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Course

On

Introductory Course in Real Analysis

By

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Lecture 68: Existence of Riemann Steiltjes Integral

Okay, so today we will discuss the result which we have stated yesterday, and which shows the existence of the Riemann Steiltjes integral and as a particular case or Riemann integral. R(alpha) we have taken the class of all Riemann Steiltjes integrals, so if F belongs to R(alpha) means F is an element of the class R(alpha), that is a Riemann Steiltjes integral of functions if and only if for every epsilon greater than 0 there exists a partition P such that the upper sum of the function F with respect to alpha minus the lower sum of the function F with respect to alpha over this partition is less than epsilon.

So for every epsilon greater then there exists a partition which this condition holds,
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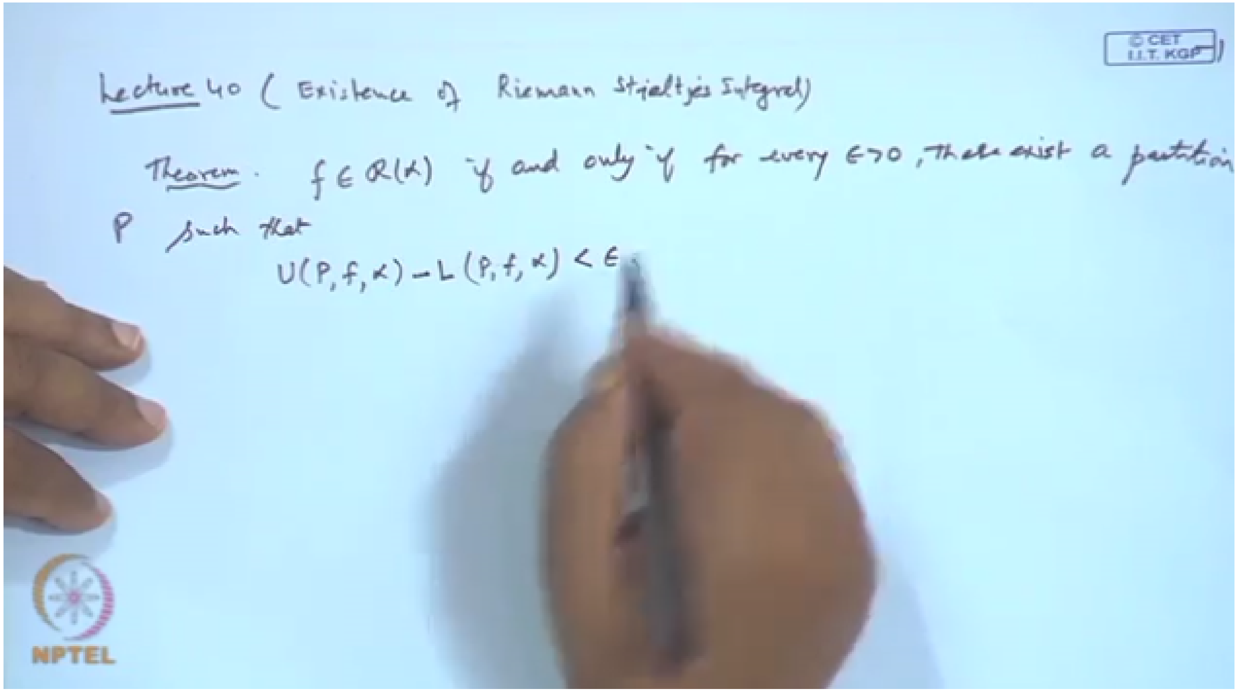

so this condition is if and only if condition let it we put it the condition as say 1, so let's see the proof suppose the condition holds, suppose 1 hold that is this for some partition, for a given epsilon we can identify a partition so that this is that upper sum - lower sum is less than epsilon, okay.

Now for any partition P, we know or we have the lower sum of the function F with respect to the alpha, where alpha is a monotonic function defined over the interval A, B, okay, so this is less than equal to the lower sum of the function, lower integral of the function F which is less than equal to the upper integral, upper Riemann Steiltjes integral of the function F which is less than equal to the upper sum of the function F over alpha, this we have already discussed because,
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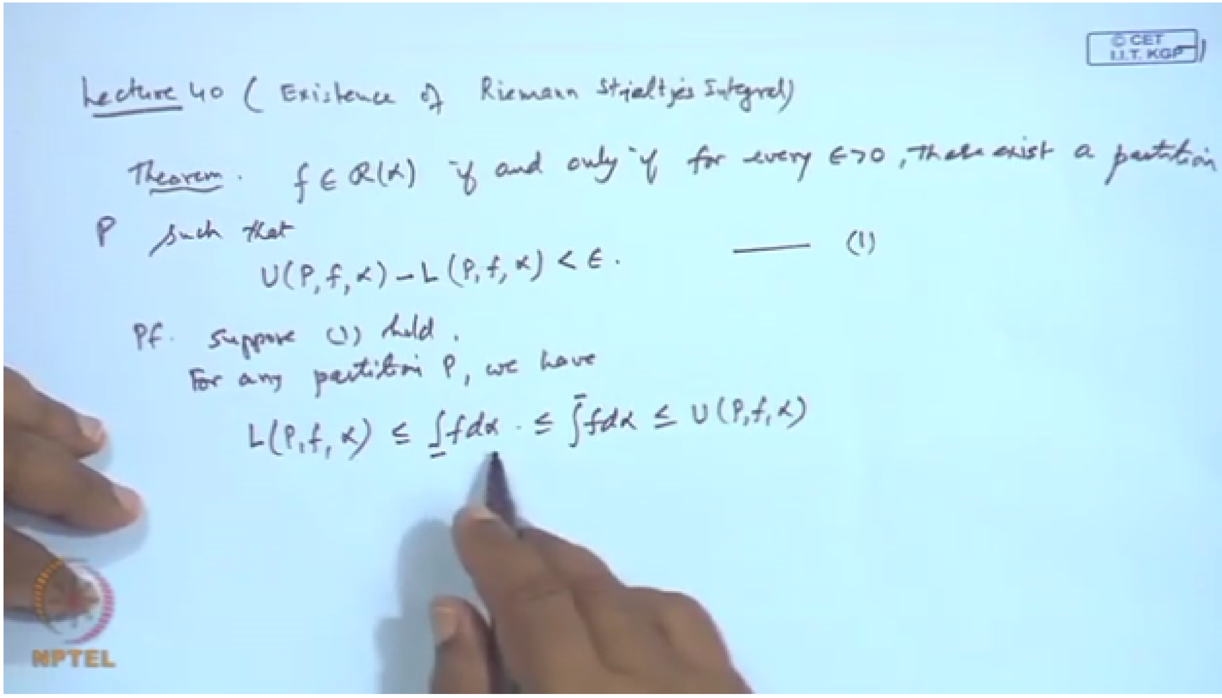

how to define the lower integral and upper integral of the lower Riemann Steiltjes integral of this is defined at the supremum value of the lower sum F with respect to the alpha, but the upper Riemann Steiltjes integral of the function F is defined at the infimum of the upper sum of F with respect to alpha, so if I remove infimum and supremum then this quantity will always be greater than, the upper sum will always be greater than or equal to the upper integral,

