NPTEL

NPTEL ONLINE CERTIFICATION COURSE

Course

On

Introductory Course in Real Analysis

By

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Lecture 66: Tutorial XI

Okay so this is the continuation of my tutorial classes, so today we will discuss the problems based on the increasing decreasing functions, as well as that some Maclaurin series expansion or Taylor series expansion, so this is the tutorial 11.

So let's take one example, first show that the function F defined, the function F defined as F(x) =X cube -6X square + 12X – 4, when X belongs to the real number is increasing function in every interval of real line, whatever may be the interval is real.

So let's see the solution, here we will make use of the Lagrange's Mean Value theorem, what the Lagrange's Mean Value theorem says? That if the function Lagrange's Mean Value theorem says that suppose a function F is a continuous function over the closed interval say A, B differentiable on an open interval A, B, then there exists a point C in the interval such that F(b) –F(a)/B-A is the derivative of the function at a point C, where the C lies between A and B.

Now if I take the two point X2 and X1 in the interval A, B, in the interval say A, B where X1 is less than X2 and over the interval X1, X2
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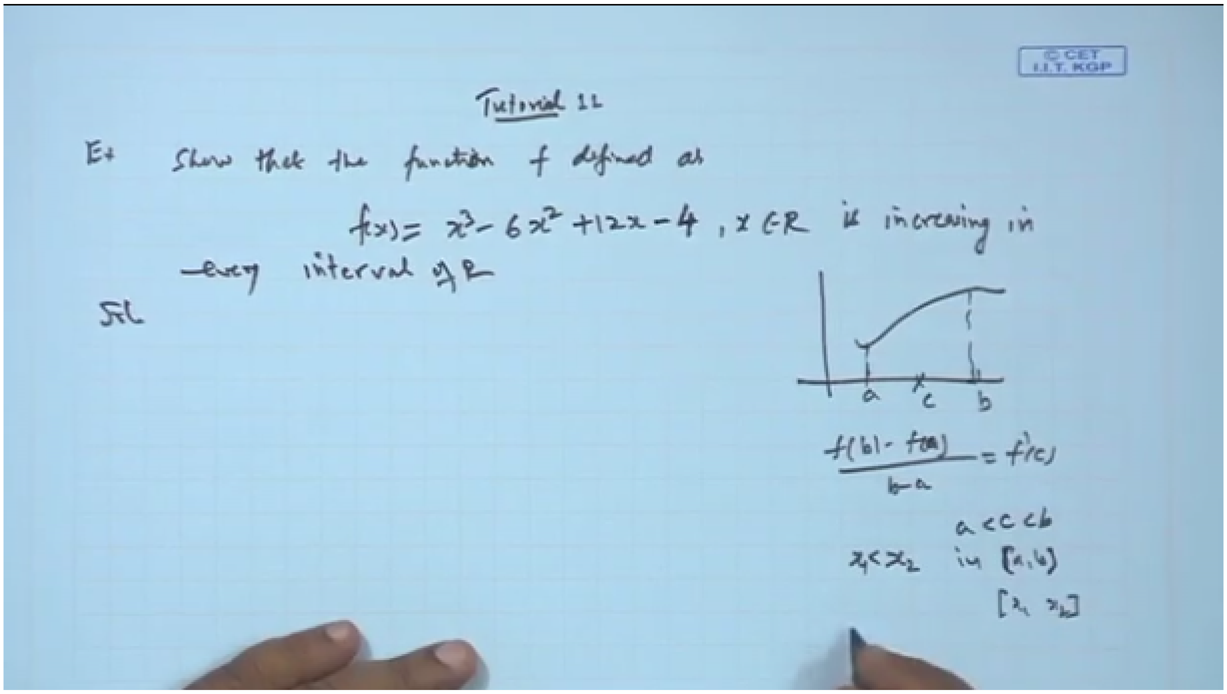

if we apply the Mean Value theorem then we get F(x2) –F(x1) divided by X2-X1 is nothing but the derivative of the function at certain point xi where the xi lies between X1 and X2.

Now from here it implies that the F(x2) –F(x1) = X2-X1 into F prime (xi). Now if our F prime(xi) is positive, then X2 is already greater than X1, so right hand side is positive, therefore F is an increasing function, F(x2) is greater than F(x1) when X1 is less than X2, so F is increasing, F is an increasing function, so this is the Lagrange's Mean Value theorem,

