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Course

On

Introductory Course in Real Analysis

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Lecture 67: Riemann/Riemann Steiltjes Integral (Contd.)

See what all the properties enjoyed by this partition, so let's see properties of partition P, so we say the refinement of the partition, refinement of P, P is the partition, so by the partition P star is a refinement of P, of the partition P, if P stars covers P totally, that is every point, if every point of P is a point of P star, then we say this, because what is the partition? Because the partition P, this is our P partition means is the collection of the points X naught, X1, X2, XN such that satisfying this condition, that's all, so if we take any other partition that will also contain the points of this.

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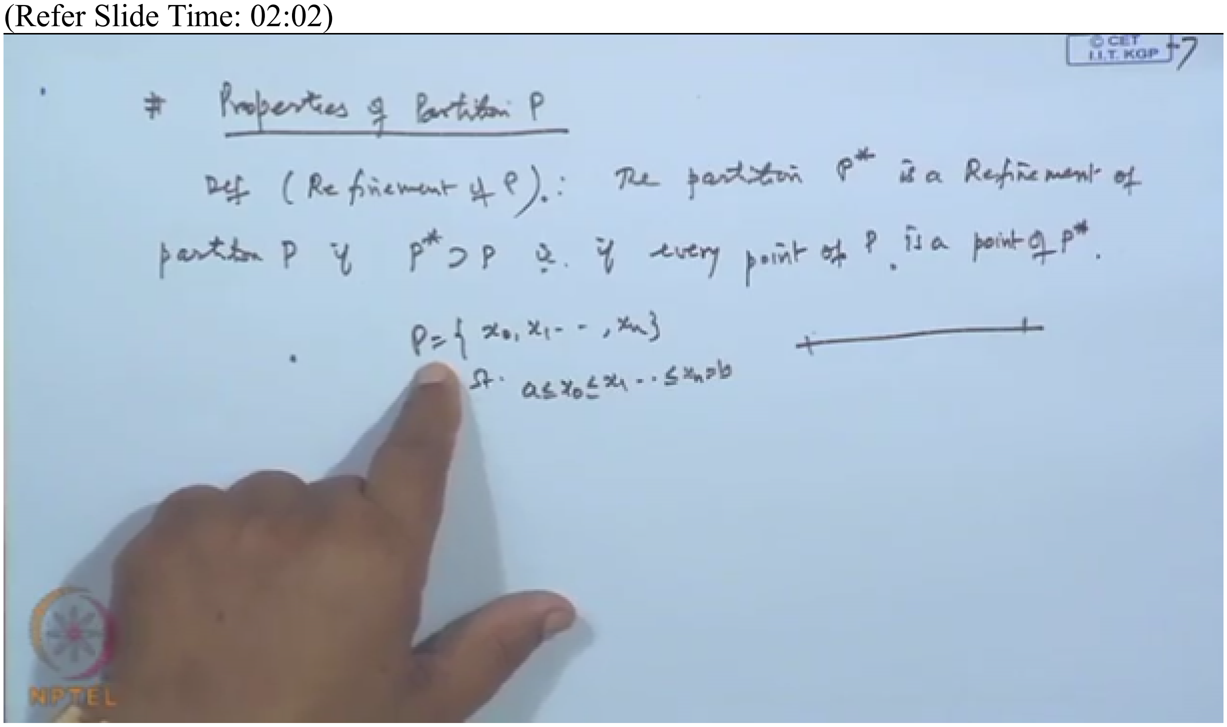

Now if this partition, if every point of P is also a point of P star it means when you take the P star then these points will definitely there apart from there, there may be some more point included here say X star, X double star and so on, this point may also be included apart from this, so that's why then every point of the P becomes the P star + some extra point are available in this, so instead of partitioning this into the N's of interval, we are partitioning N+1 sub interval, N+2 intervals by introducing more point in between, but earlier partition written as it is then it is called the partition or refinement of the partition, okay, because the partition may be different the P1 and P2 are two different partitions, suppose this is our P1 okay say one of the partition X naught, X1, X2 and say XN = B this is one partition, I take another partition A, B say P2 which is entirely different say by naught less than by 1, less than by 2, less than by N which is B, now this X naught, X2 and XN, by 1, by 2, by N may not be the same point, but if we take the partition P1 union P2, this partition is P star.

