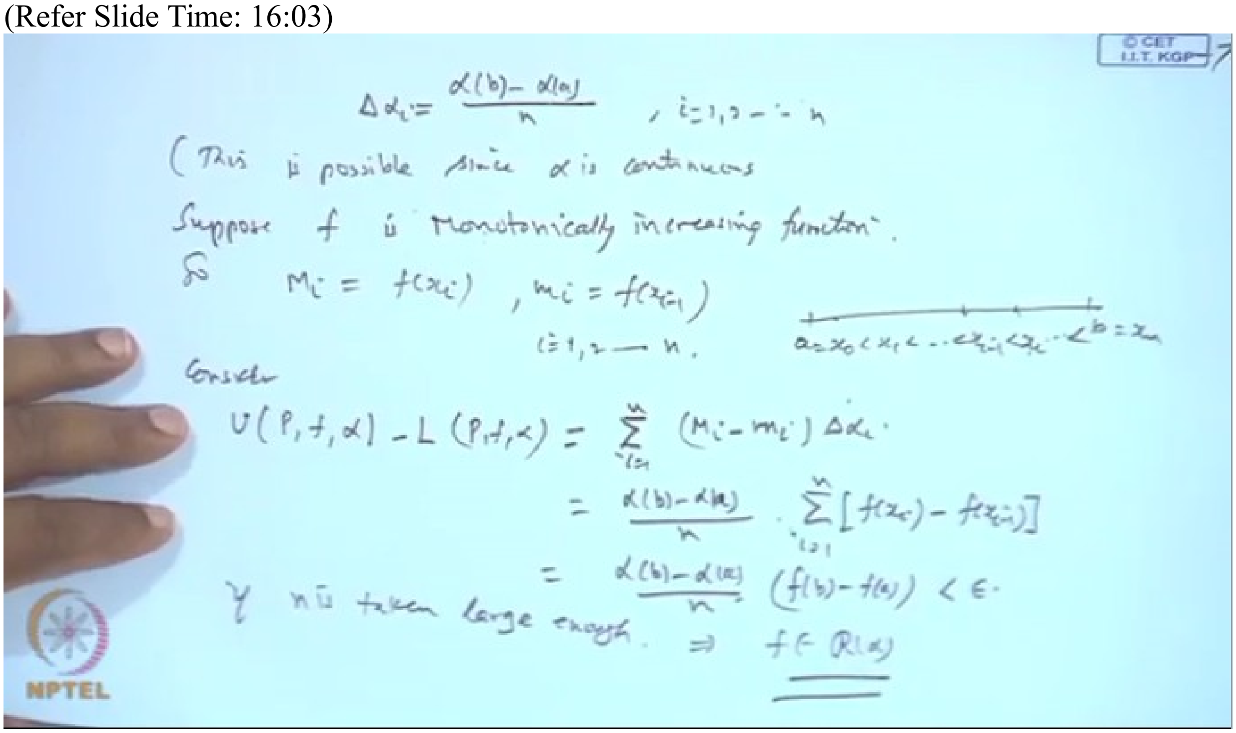

Another results which we so every monotonic functions now we come to the F is bounded function, suppose F is bounded on a closed interval A, B and F has only finitely many points of discontinuity on the interval A, B and alpha is continuous at every point at which F is discontinuous, then F will be an element of R alpha, that is F will be a Riemann Steiltjes integral, so what it shows is A, B interval is given, the function is given to be bounded on this, but at a point of and say these are the points, this is the point of discontinuities, these are the points of discontinuities of F, but and finite number, these are finite points of discontinuity,