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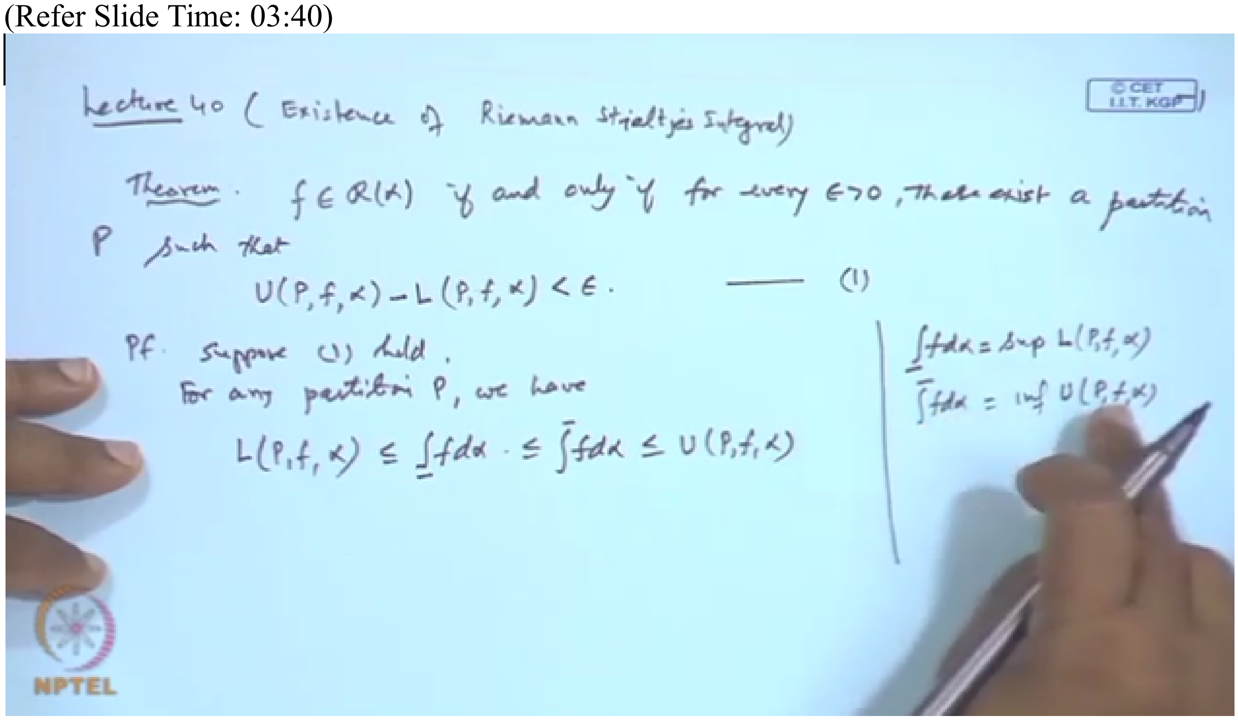

while lower sum will always be less than equal to the lower integral, so for any partition P this result holds, okay.

Now using the condition one, what the condition one says that there exists a partition P such that the difference between the upper sum and lower sum is less than epsilon this is given, so since this result 2 is true for any partition, so in particular this particular partition the upper sum - lower sum will remain less than epsilon, therefore using this we see here that 0 is less than equal to upper integral Riemann Steiltjes integral - the lower Riemann Steiltjes integral this remain less than epsilon, because this difference is less than epsilon so this.

And this is non-negative because the upper sum and lower sum is defined the maximum value of MI and delta, alpha and so on where alpha is a monotonic function, okay, so we are getting further, clear, so we get less than epsilon, but epsilon is arbitrary number, so once it is arbitrary it can choose any, for any epsilon very small this result hold, so this shows that lower integral and the upper integer Riemann Steiltjes integral coincide, and once they coincide then this implies the F belongs to the class R(alpha) the Riemann Steiltjes integrable functions, so this one.

Conversely suppose F belongs to the class of all Riemann Steiltjes integrable functions then we have to show that condition one holds.
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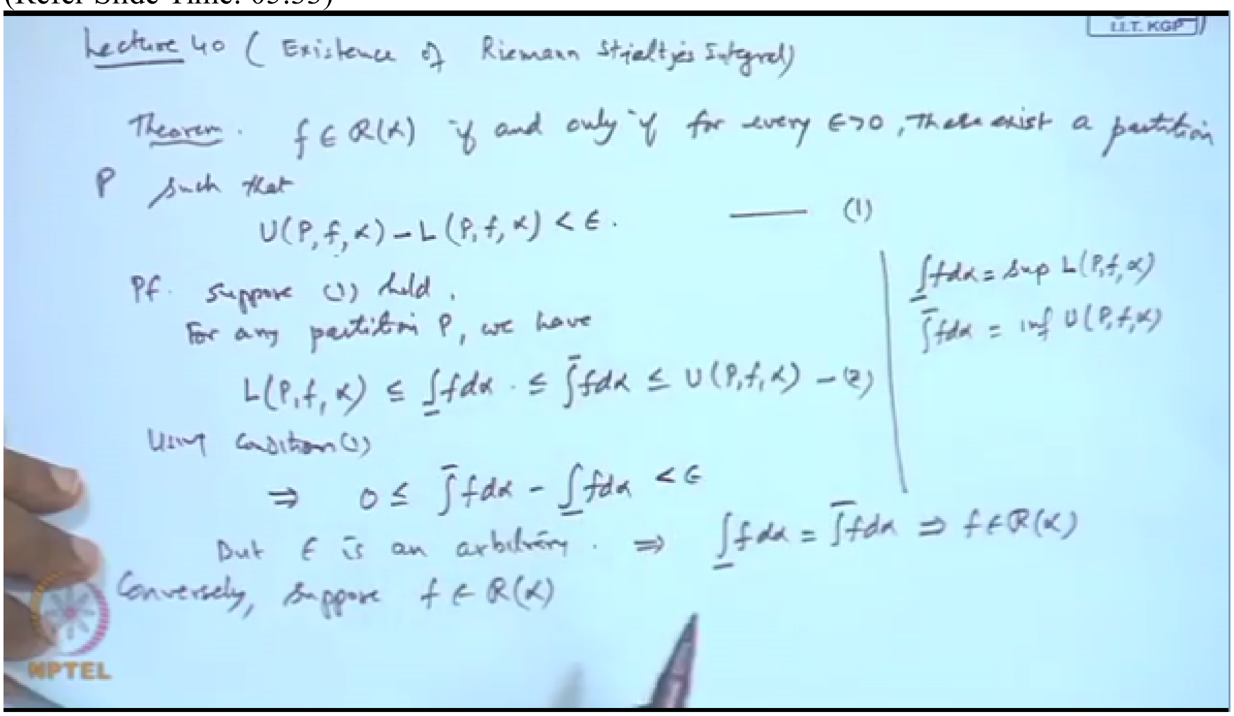

Now once F belongs to Riemann Steiltjes integral it means upper sum and lower sum coincide and equal to this, so what we get? From here the upper sum is the infimum value of this, okay, and lower sum is the infimum and lower integral, upper integral is the infimum value and lower integral is the supremum value of this.

So suppose I take this F belongs to R means that lower integral and the upper integral both coincide first thing, okay they are equal, now use this thing if I remove the infimum then what happen is, because this is the infimum value so this will remain less than this value, so if I take a number slightly higher than this then there will exist a partition P2 where this value will +epsilon will be greater than this number.