Then what happen is A and B are there, then all the points X naught then maybe the Y naught then X1 and continues like this, XN = BN, YN is this, so all the points are taken together which are partitioning the interval in A, B into 2N subintervals particularly, so this becomes the refinement of the partition P1 as well as you can say refinement partition P2,
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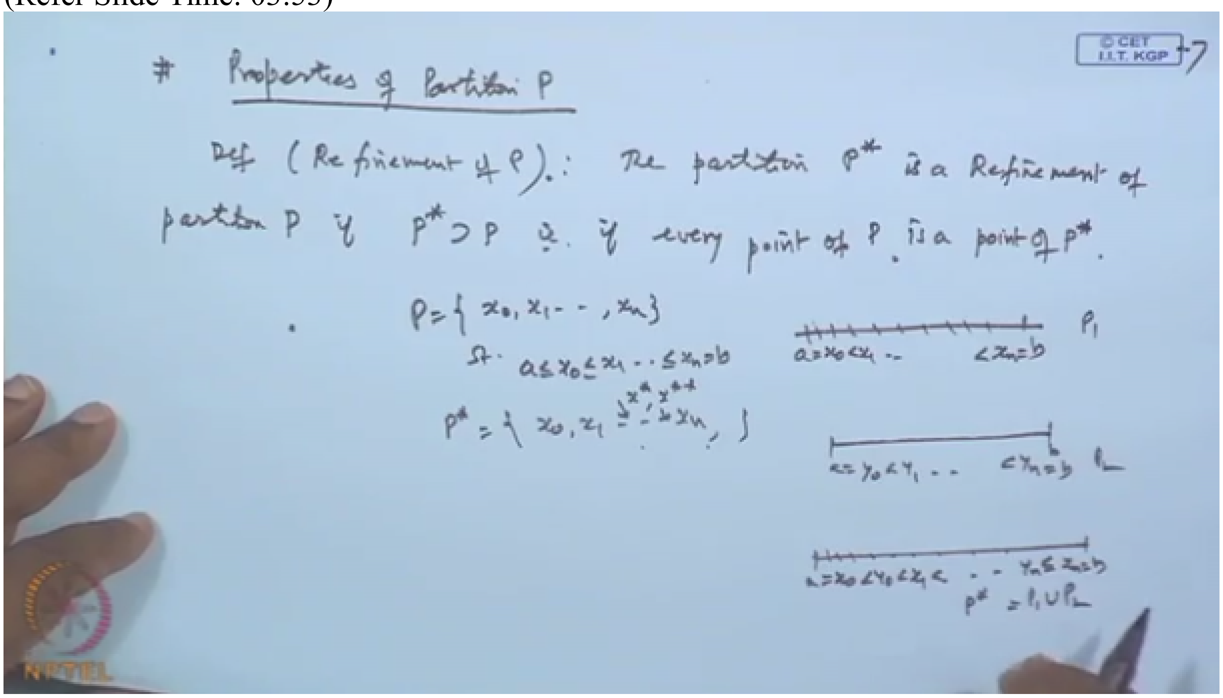

so in particular even if the partition P is there, if I just encode one more point in it then obviously the number of subinterval increases, and in that way we are getting a refinement of the previous partition P, okay, so this is the concept of the refinement, clear?

So obviously here, so clearly if P1 and P2 are two different partition of the interval A, B, then their union P1 union P2 is the say equal to P star is the common refinement of this P1 and P2 provided of course P star is this, union of this, okay. And if P star this, then P star is the common refinement, if P star is this then this common refinement so this is what there, okay.
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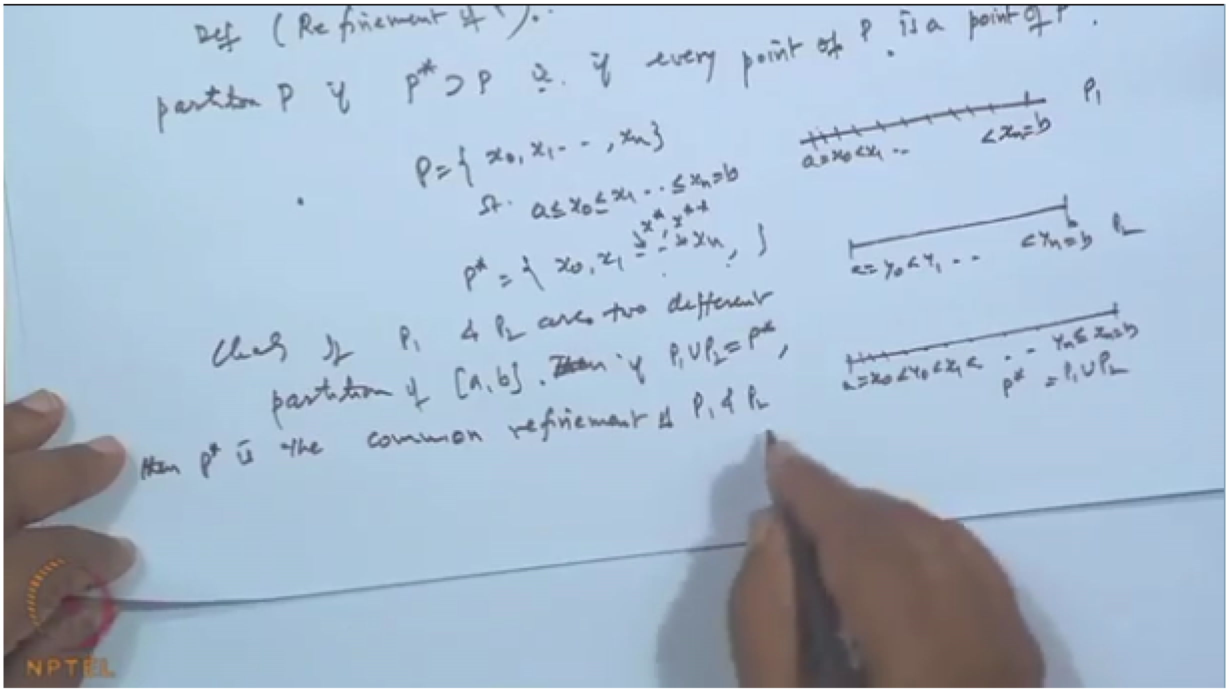

Now result which we were talking is, so let's the first result is if P star is a refinement of the partition P, if P star is a refinement of the partition P, then the lower sum increases, then lower sum L, P, F alpha will be less than equal to L, P star, F alpha, while the upper sum P, F alpha, this upper sum decreases, that upper sum of P star, F alpha, so lower sum increases and upper sum decreases, so when you divide the interval A, B into a partition P, and if I include few more point, introduce few more point in it and getting the partition P star which is the refinement of P, then in that case the lower sum will form an increasing function, will be an increase or lower sum will increase, but the upper sum will decrease, so that's to prove, so proof is this, okay.