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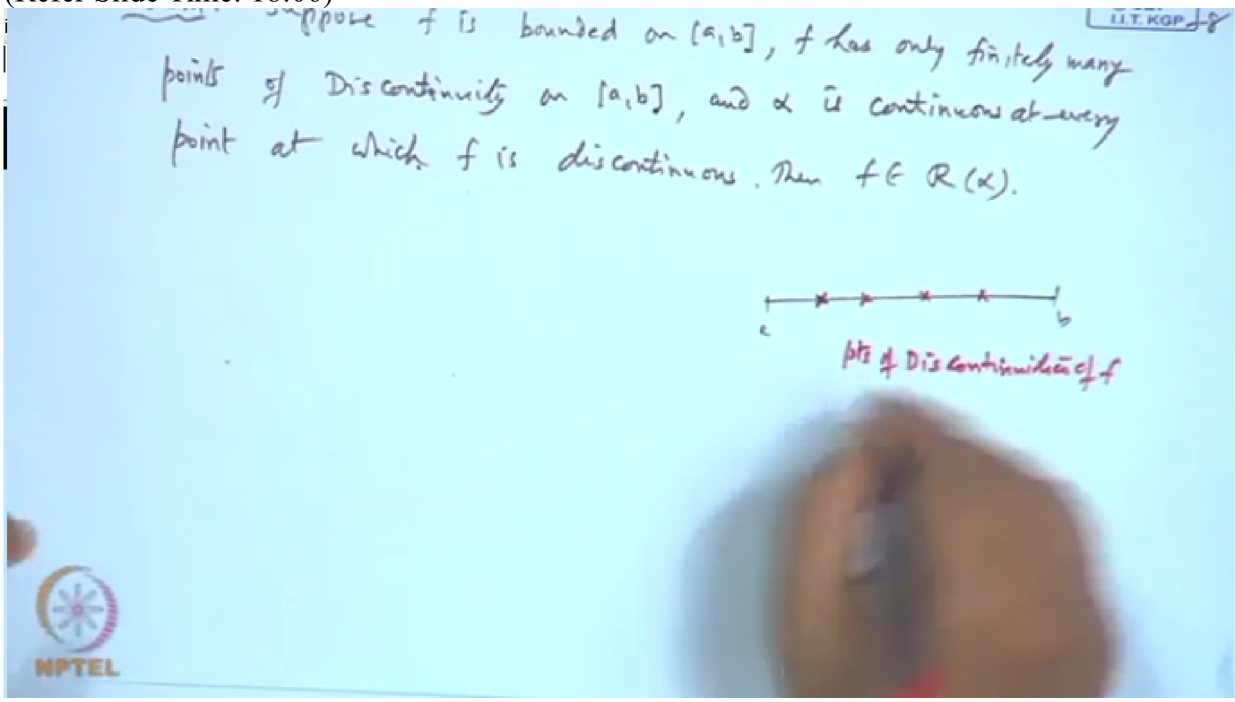

not in finite number, it's a finite number of points are there which is a point of discontinuity.

And what is given is at these points alpha is continuous, that is very important part, if alpha is also discontinuous at this point then this result will not hold, that we will see while proving we can easily see the region, if both the alpha and F has the same point of discontinuity over the interval A, B, then the F cannot be in R alpha, this result will not hold, only this result hold when the function is discontinued at the point alpha must be a continuous at that point, so this is the very important point here to prove okay.