Similarly here if I remove this supremum because this is the largest value, so this is greater than equal to this number, so if I take a number slightly lower than this then there exists a partition where this number is greater than the number this - some number epsilon/2, so there exists by the definition I am calling definition A, okay, definition we can say that let epsilon greater than 0 is given, so with this epsilon there exists or there exists partitions say P1 and P2 such that

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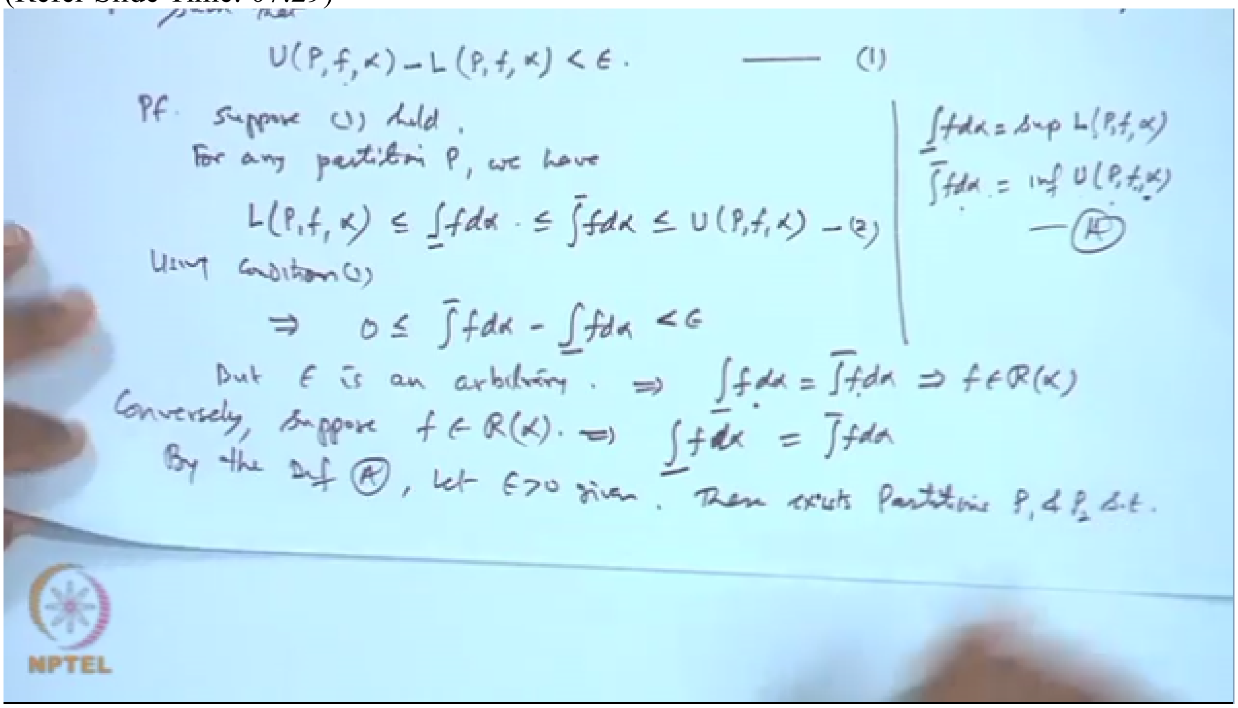

the upper sum of the function F with respect to alpha over the partition P2 - integral FD alpha is less than epsilon/2, and lower sum of this integral FD alpha - lower sum of F with respect to alpha over partition P1 is less than epsilon/2, this we can get in this way, so let it be equation say third.

Now if we take the common partition let P is the P1 union P2, okay, so if we take the refinement of this, this is the refinement of P1 and P2, so if this is a hold for this then for the refinement also we can get this general result, what we get? Upper sum since P is the refinement of P1, P2, so upper sum decreases than lower sum increases, this we have already discussed so you start with this, upper sum of the function F with respect to alpha over the partition P which is the refinement of P1, P2, since upper sum decreases so this is the less than equal to upper sum of the function F with this alpha over the partition P2 further using the three, the upper sum of this less than integral FD alpha + epsilon Y2, is it not?

Now using the second part of this three integral FD alpha is less than further lower sum of P1, F alpha + epsilon/2 so this is less than +epsilon okay.

So what we get it? From here we get this that U(P, F, alpha) -L okay, one more thing is because this lowers partition, and lower sum increases for the refinement, so this is further less than L(P,

F alpha) + epsilon, so this minus this F alpha is less than epsilon, at least this is true for this partition P, so there exists a partition P for which condition 1 holds, and that proved the results okay, so this.
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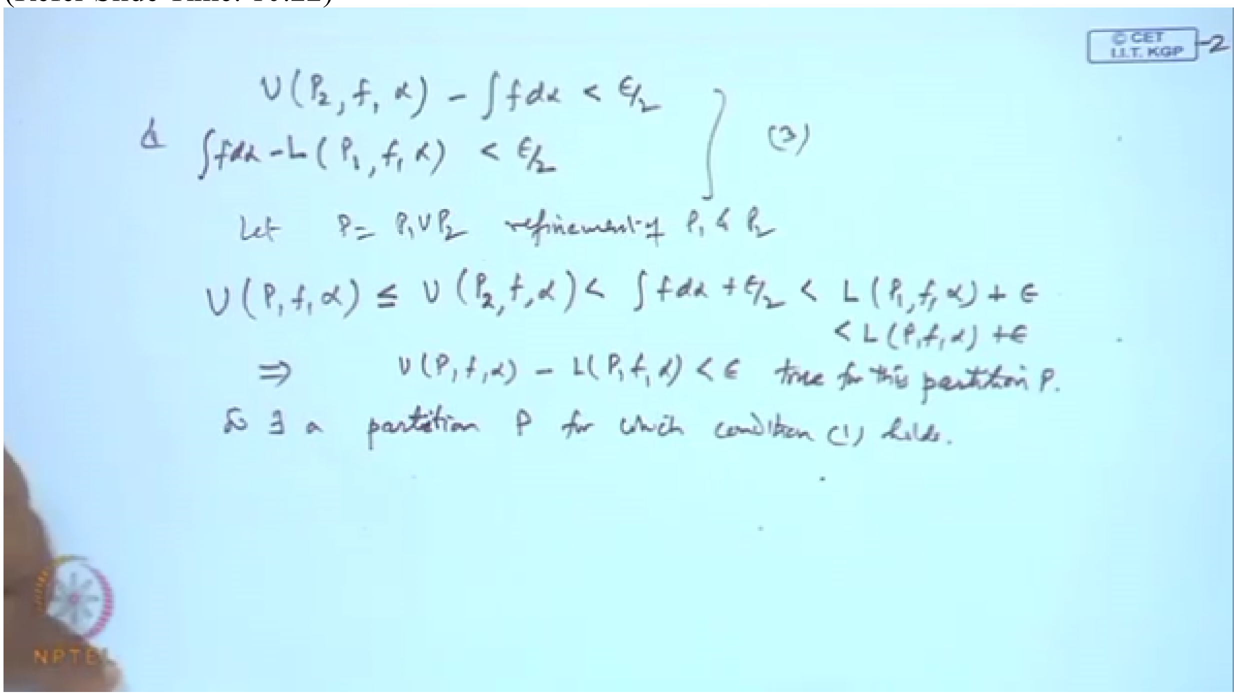