Let's assume let P be the partition say A is X naught less than equal to X1, less than equal to X2, XN equal to say B, be the partition of A, B. And P star is supposed contains just one more point, say one more point, just one point more than the partition P that is suppose there exists or there is an extra point X star lying between in this sub interval XI-1 to XI lying this where XI-1 and XI are two consecutive points of P,
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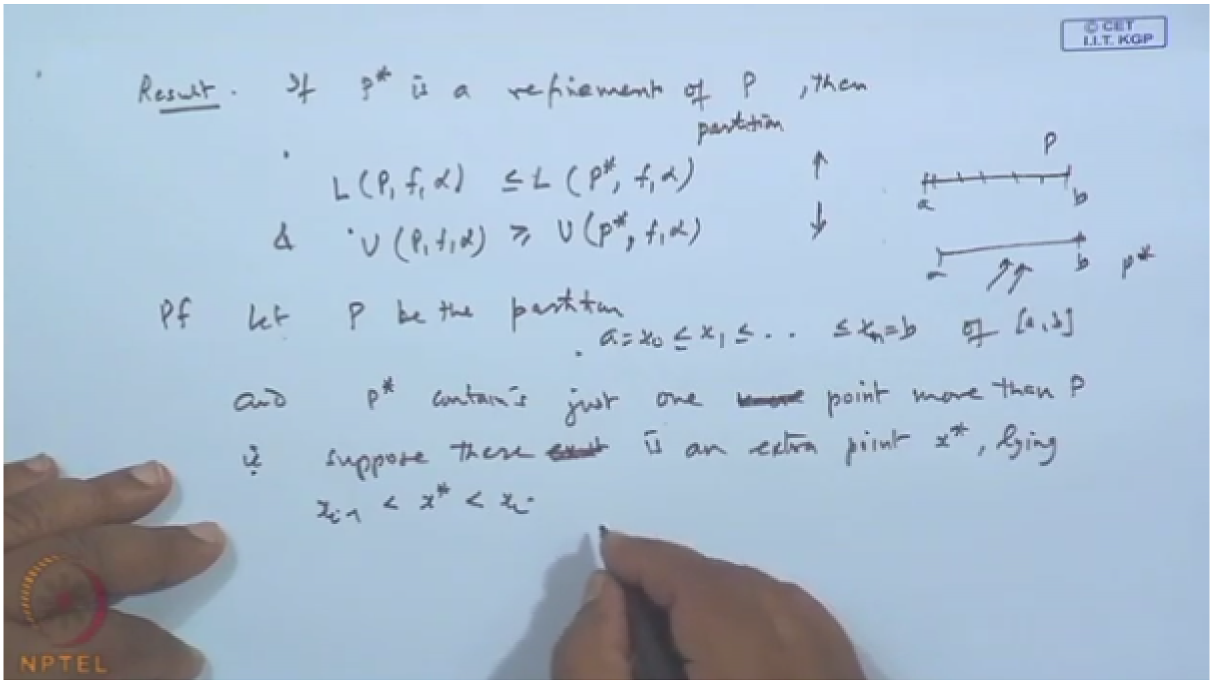

so what we are doing is that this is our interval A, B here we are having the partition X1, X2, XN, here is XI-1 this is say XI, and then XN is this, okay like this.

Now what we are doing is, this is our partition P,
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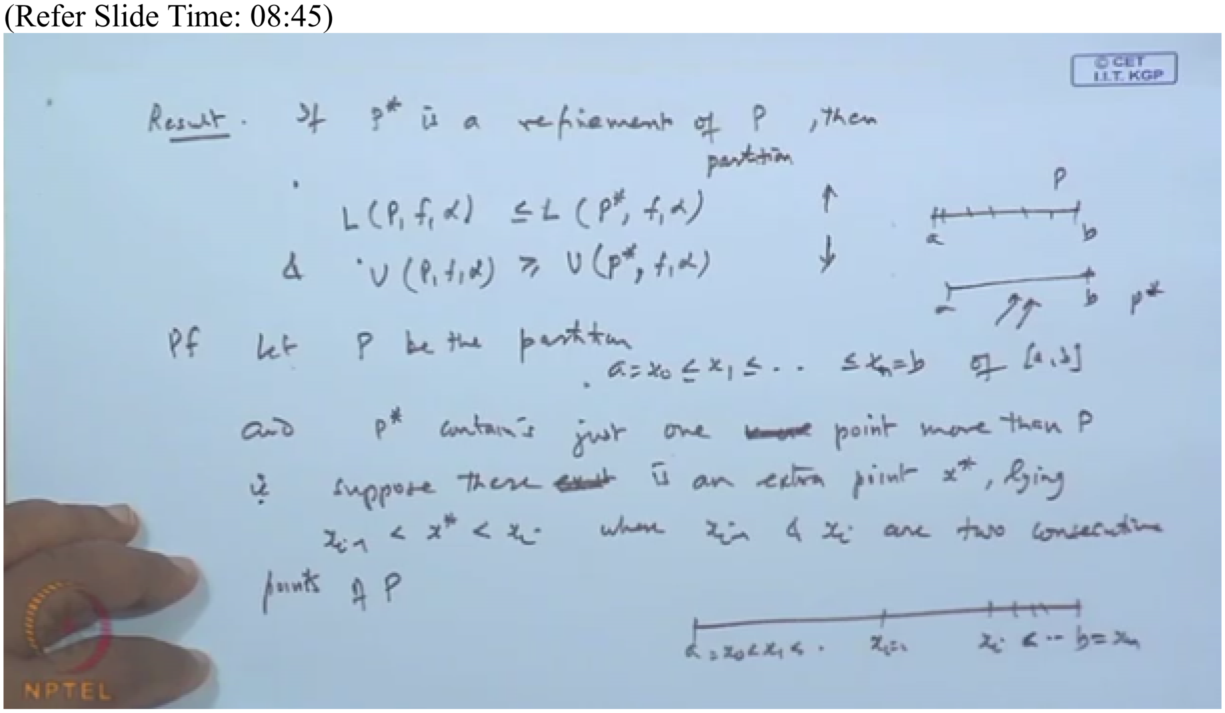