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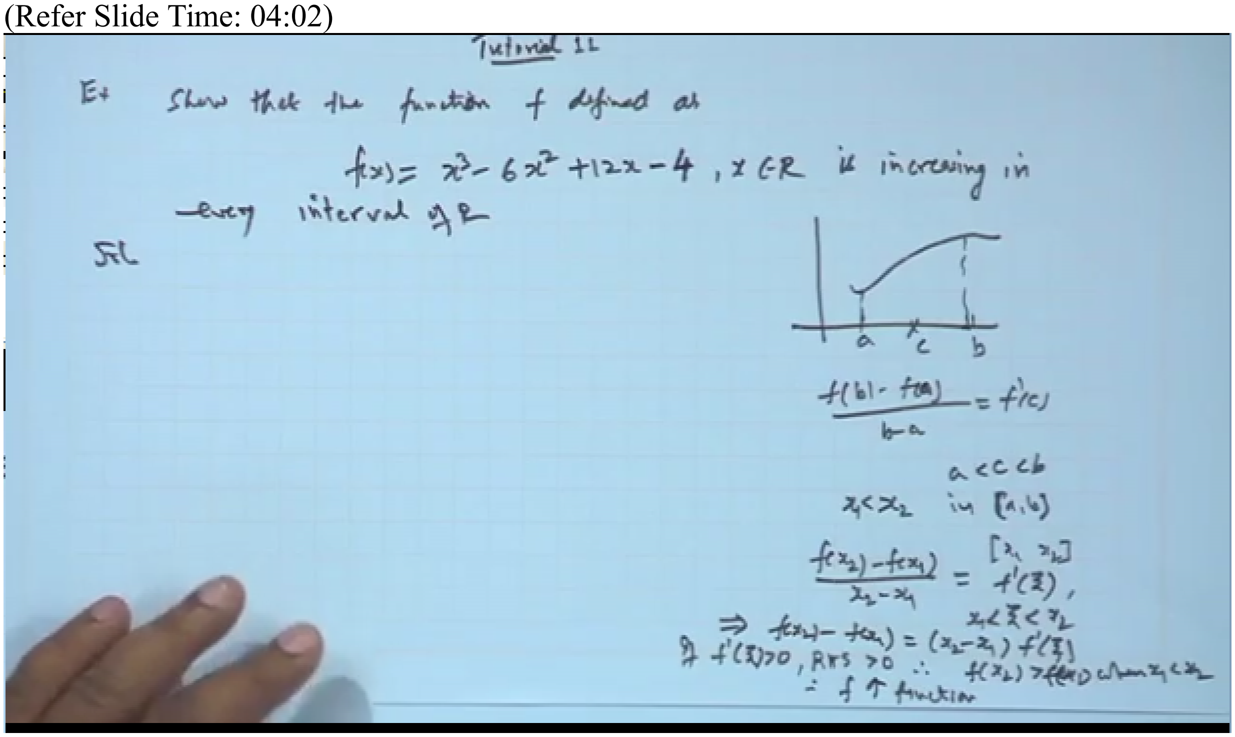

we will use this result to establish that this function is an increasing function, so what we have to do is we have to differentiate the function F with respect to X and see the behavior of F dash(x) over the any arbitrary interval of R, whether it is strictly positive or strictly negative or maybe the oscillating, so if we look the function F dash and differentiate that the derivative will come out to 3X square - 12X + 12, and that is the same as 3X square - 4X + 4 that is equal to X-2 whole square into 3.

Now whatever the X may be, X is any arbitrary interval A, B of R this quantity will always be positive, therefore the derivative of the function is positive, hence by Lagrange's Mean Value theorem we can say that the function F(x) is an increasing function over any interval A, B of R and that completes the proof because of this discussion which we have made it, okay so this one.
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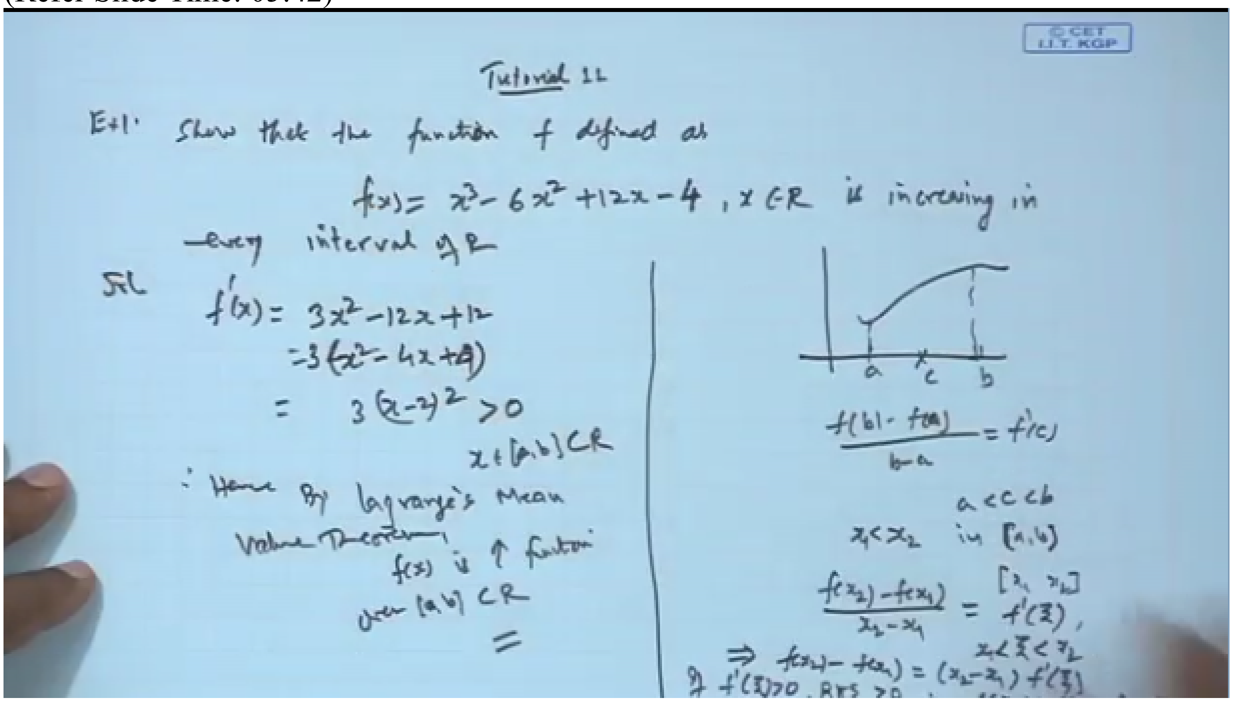