Now this results gives a guarantee or criteria for a function F to be Riemann Steiltjes integrable function, if the upper sum - lower sum is less than epsilon for some partition P. In particular when alpha X = X then you can get the corresponding corollary, right, as a corollary if alpha X is identically X, then F belongs to R, the Riemann Steiltjes integral, sorry Riemann integral if and only if there exists a partition P such that for given epsilon greater than 0, there exists a partition P such that a person of the function P, F - the lower sum of the function with respect to partition P is less than epsilon, and this is F, N only part for the existence result for the Riemann integrable function so this one, proof is okay, okay as a corollary we can get.
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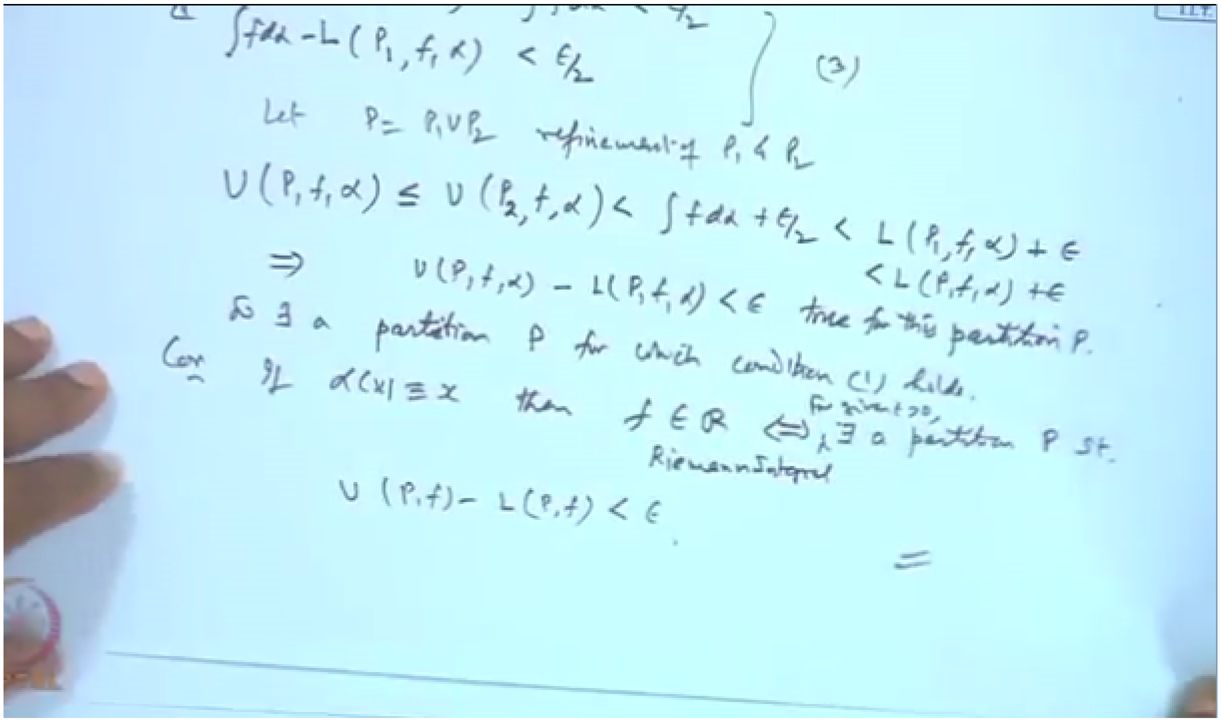

Now from this result we can drive few more conditions, we can say is sometimes sufficient conditions, sometimes necessary condition for the functions to be in Riemann Steiltjes integral or particular Riemann integral, so we have this theorem, the theorem says in the three parts, A, if one holds, if condition one hold, one hold means the condition that is there exists a part, that is for every epsilon greater than 0 there exists a partition P, partition P such that upper sum of the function F with respect to alpha - lower sum of the function F with respect to alpha is less than epsilon, so if this condition holds for some partition P and some epsilon, holds for some partition P and for this one, okay, and for some epsilon greater than 0, then the condition one holds with the same epsilon for every refinement of P, of the partition P means the condition one says that there exists some partition for every epsilon there exists a partition.

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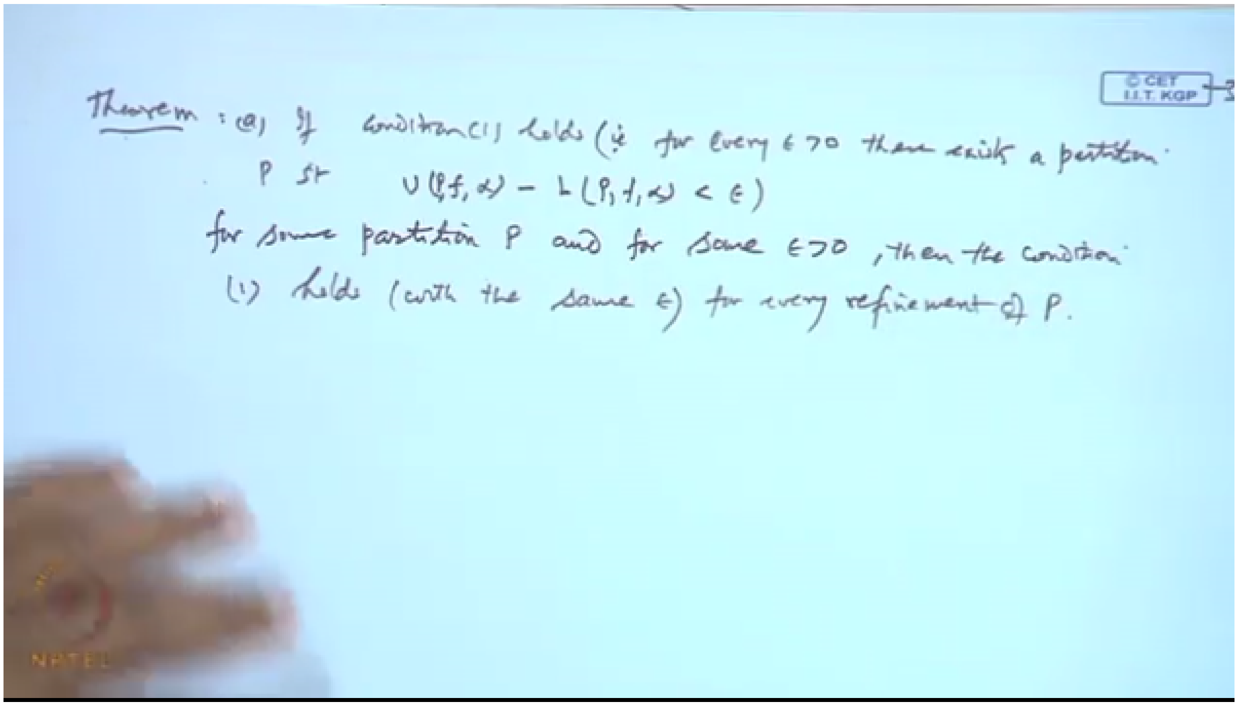

Now what it says is if suppose this results hold for some partition, for this epsilon then for the same particular epsilon this result will hold for any refinement, if we take a refinement of P, say P star then this result will continue to hold good, okay.

The proof of course is simple, let me see the proof one, in case of let P star be the refinement of P with the same epsilon greater than 0, this epsilon we are having the partition which is the refinement of P, okay, epsilon I am not changing, refinement of P but when P star is the refinement to P the upper sum decreasing, lower sum increases, so upper sum of this is greater than equal to the upper sum of this partition of F with respect to alpha over the partition P star, and the lower sum of this is increases, this decrease increases so minus of this will decrease, so L(P star, F, alpha), but this is less than epsilon it is given so this implies that upper sum of the function F with respect to alpha over the partition P star - the lower sum of the function F with respect to alpha over the partition P star is less than epsilon, where the P star is the refinement is true, so hold, so this proves that.