Let's see the proof of this, let epsilon greater than 0 be given, now since F is bounded, so let us see the supremum value of F(x) is suppose M, this is the supremum value. Now let E be the set of points at which F is discontinuous, okay, now since E is finite because it's given that function F is has a finitely many point of discontinuity, so since set E is finite so we can cover E by means of a finitely many disjoint intervals say UJ, VJ, finitely many disjoint intervals which is a subset of A, B, because these are what a point,
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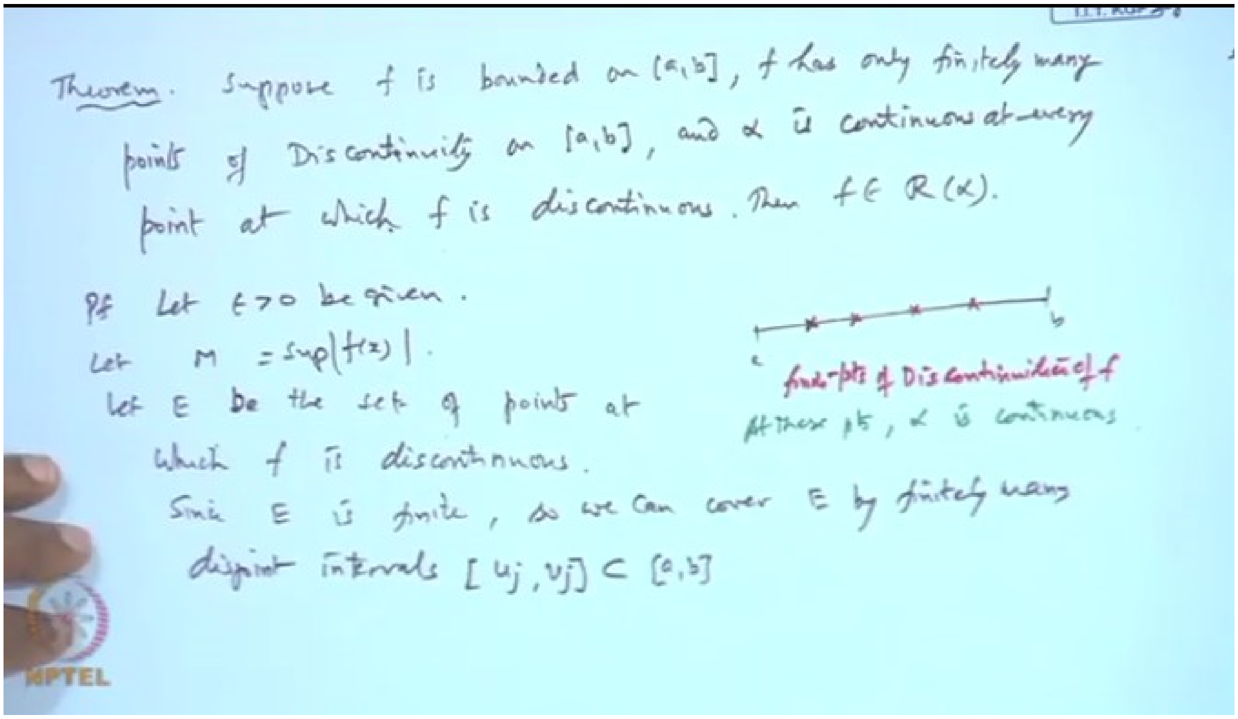

these are the point of discontinuities, so only a scattered point is isolated points we can cover it by means of these intervals, okay like this, say U1, UJ, VJ, this is the interval covering.

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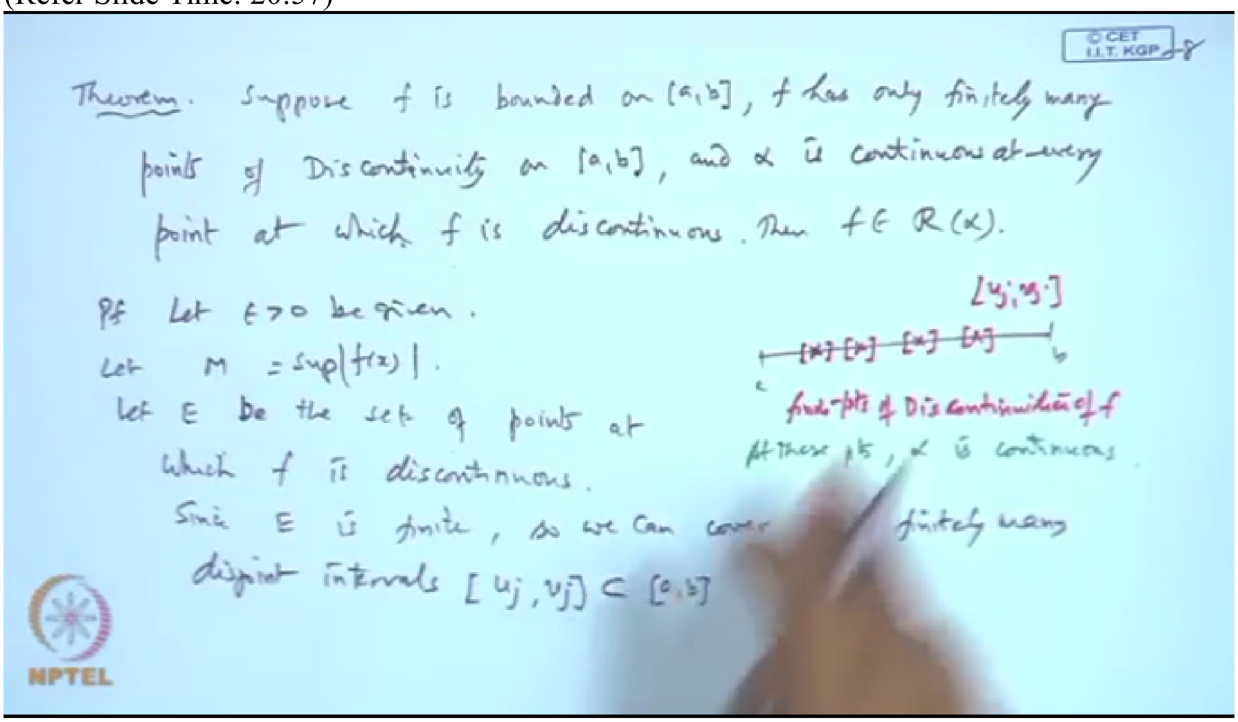