Second question is also in the similar lines but slightly different, okay, the question second says find the limit of this, find limit of X tends to 0, 1+X to the power 1/X - E divided by X, now this result if we look it waste on the L Hospital rule, why? Because when X is 0 limiting value of 1+, limit of 1+X raised to the power 1/X when X tends to 0 is always be E, this is a standard result, so when you take the limiting value or when you take the limit or the bulk numerator and denominator separately it will take the form 0/0, and 0/0 is an indeterminate form,
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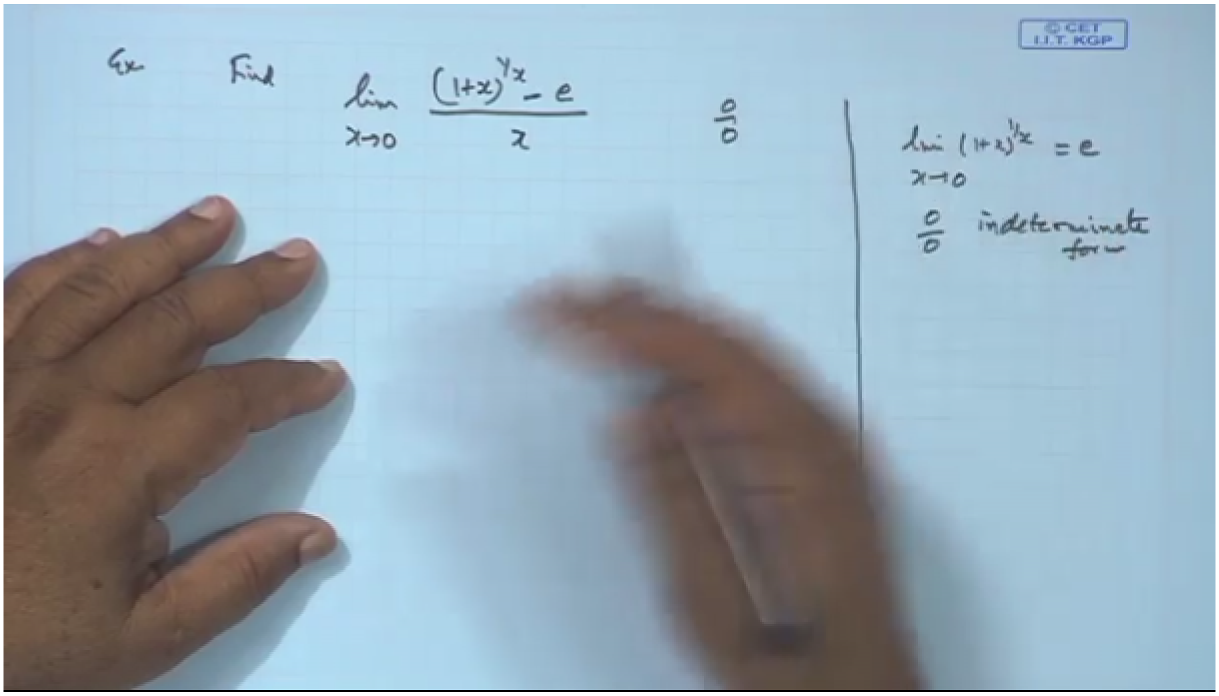

so we cannot say the value is 0 or value is 1 or like this, it depends on the behavior of the function when X tends to 0, so in order to solve such a problem we used the L Hospitals rule, again the L Hospital rule is a consequence of our Lagrange's Mean Value theorem, we are not worrying we have discussed this in the theory part, what L Hospital rule says that in order to find the limit of the function F(x)/G(x) when X tends to say A, we differentiate the numerator and denominator separately and take the limit when X tends to A, so if this is of the form 0/0,

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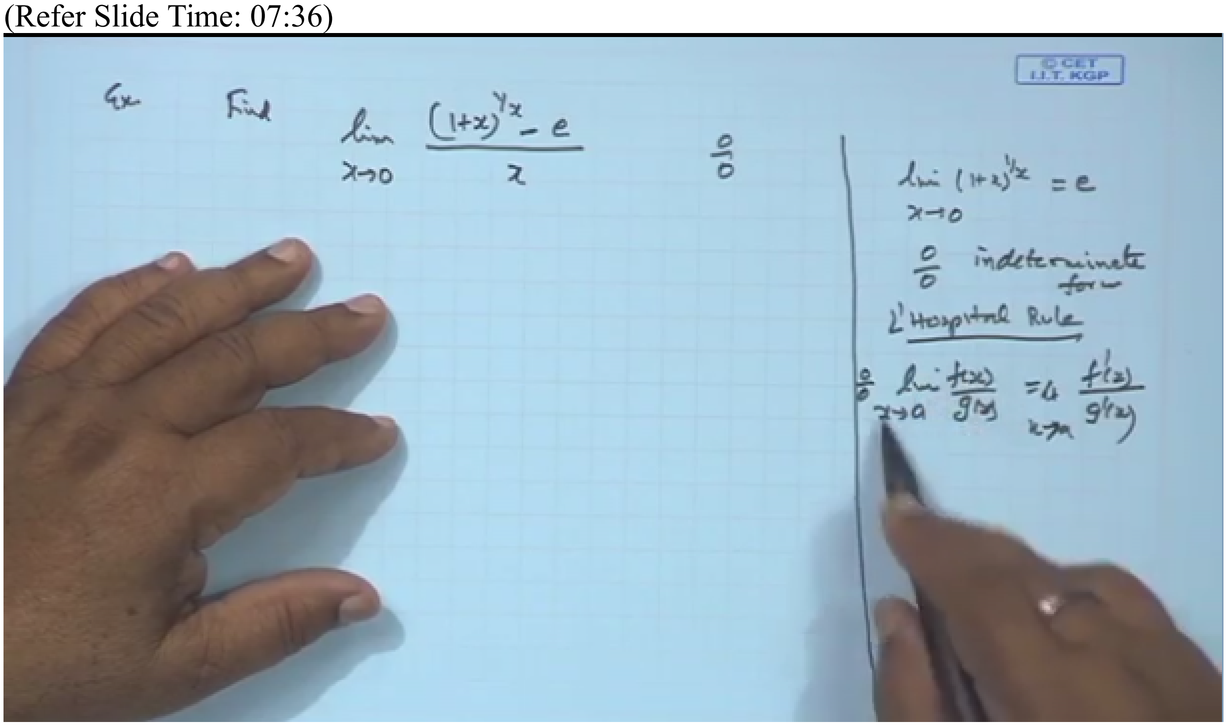