Second result says if one holds, if the condition 1 holds for the partition P, say X naught, X1, X2, say XN and if SI and TI is suppose are the arbitrary points in the interval XI-1 XI then the sigma of this F(SI) – F(TI) under modulus sign multiplied by delta alpha I is less than epsilon, I is 1 to N, this is true,
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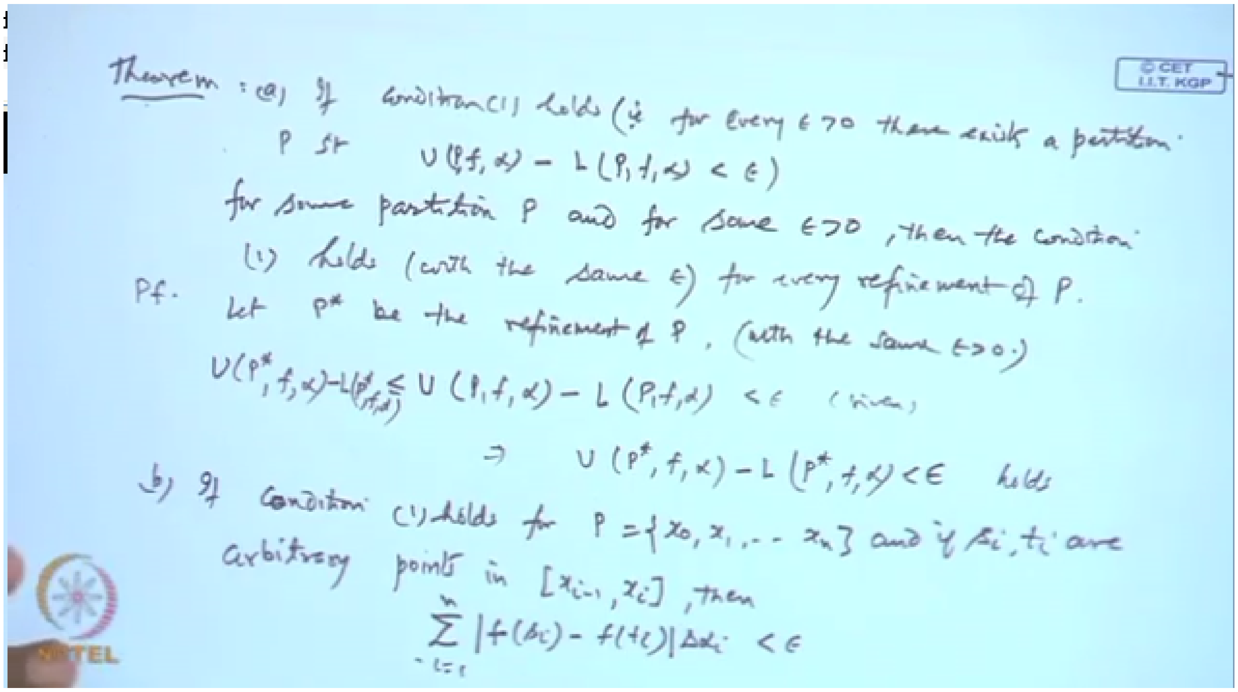

so it means if this is the partition of A, B interval, and if this is the point say XI-1, this is the point say xi and if we picked up the two point SI and TI in this sub interval XI-1 to XI then the
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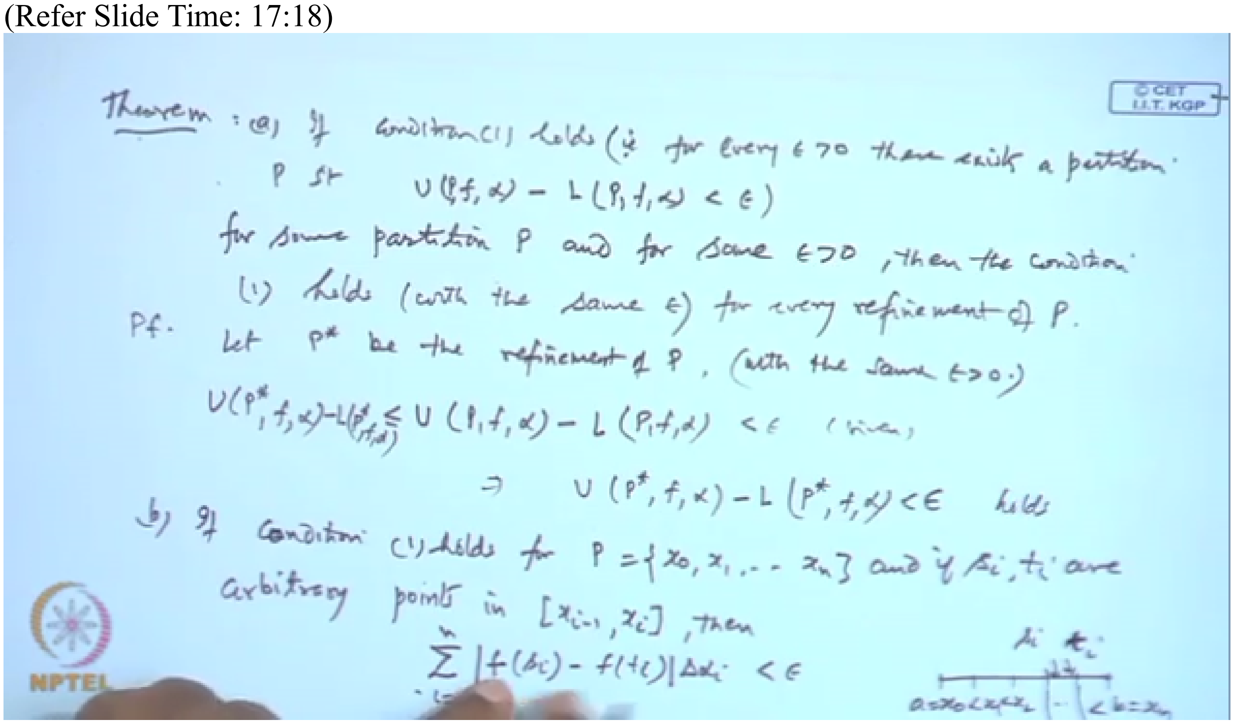

functional value at the point SI-TI sigma of this multiplied by delta alpha is less than epsilon, that's what it says, that if the function, if the condition 1 holds that is if they'll exist for a epsilon greater than 0, if there exists a partition such that this upper sum - lower sum is less than epsilon then it will also imply that this condition will hold, the reason is again very simple, the proof is giving like this, what is our F(si)? Since F(si) and F(ti) both are, both belongs to the interval MI, into M capital MI, here what is mi and capital MI? Where mi is the infimum value