now P we are just increasing one more point say here X star, so this new partition becomes P star and it is the refinement of P, so clearly P star is the refinement of P. Others I am not changing only in 1 in sub interval I am taking one extra point that's all,
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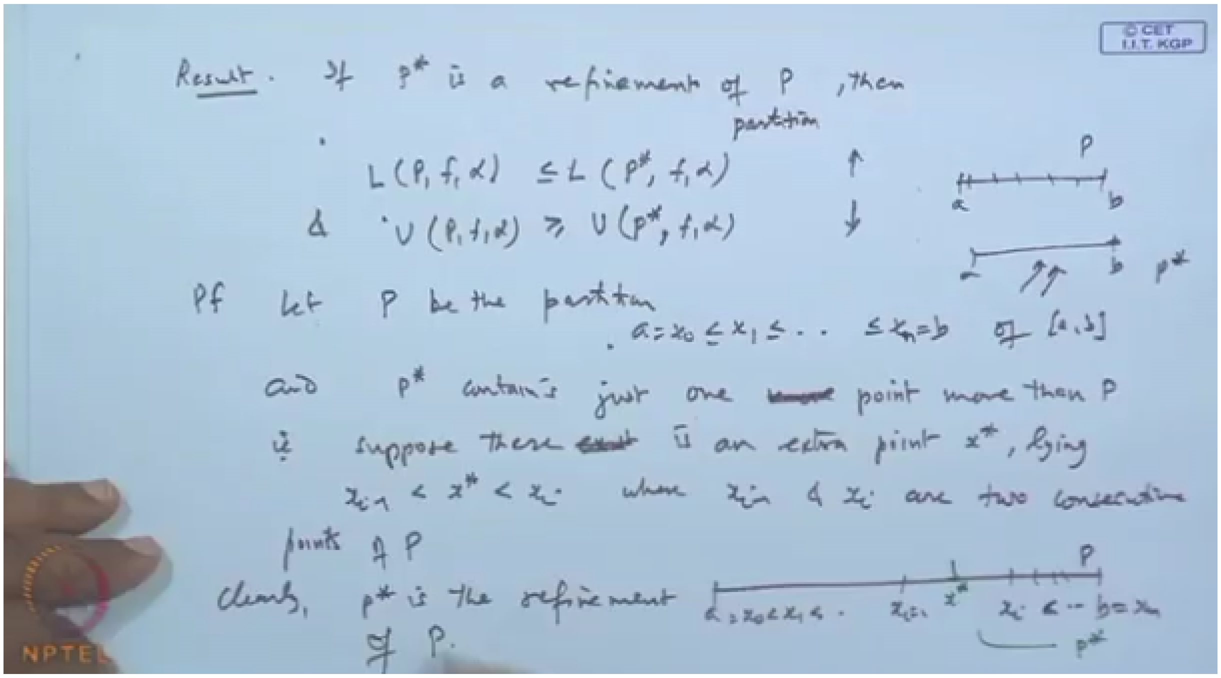

so it becomes a refinement of this.

Now with this point X star what we want to claim that lower sum will increase and upper sum decrease, so let us prove first for the lower sum, okay, so let W1 is the infimum of the function F(x) over the interval XI-1 to X star where the W2 is the infimum value of the function F(x) when X lying between X star to XI, okay, and MI is the infimum value of the function F(x), when X is varying over XI-1 to XI, so clearly this W1 will be greater than equal to MI, and W2 will also be greater than equal to MI, because this MI is taken over the whole interval XI-1 to XI as the infimum value is ,taken and then these are the infimum value over, the partition of this subinterval, so obviously this infimum value maybe more, so if W1 is greater than equal to MI, W2 is greater than equal to M to N, and hence consider the lowers sum with respect to the partition P star of the function F with respect to alpha minus the lower sum with respect to the partition P and F and alpha, so what we get is over this interval P star is the sum of the two intervals, this is our X star, so first you take over this and then this, so here the infimum value is denoted by W1, so it is the W1 and then value at a point alpha X star minus the value of the function alpha at the point XI-1 so this is the value of this lower sum over this hint, minus and then for so this will be multiplied by alpha, okay, W1 multiplied by this.

Then over the second interval W2, this is alpha XI-alpha X star so this is the lower sum of P, over the partition P star, okay this plus this, minus the lower sum of the partition function F with respect to the partition P so that is equal to MI into alpha XI - alpha XI-1 with respect to the alpha.