then in order to find the limiting value of this ratio we differentiate the numerator with respect to X, denominator with respect to individually separately and then substitute X = A.

Now if this limit exists, if the right hand side limit exists then we say the left-hand limit will exist, and basically an equal to the value, limit of writing, limit of F prime(x)/G prime(x) when X tends to A, so this is known as the L Hospital, so here it is of the 0/0 form hence to find the limit of this apply the L Hospital rule and differentiate numerator and denominator separately, so this is the same as D/DX 1+X raised to the power 1/X - of course, -E divided by derivative of this X and then take the limit as X tends to 0.

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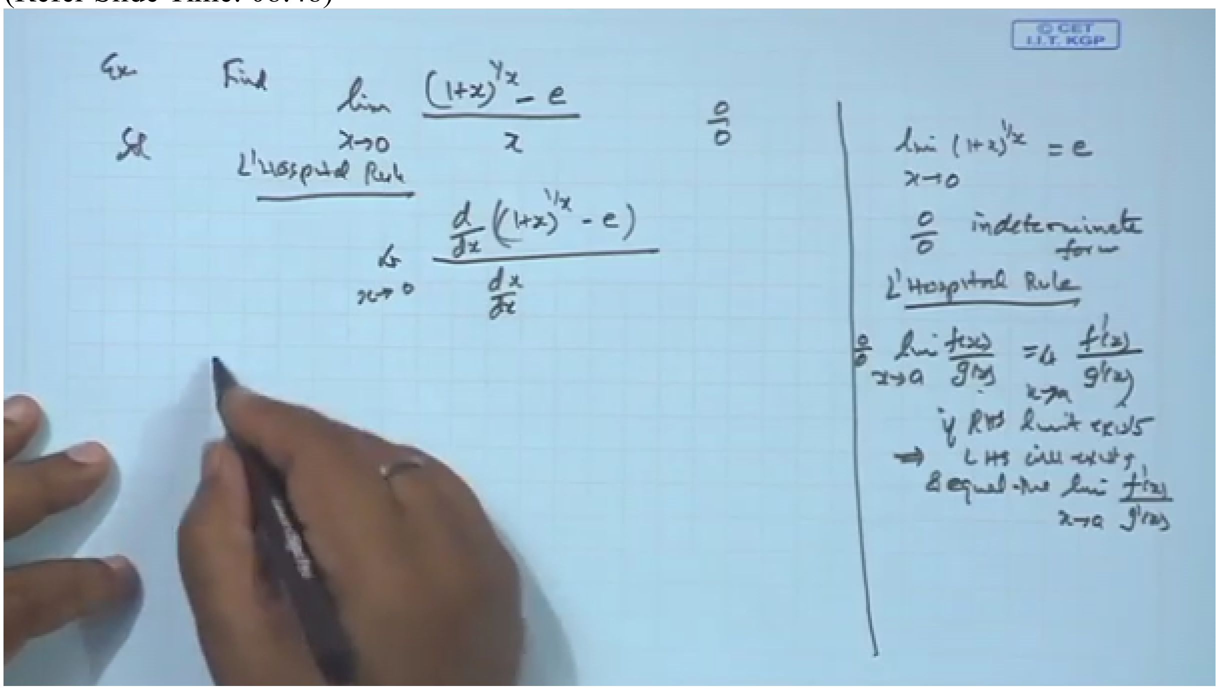

To find the derivative of this we get let Y = 1+X raised to the power 1/X, so take the log by becomes log of 1+X/X and then differentiate it with respect to X, so when you differentiate with respect to X this is 1/Y DY/DX this is denominator into derivative of the numerator - numerator into derivative of the denominator divided by X square, so basically DY/DX becomes by, by means 1+X to the power 1/X and then this will be equal to X-1+X log 1+X divided by X square 1+X, so this is given,