of the function F(x), when X greater than equal to XI-1, and less than equal to XI, and capital MI is the supremum value of the function F(x), when X lies in the interval XI-1 to XI,
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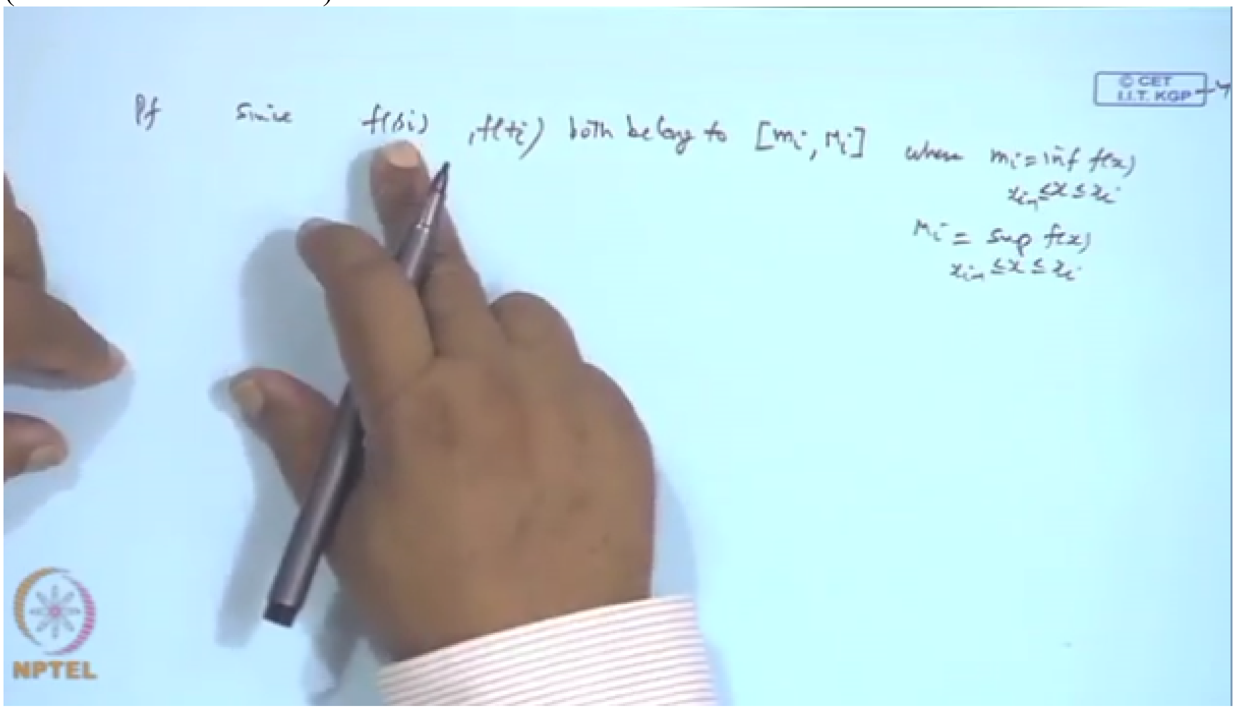

so both these values are in the interval mi and capital MI.

So if we start with this say upper sum and the lower sum, what is the upper sum? Upper sum is F(si) - lower sum, so what we get it? F(si),sigma so consider this thing, I is 1 to N mod of SI – F(ti), consider this multiplied by delta alpha I.

Now FI and SI is one of the value, is value lying between this so both are lying between MI and capital MI, so clearly F(si) and –F(ti) mod of this will remain less than or equal to MI - small mi, is it not? So we can say this is less than equal to sigma Mi - small mi, I is 1 to N delta alpha I, but this is nothing but what? The difference of the upper sum - lower sum with respect to the partition P, upper sum - lower sum of the function F with respect to alpha over the partition P, okay, and this already given by condition 1, because condition one holds,
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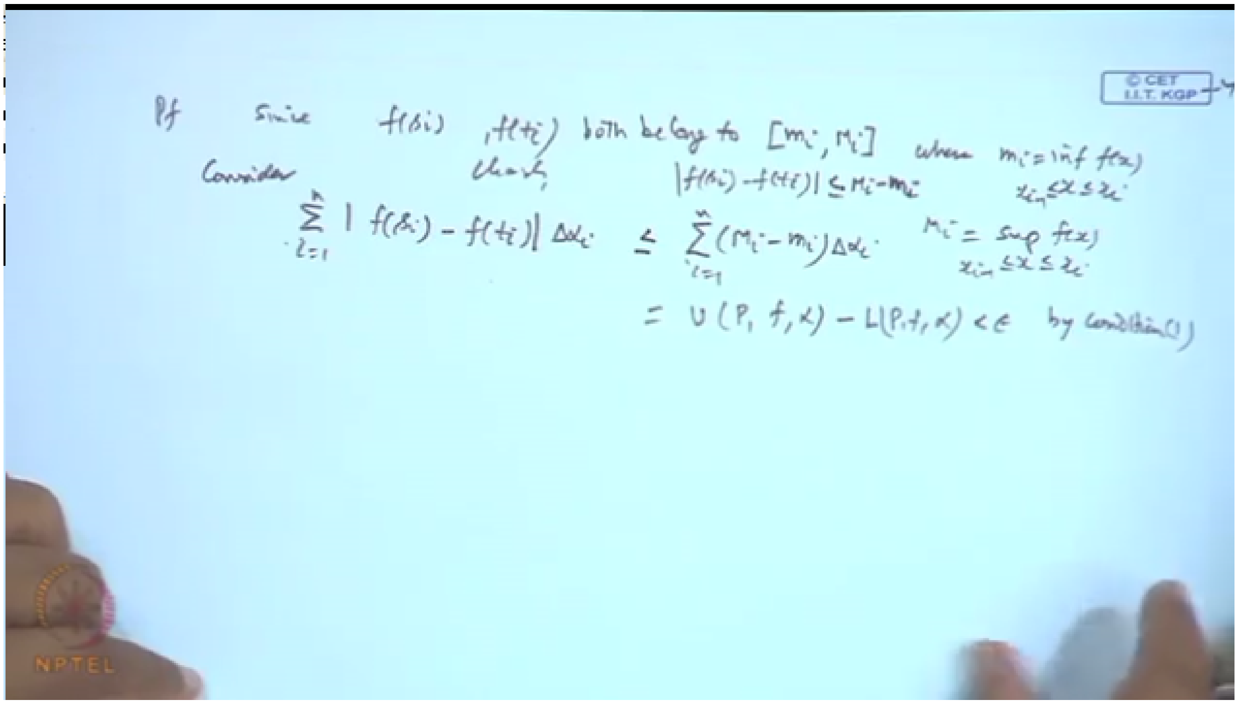

so this is true by condition 1, so this is it, therefore this sum will be less than this, so this proves that.

Now it's third portion which it says if suppose, if F is Riemann Steiltjes integral with respect to alpha, and the hypothesis of B that is sigma of this thing is less than epsilon for partition P, X naught, X1, X2 and SI, TI are the point in between this XI-1 and XI, then this condition holds, this hypothesis is given, this hypothesis hold then sigma of F(ti) delta alpha I - integral A to B FD alpha is less than epsilon, that's all, okay, I is 1 to N.