Now let's combine just, so W1-MI and within bracket you get alpha X star -alpha XI-1, is it not? Then + W2 - MI alpha XI - alpha X star, now W1 is a greater than equal to MI, W2 is greater than equal to MI so these two things are positive, alpha is a monotonic increasing function, X star is greater than or equal to XI-1, so value of X star at alpha at X star will be more than value of this one.

Similarly here XI is greater than equal to X star so this will be positive, so it is greater than equal to 0,

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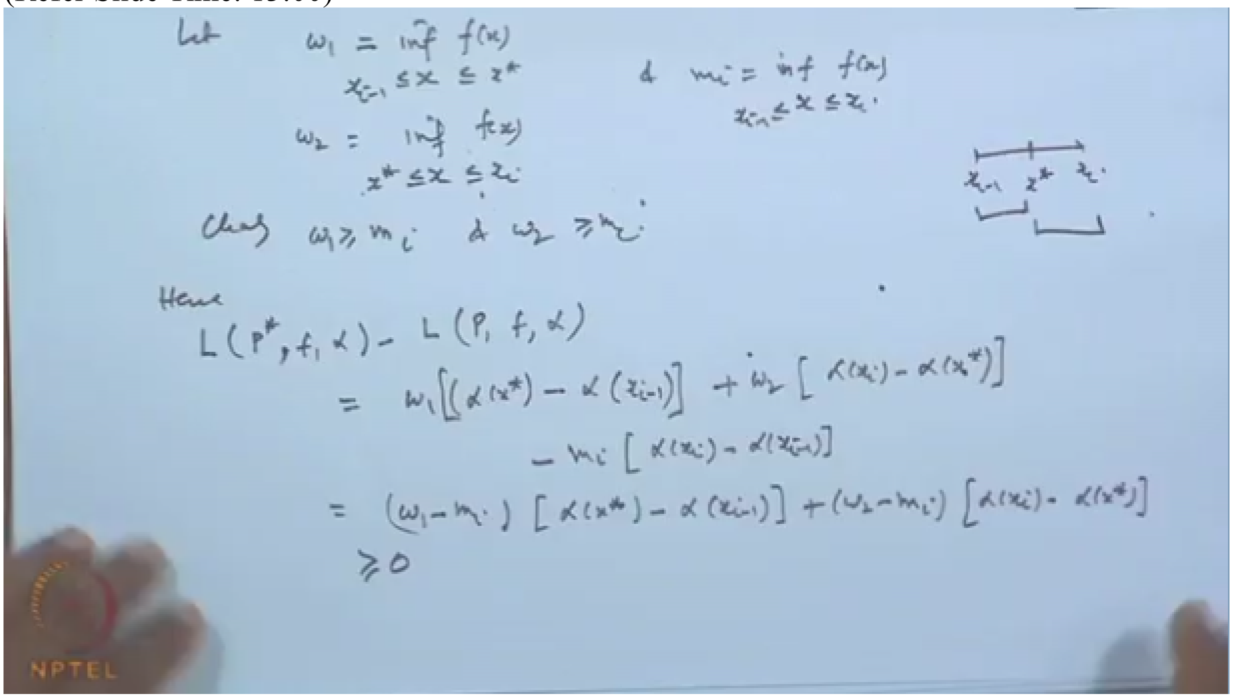

so what it shows that when we increased the partition when we get the refinement of the partition by introducing the points, more point then lower sum increases and that proof the first part of this.

Similarly so this proves that lower sum of the function F with respect to the partition P is less than equal to lower sum of the function F with respect to the partition P star which is the refinement of this increases, similarly upper sum we can so, that P, F, alpha is greater than equal to upper sum of P star, F alpha, so this is what we proved, okay, so this is what, okay.

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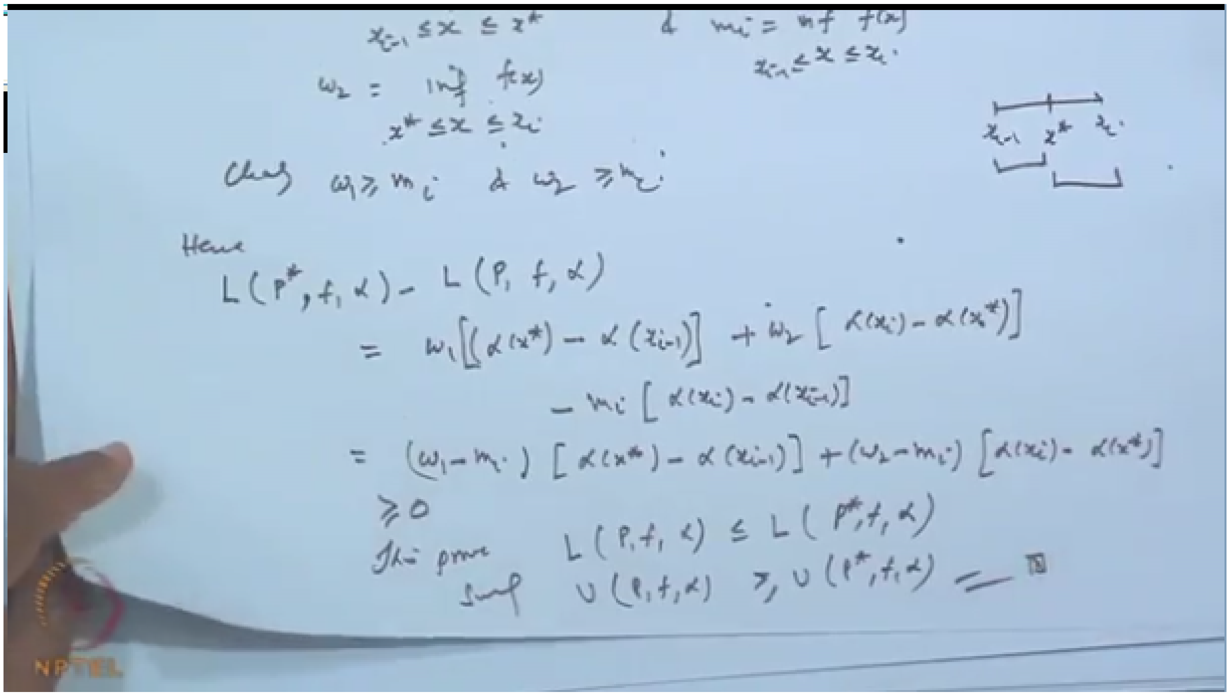