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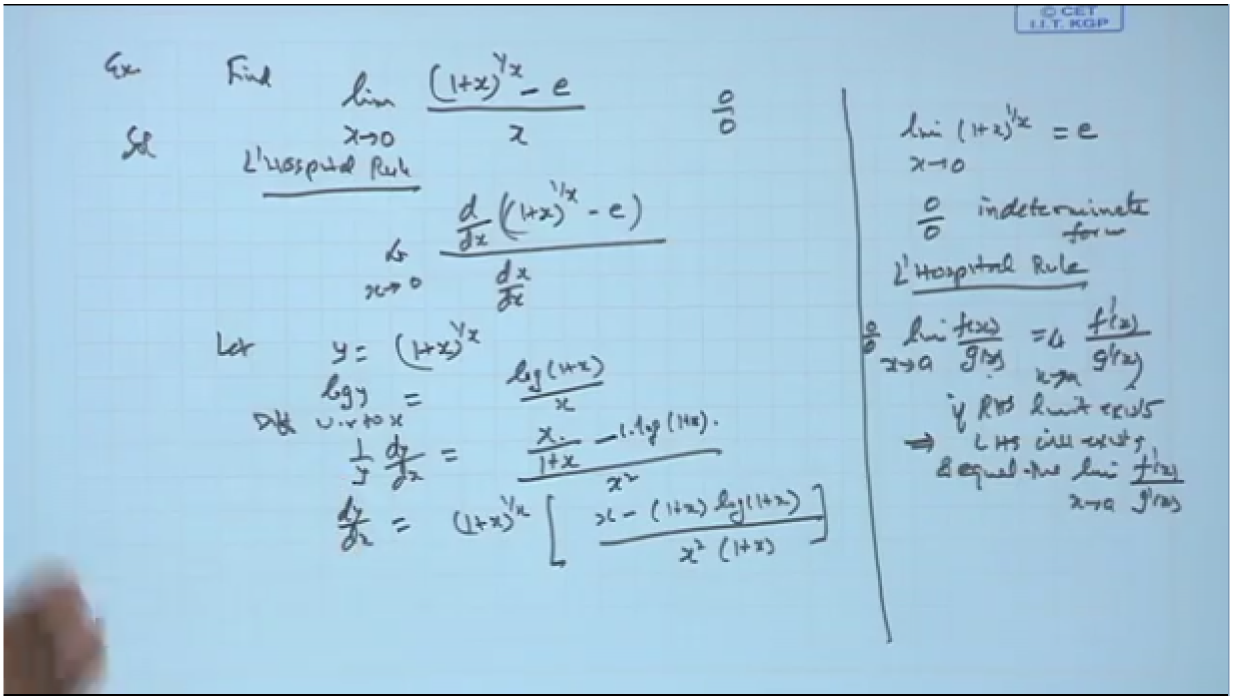

so this limit is suppose L then we get L = limit X tends to 0 1+X raised to the power 1/X into X- (1+X) log of 1+X and divided by X square 1+X, now limit of this E so we can simply break,

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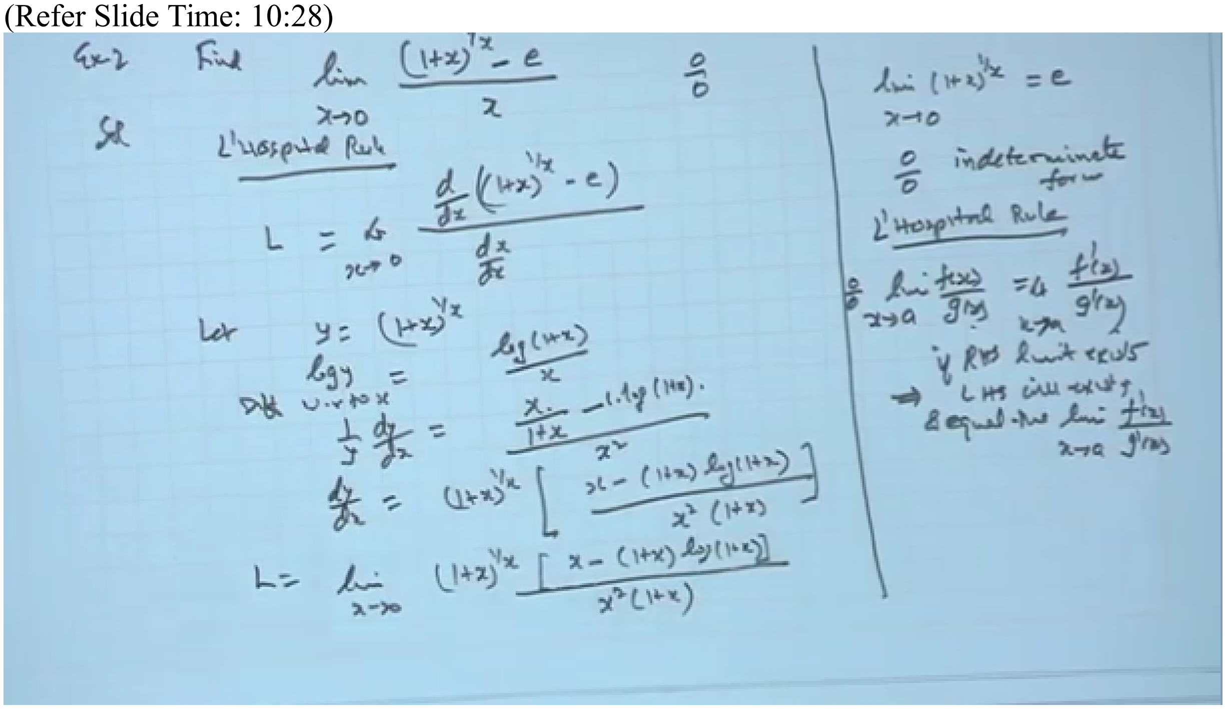

write down this thing H limit 1+X raised to the power 1/X as X tends to 0 into limit, as X tends to 0 X- (1+X) log (1+X) divided by X square (1 +X), now the limit of this E, and this limit is of the form 0/0, X tends to 0 so it is 0/0, so again apply the Hospital rule, so L Hospital rule says differentiate the numerator and denominator separately so we get 1 – (1+X) divided by 1+X - log(1+X) and this is equal to X square, so X square means 2X + X cube 3X square, this is okay, so differentiate this and then take the limit as X tends to 0, so this will get cancel and we get