So proof is let me see, the proof we will start with again similar way, but start with the lower sum of the function F with respect to alpha over the partition P, this is equal to sigma, I is 1 to N, small mi delta alpha I, but mi is the infimum value of this function, so obviously this will remain less than equal to the value of the function F(ti) where ti lies between XI-1 to XI, this is the point, so I is 1 to N, and then delta alpha I, but again this value is less than equal to is maximum value, so it will be less than equal to sigma I is 1 to N, M of I delta alpha I, because it is the supremum value of this function, and which is nothing but what? Upper sum of the function F with respect to alpha over the partition P, so this is the one result, so this shows that is the lower sum of F with respect to alpha this satisfy this condition, let it be 4, okay.
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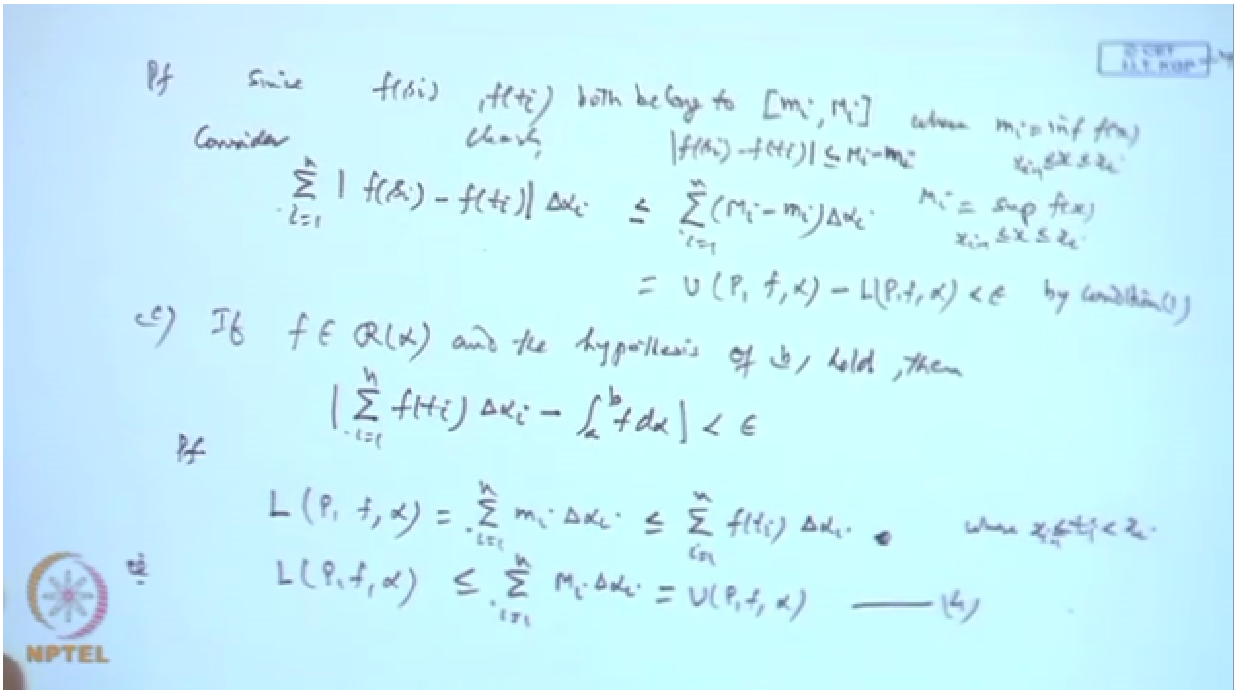

Further the lower sum of this, because F is given to be Riemann integrable function, so the supremum value of this will coincide the integral FD alpha, so if I remove supremum which is less than equal to this, and again the infimum value of the upper sum is this, so this is less than equal to upper sum of F with respect to alpha, this holds for any partition P is it not? So if I take the 4 and 5 together, then what we get? If we take the 4 and 5, so 4 and 5 together imply, this difference if I take, then what happen is the difference is coming to be 0, in fact tending to 0 is very very small, because U, P alpha – L alpha is less than epsilon, the condition in the hypothesis two, hypothesis two holds means the condition one holds, one holds means difference of U -L is less than epsilon, so difference of this minus this is less than epsilon in absolute value implies I is 1 to N, then F(ti) take this difference, okay, mi and this I will say here this part you can take it here, okay. And then again this is less than so we need this part so less than F(ti) - integral FD alpha modulus of this that is sigma of this under this, modulus of this is less than epsilon holds, okay.

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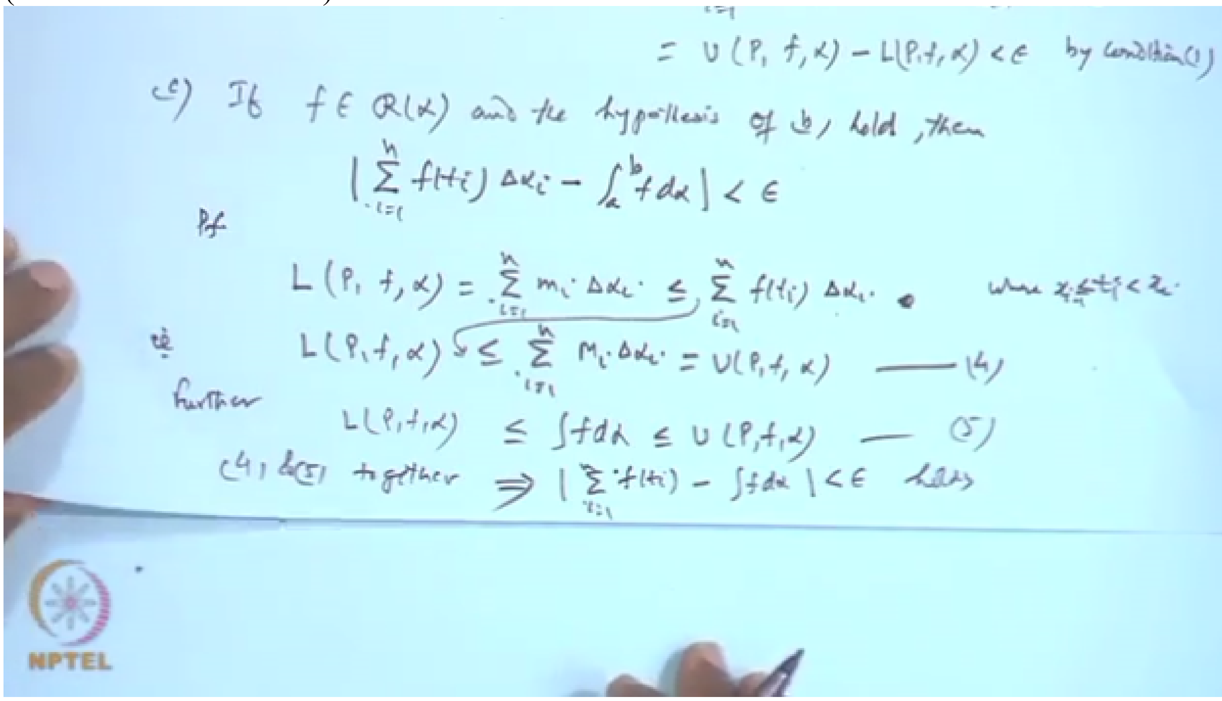

Now this result also suggest the way to define the Riemann integral, Riemann Steiltjes integral, because we have discussed that definition of the Riemann integral and Riemann Steiltjes integral by constructing the lower sum and the upper sum and then finding that supremum infimum value, and if the lower integral and upper integral coincide then we say F is Riemann integral function, Riemann Steiltjes integral and when alpha X = X then it is the Riemann integral function.

Now in most of the books earlier if you see they don't take it to the upper sum and lower sum, what they start with the function F defined over the interval A, B, okay and then functions we assume to be a continuous function bounded function subtract, and then what they do is the partition the interval find out the point in the sub interval take the value of F(ti) and then multiply this by the delta alpha I, so the delta alpha I is left, so multiply delta alpha I and taking the limit when N is sufficiently large it means when the number of the partitions are more, I mean infinity and for so when the points are very close to each other, then this integral limit exists, we call it this integral of the FD alpha and existence and we say that integral exists, so this is the way of defining the Riemann integral or Riemann Steiltjes integral, clear? So however the both way, equivalent way of defining but here this is a more better way because you are explicitly you are constructing the sum and then seeing the things because this will go like this, if suppose I have a graph of the function say like this, suppose this is our A, this is our B and let us partition it, okay, so this will be our partition and like this.