Now since function F is given to be continuous at this point, so it means by definition of the continuity if I look that definition of the continuity what happens? Say this is the one term, okay, so at this point the function is continuous it means for a given epsilon greater than 0 if I check say alpha is this point, then at this point alpha and for any number say epsilon/N suppose these are N points are there, so I can choose the epsilon by n say interval length, so that all the points image of any point inside this interval is falling this, so corresponding to this we can identify a length, a delta neighborhood such that the image will fall,

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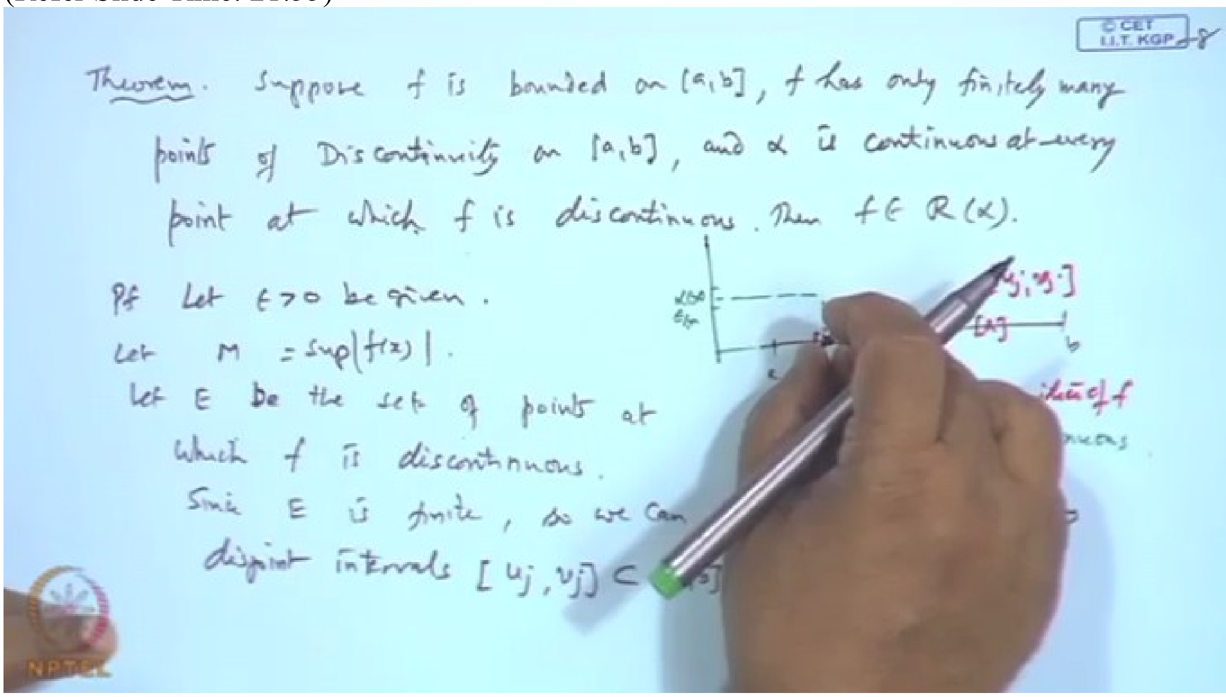