from L is E minus, this is minus, first function derivatives for a second function, derivative of this and then this is,
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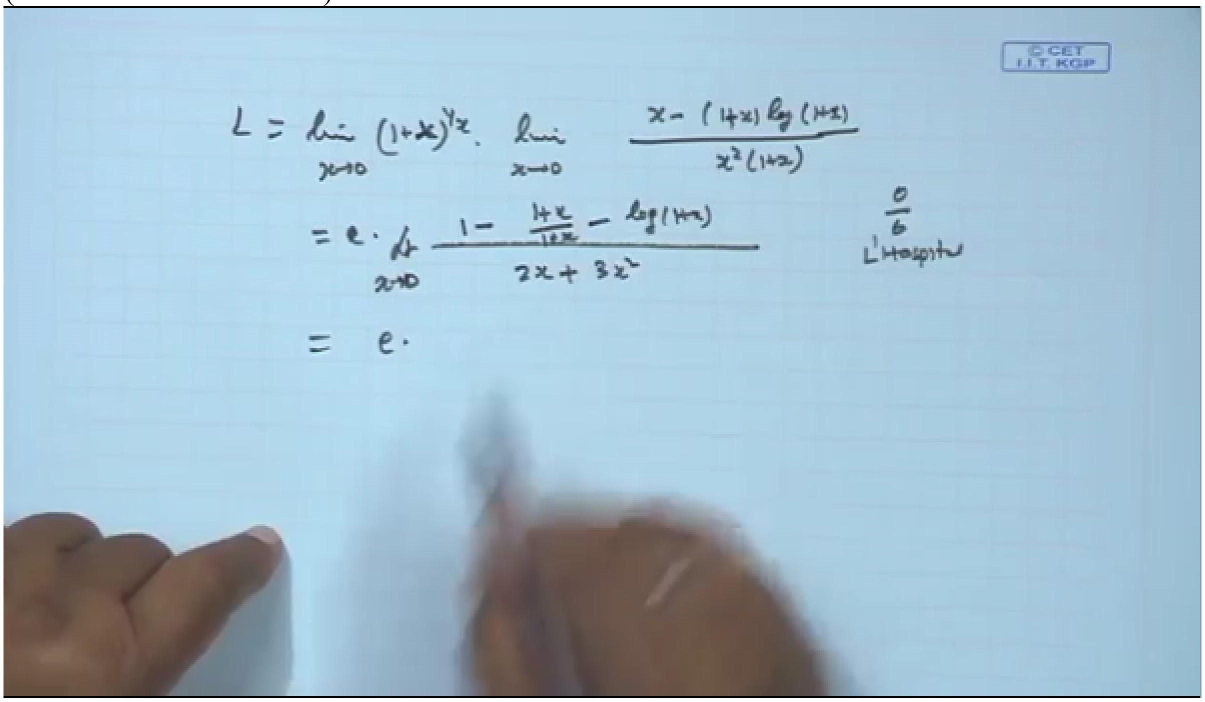

so here we are getting minus outside limit X tends to 0 log(1+x) 2X + 3X square where again when you substitute X = 0 is again the indeterminate case, so again apply the L Hospitals rule and when you get the L Hospital rule you are getting E, differentiate this 1+X 2+6X, limit X tends to 0 and that is the answer is –E/2, so answer will be that is –E/2, okay so that will be equal to answer, clear?

So we can repeatedly we can apply the L Hospital rule to get the limiting behavior of this, now same example,
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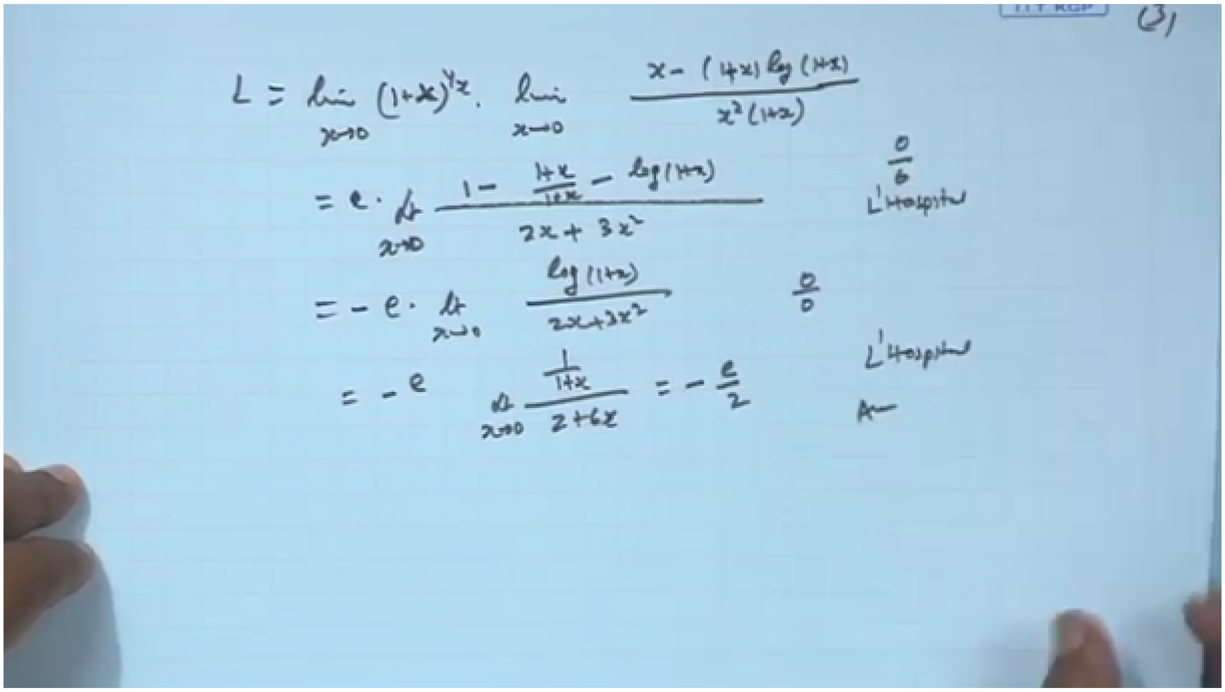

another example based on this again Hospital rule evaluate limit X tends to 0, 1/X –cot X, now this is of the form infinity – infinity, so again it is an indeterminate form, so in order to solve it we have to apply the L Hospital's rule but when we apply the L Hospital's rule there is no