Now when you are choosing the value then what you are doing is you are constructing
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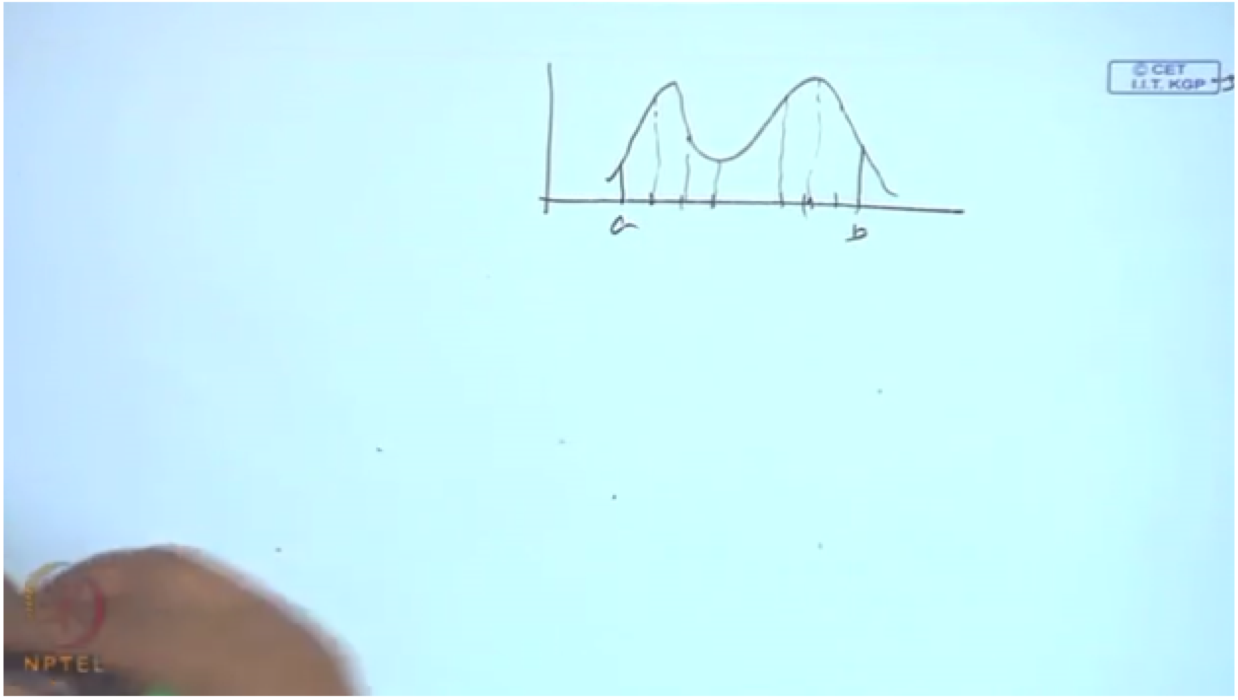
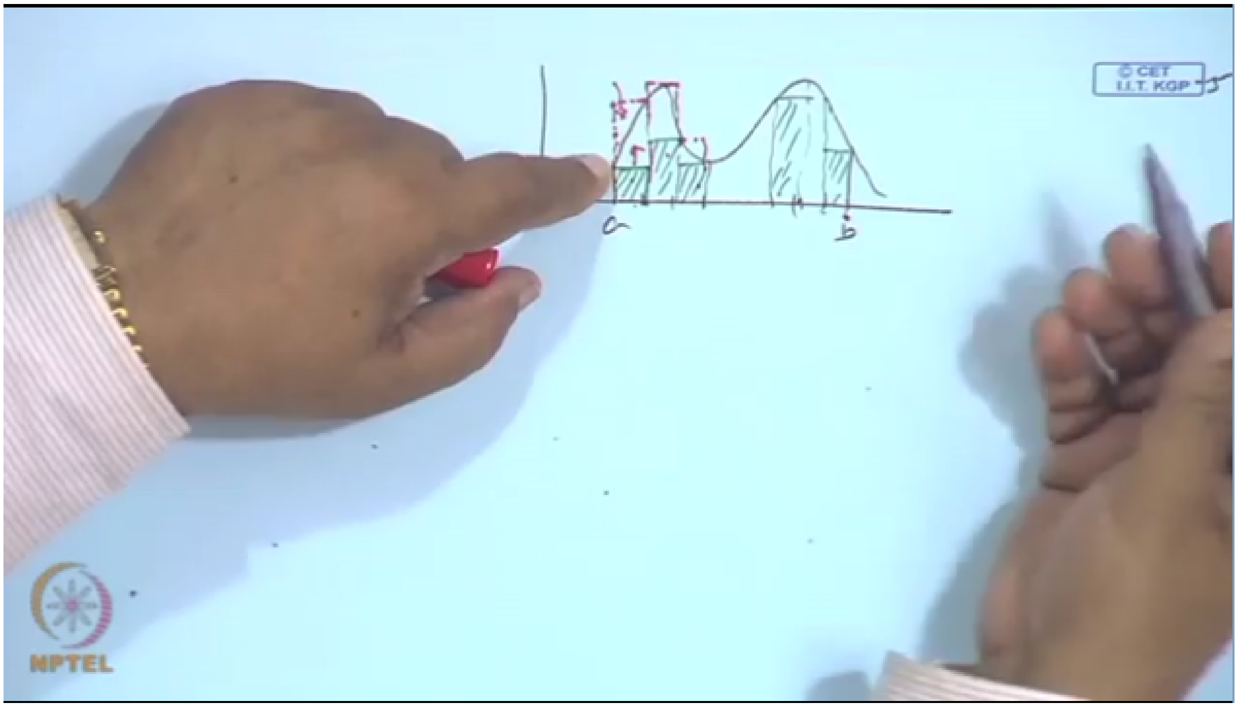


this sum in this interval you are taking the infimum value of the function, so this is the infimum value of the function over this interval so you are taking this, then infimum value in this interval you are taking this sum, infimum value on this and total sum you are choosing, and then changing the partition over this and basically after this changing etcetera we will get this limit, when that supremum and infimum value coincide then we say the integral function is integrable.

And what is the upper sum? A person in this case will be this much when you take second upper integral this is the lower integral this give, an upper sum will get this part, okay, so here also this is the, in this interval the upper sum is this so we can take this one, okay, like this and continue this one, so when you choose the limiting value they basically the upper sum decreases and lower sum will increase and they will coincide with this total area bounded by this curve, (Refer Slide Time: 28:39)



if the function is continuous it represents the area bounded by the curve Y = F(x) and the coordinate X = A and X = B with the axis of X.

In case of the bounded functions we say it is a enhancement of the definition integral and we call it as a Riemann integral for this, okay, and when we take up the alpha X as a monotonic function and then choosing the values of XI manual at the point with respect to alpha then you get the Riemann Steiltjes integral, so that's the difference between these three concepts, however the way in which is defined is basically gives the other definition whatever the in terms of the upper sum, lower sum or directly also as the limit of this sum, but since limit of the sum is difficult to compute any integral because the partition is more important when you take the limit of the sum of the right hand side because it becomes infinite series, so once you get the infinite series, then what happens is the sum of the infinite series will be difficult unless you choose your properly partitioning point, and looking the proper partitioning point with respect to the function is not so easy, so that's why we try to avoid that part limit of this sum to can prove the Riemann Steiltjes integral or Riemann integral, we take up upper sum and lower sum and taking the infimum is supremum value, so that's what is that, okay.