so some of these values alpha X minus this alpha values at this interval can be made less than epsilon because of continuity, so that is the advantage of alpha, we're taking to be as a continuous at the point where the function has a discontinuity, so that's what okay, so let’s take the difference of them.

Now such that the sum of the corresponding difference is there, okay now since alpha is continuous at these points, okay, so for a given epsilon greater than 0 we can choose delta J such that the sum of this, such that the corresponding difference is less than epsilon, such that the sum of the corresponding difference alpha VJ - alpha UJ is less than epsilon, let’s see why? I will again repeat suppose these are the points, so let this point is covered by this interval U1, V1 here we are taking say U2, V2, here this is the point then U3, B3 and like this, alpha is continuous at this point, so we can for a given epsilon greater than 0 we can identify here delta 1, delta 2 and so on such that image of this for this epsilon, image of any point will fall within the epsilon neighborhood of this, so what I am choosing is the total sum of this difference is less than epsilon, it means if the number is N then each one we can take epsilon by N, so the total multiplied by N will give this, so that's what is getting, okay.

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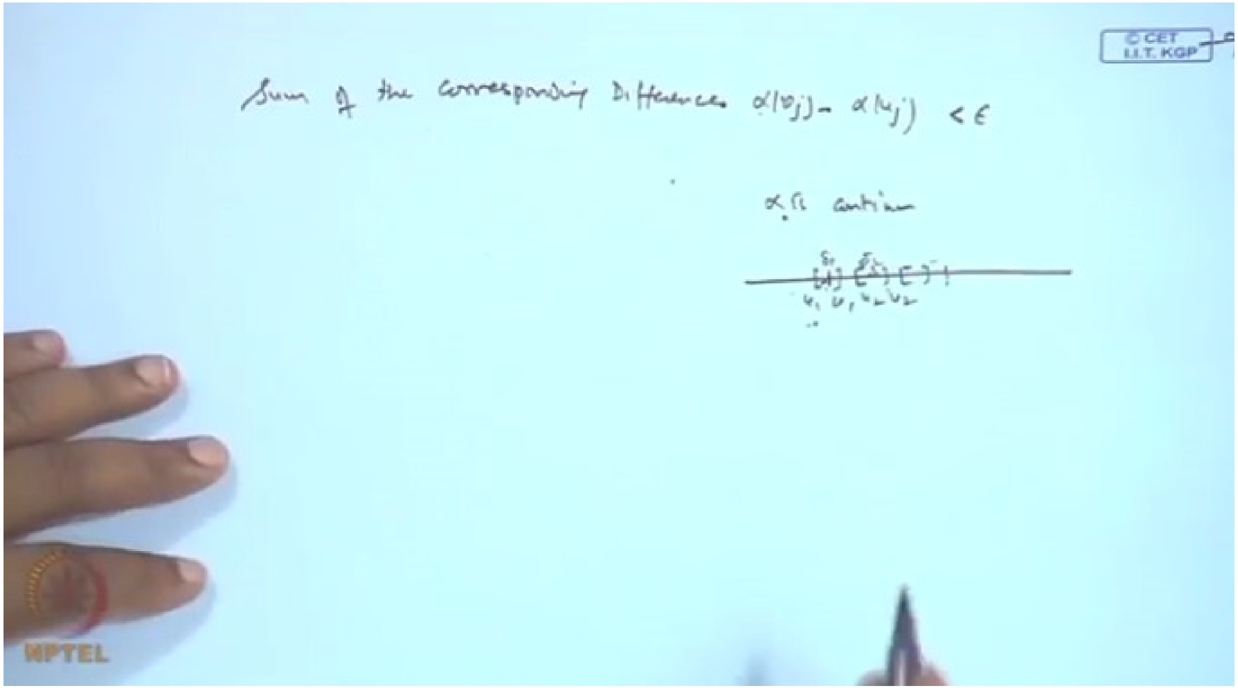

Now we can replace these intervals in such a way, so okay.