Now we come to property which is related to using this partitioning interval, and property of this I mean lower sum integral and upper integral, so this we put it as a form of theorem, what the theorem says is the lower integral of the function F and lower Riemann Steiltjes integral of a function F over the interval A, B will always be less than equal to the upper Riemann Steiltjes integral of the function F with respect to alpha over A, B, so obviously this result is also valid for a Riemann integral because when alpha X becomes X, then lower Riemann integral is always be less than equal to upper Riemann integral, okay the proof of this.

Let's see the proof, let P star be the common refinement of two partitions P1 and P2, okay, so let us take the partition P, so let P star is the union of P1 and P2, so obviously, so P star = P1 union, because it is a common refinement, so we can take P star is the P1 union P2, so it is the refinement for P1 as well as refinement for P2, okay.

Now since we have already proved that when you have a refinement of a partition then lower sum increasing and upper sum decreasing, so using this one we get, so we get the lower sum of the function F with respect to the partition P1, over the partition P1 and with respect to alpha is less than equal to the lower sum of the function F over the refinement P star with respect to alpha, because P star is the refinement so lower sum increases, but this is always be less than equal to upper sum, because lower sum is always be less than equal to upper sum with respect to the same partition, however we will prove it this is also true for a general, in general lower sum will always be less than equal to upper sum, whatever the partition is choose that we will show it next.

And F, and then alpha, and then now upper sum decreases, upper sum decreases means it is less than equal to P2, F, alpha, because P star is the refinement, so obviously the upper sum respect to P star is less than or equal to the upper sum of this, okay. So from here what we conclude? This implies that the lower sum with respect to the partition P1, F, alpha is always be less than equal to the upper sum with respect to the partition P2, F, alpha, it means if we take any two arbitrary partition P1 and P2 then always, and function F is fixed, alpha is fixed then lower sum of that function F with respect to alpha will always be less than equal to the upper sum of the function F with respect to alpha whatever the partition is choose,

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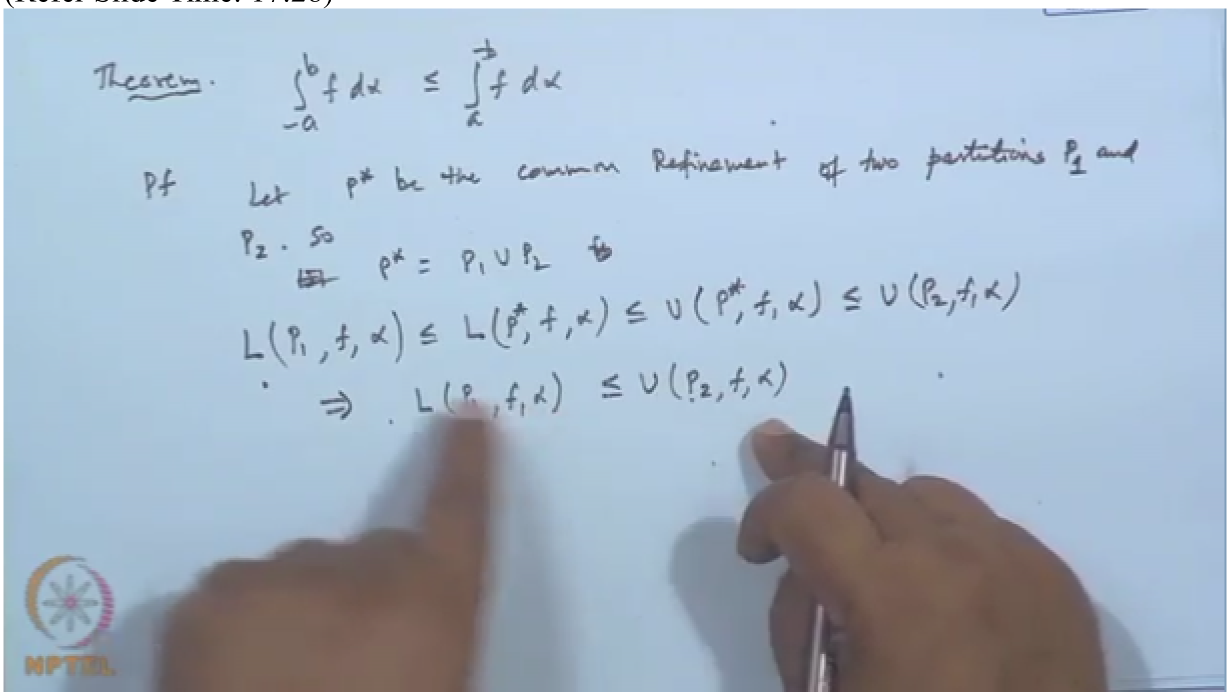

for a same partition this is true but for in arbitrary partition also we have shown that this lower sum is always be less than equal to upper sum, is it okay?