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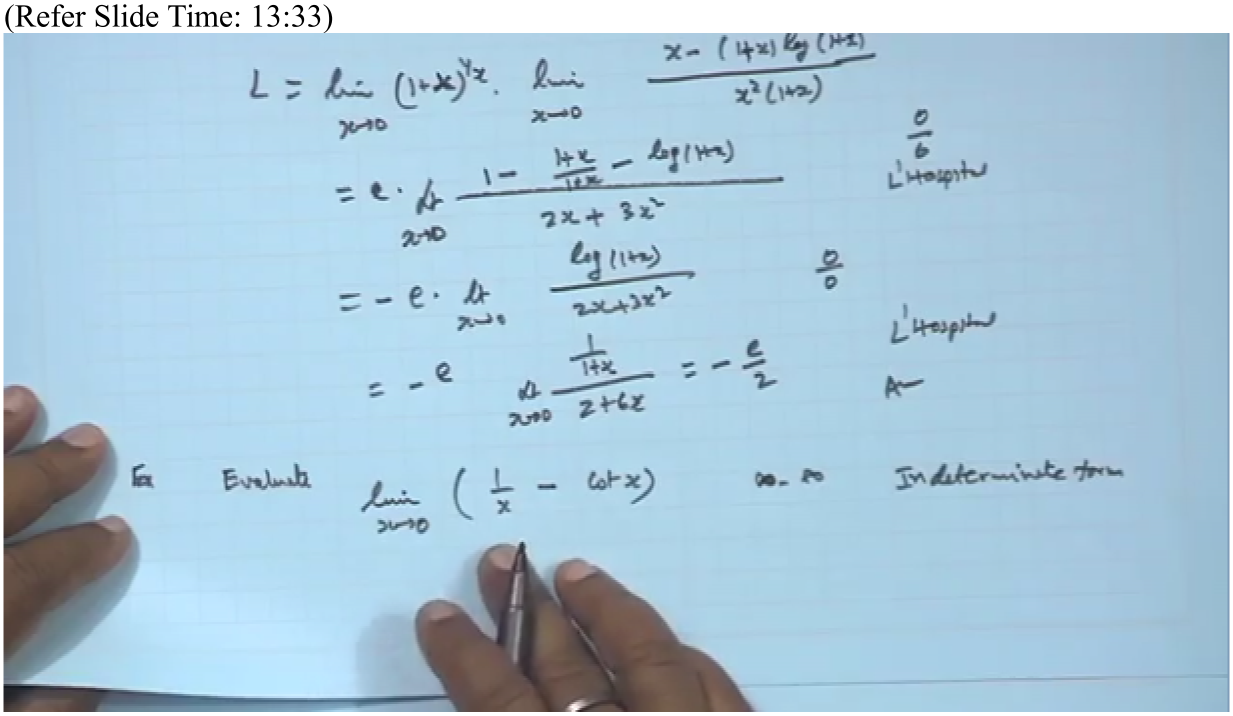

numerator denominator so we will try to put it in the form of 0/0, and this can be written as cot, means cos by sine, so we can say sine X – X cos X divided by X sine X, and limit X tends to 0, so let this limit is L like this, now it is 0/0 form, apply the L Hospital's rule, so when you apply the L Hospital's rule the value differentiate the numerator and denominator separately and we get from here is L = limit X tends to 0, the derivative of sine X is cos X - cosine X - X into cos