Now second step what we do is this is the first step, second step what we do, we can place these intervals in such a way that every point of E intersection A, B lies in the interior of some interval UJ, VJ, what's the meaning of this is, E intersection A, B, this is the set of those points say YI such that F is discontinuous at YJ, okay, so now you are taking these points are enclosed by U1, V1 in such a way that each point of this lies in the middle of this, in the interior of this we can take up U1, V1 in such a way that the first point lies in Y1 in this, E2, V2 so there is a second lies in it and like this,

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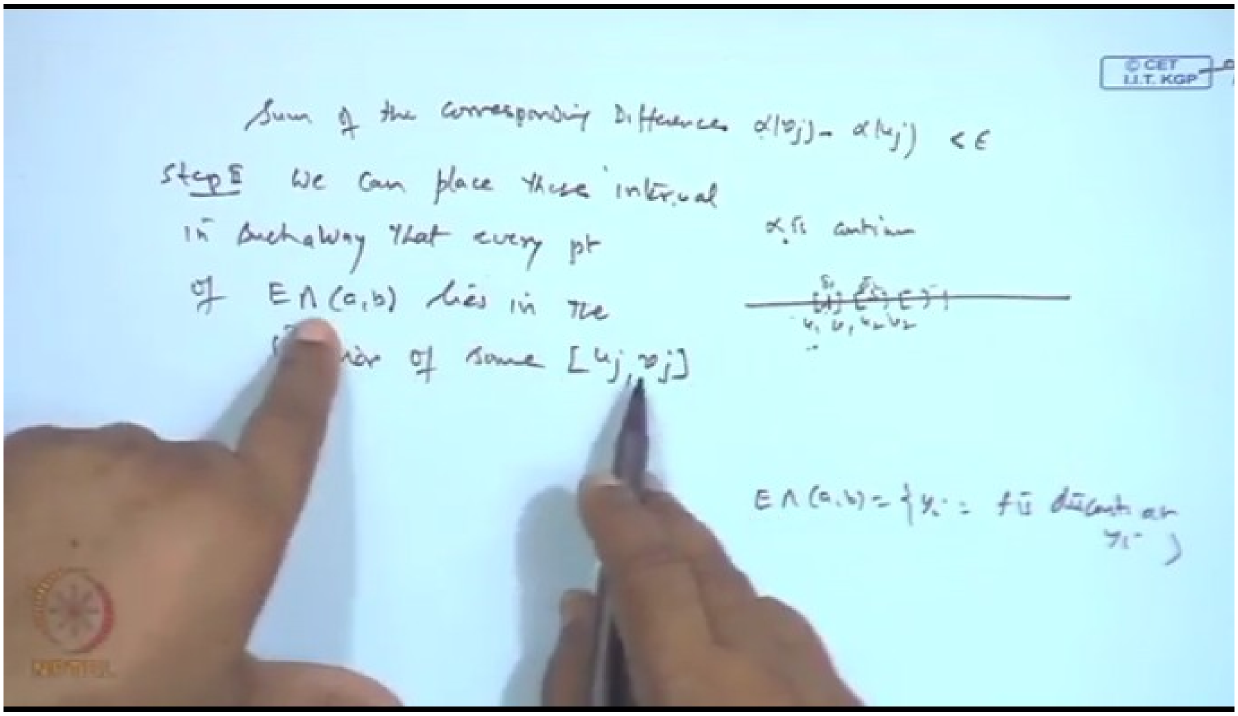

so this is the way we will construct, and then proof will go, so next time we will continue this proof, okay. Thank you.