Now what we do is here let us fix up the P2, fix partition P2, okay, and let take the supremum value, supremum is taken over, supremum is taken over all P2, let us seek first that, or assume is all P1, this I am fixing and here I am taking the supreme over P1, so when you take the supremum of all P1 then this will give the what? It will give the lower sum so we get from here is A bar lower b, FD alpha, this is the lower Riemann integral, Steiltjes integral will give and then less than equal to U(P2, F, alpha), now again you take the infimum value, then in the right hand side take infimum over, supremum is taken over all P2, so we get from when you take the infimum then it will give the upper sum, we get from here is FD alpha is less than equal to upper sum of this P2 FD alpha, I mean this proves the result, okay, so this proves the result, so it is in test results, okay.

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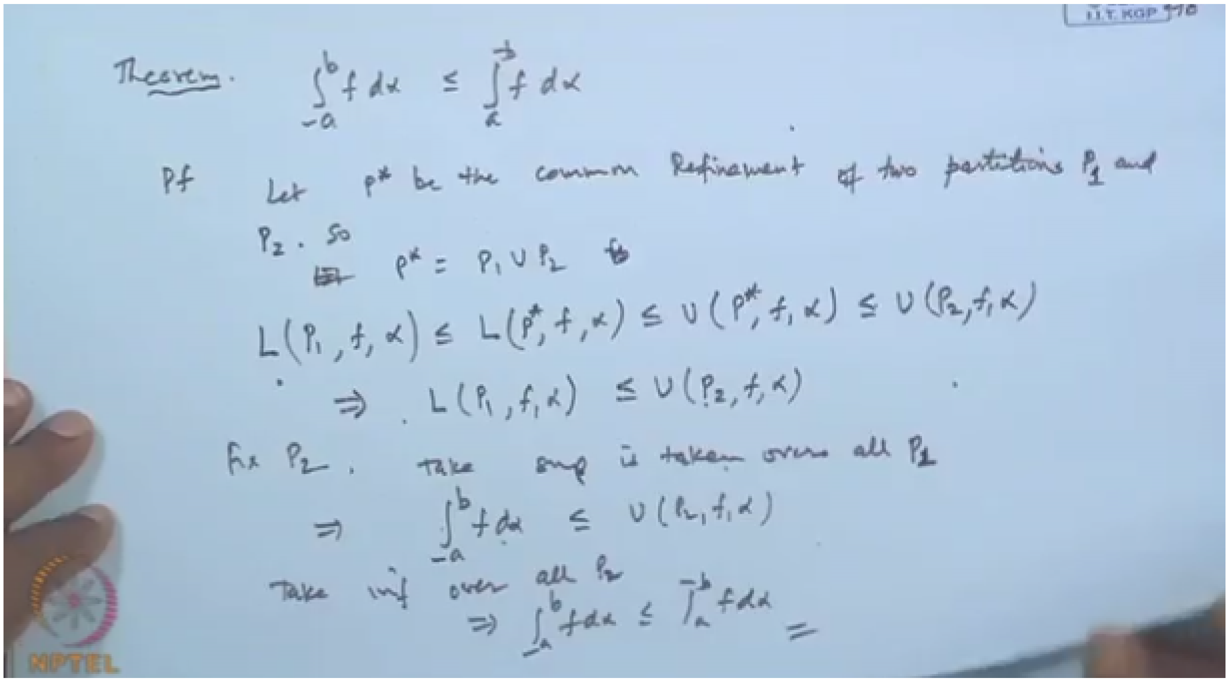

Now the question arise is, as we have seen in the first page that what is the guarantee if F is a bounded function, okay we have defined the Riemann integral, we have defined the Riemann Steiltjes Integral, then what is the guarantee whether they are equal or not, because if they are not equal there are no point of us going further, so the existence of their integrals is important, then the both the integral coincide, both will have a same value, so under what condition both these integral lower integral and upper integral exists and have a equal value so that we can say F is Riemann integral function or F is Riemann Steiltjes integral, so this theorem gives a little bit about the existing part of this, existence of Riemann Steiltjes or Riemann integral, because both I'm dealing at a time on Riemann integral, okay, that's what.

The theorem is let F belongs to the Riemann Steiltjes integral with respect to alpha on A, B, okay, then we say F is Riemann Steiltjes integral or let remove it, F is Riemann Steiltjes integral function over the interval A, B with respect to alpha, if and only if for every epsilon greater than 0 there exists a partition P such that the upper sum P, F, alpha, upper Riemann sum - upper Riemann Steiltjes lower sum L(P, F, alpha) is less than epsilon, so this condition is the necessary as well as sufficient for the existence of a function to be Riemann Steiltjes integral,
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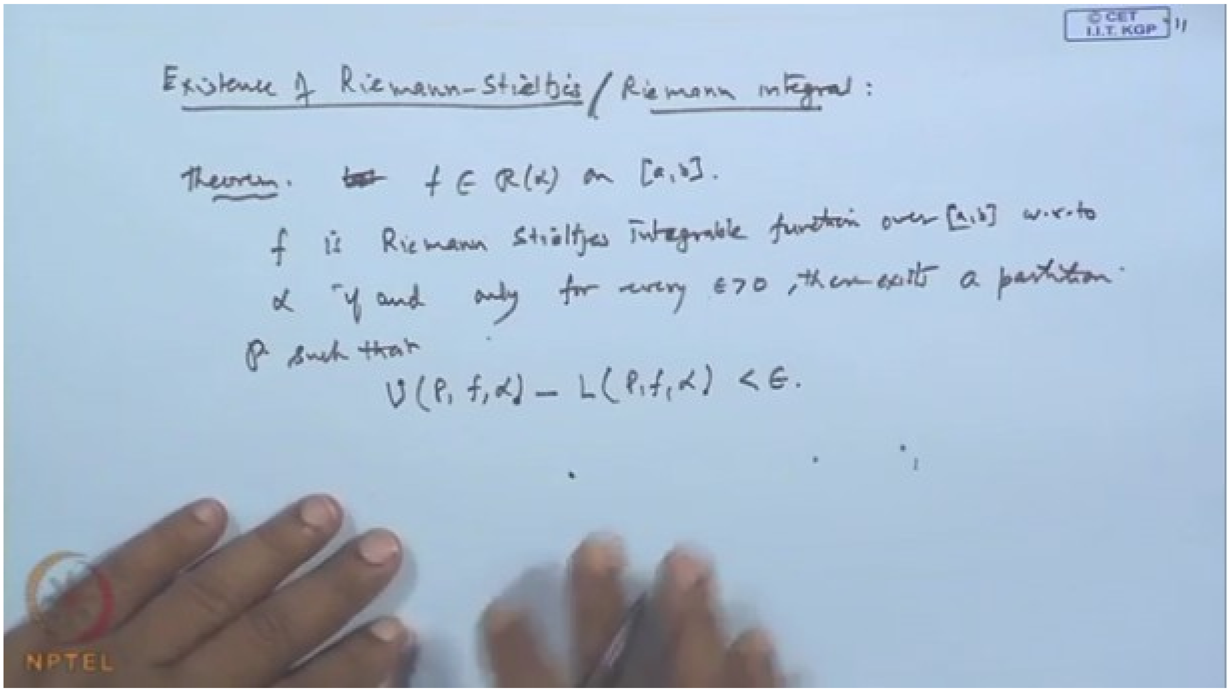

so in particular when we take alpha X = X, so we can say as a particular or as a corollary we can say F belongs to the Riemann integral on A, B, if and only if for every epsilon greater than 0 there exists a partition P such that the upper sum of the function F with respect to, partition P -lower sum of A, B respect to partition P is less than epsilon, so okay,

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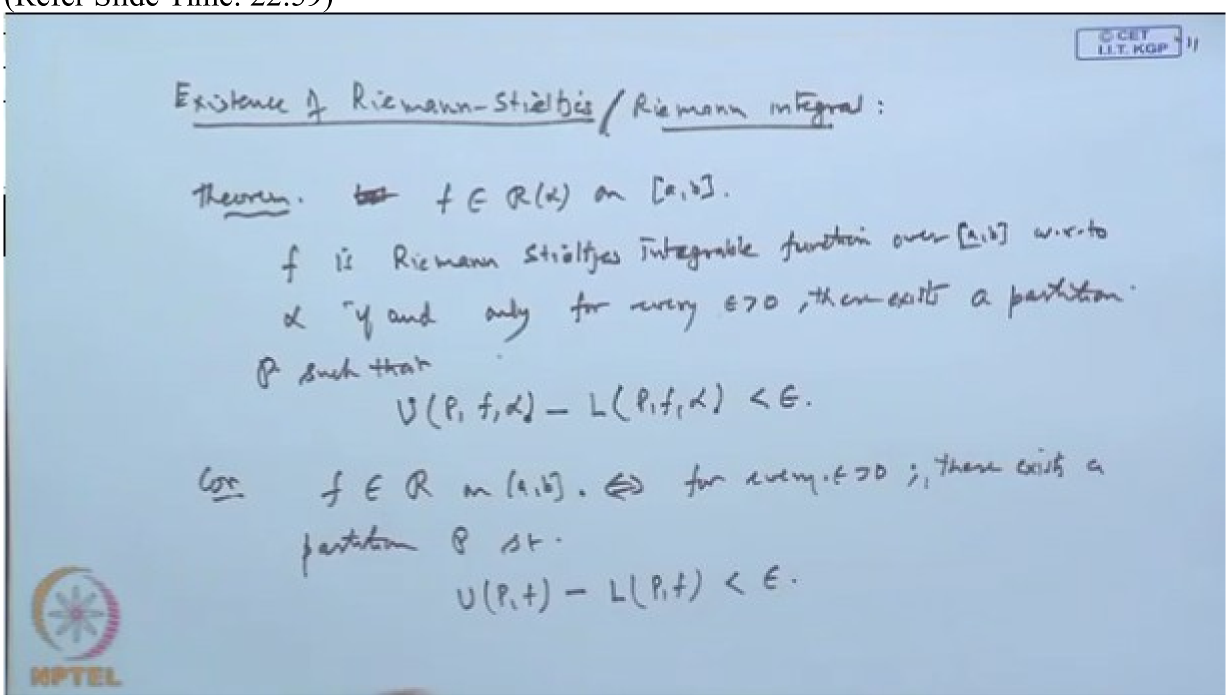

so this is for the Riemann necessary and sufficient condition for Riemann integral, this is for necessary and sufficient condition for Riemann Steiltjes, okay. Now we will see the proof next time. Thank you very much, okay. That's all.