is sine X with minus sign divided by first function sine X derivative X and then + X cosine X, so when you take the value X tends to 0 it is basically 0/0 again, so further we apply this result we get L Hospital rule and we get first is X cosine X + sine X divided by cosine X + X sine X with minus sign and then + cosine X, so again X tends to 0 this part is 0, this is 0, this is 1, so 1+1, 2 and 0 so basically the limiting value of this when X tends to 0 is 0, so that is what this get L becomes 0, so that's the answer for this question.
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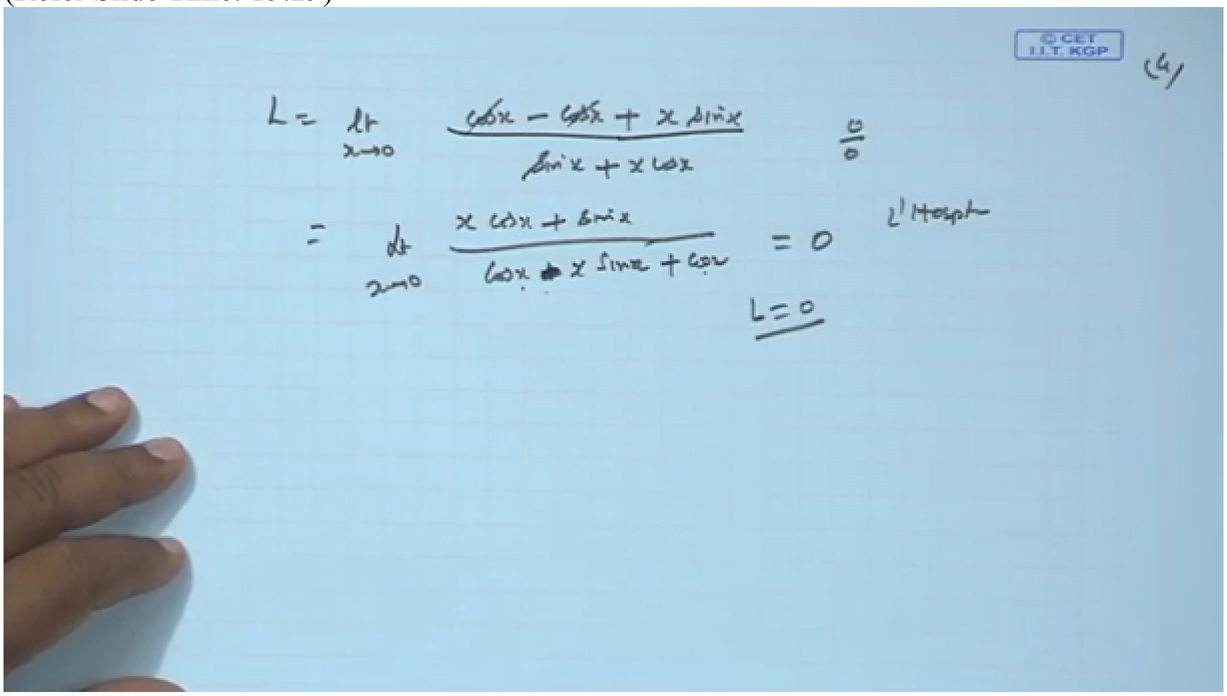

Exercise 4, say evaluate this limit, limit X tends to 0, sine X/X raised to the power 1/X, now we know the limit of this X tends to 0 sine X/X is 1, so basically this is of the form 1 to the power infinity, again it is an indeterminate form, so in order to prove, well find the value of this, we have to use the L Hospital rule and then bring it either in the form of 0/0 or infinity/infinity, now this can be done as follows, let Y is sine X/X to the power 1/X, then what will be the log of Y? Log of Y will be log of sine X divided by X and this is log of X, 1/X into log.

Now if we take the limit of this as Y tends to 0, log of Y this is the same as limit X tends to 0, sorry limit X tends to 0 is the same as limit log of limit X tends to 0 of this term, is it not? Log of Y means log of this part,
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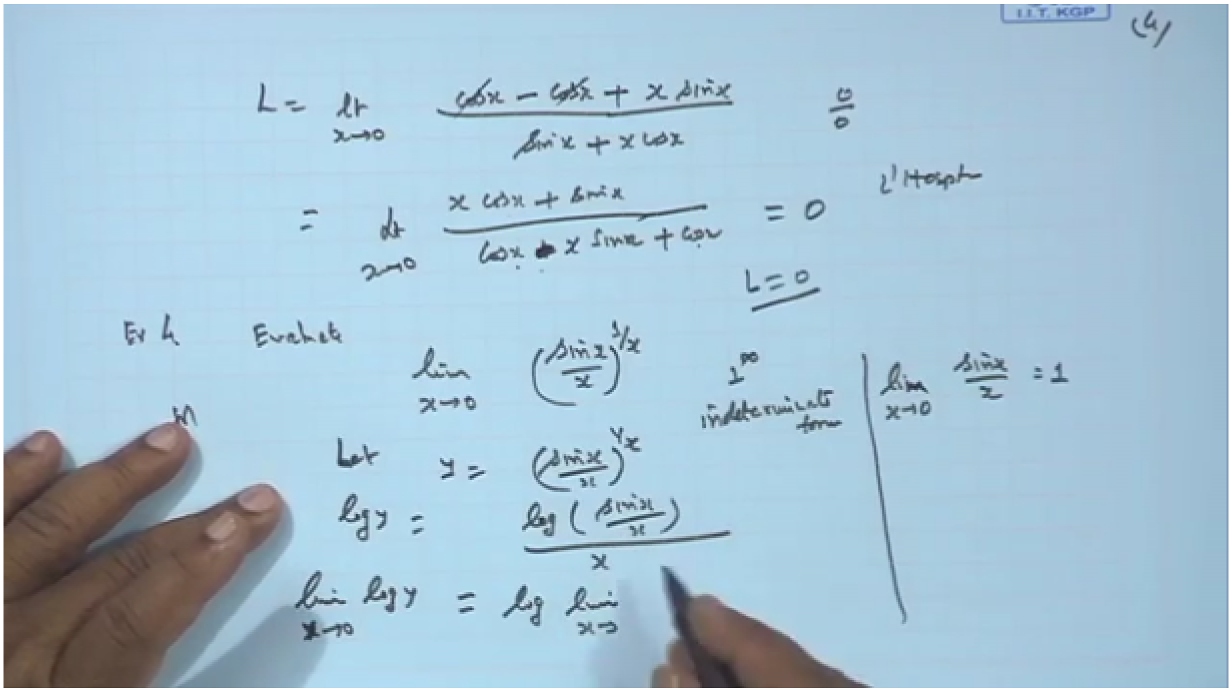

so limit of this part sine X/X or you can say limit of log sine X/X divided by X this one.

Now when X tends to 0 sine X/X is 1, so numerator is 0, denominator is also 0, so it is basically 0/0 form, therefore to apply the L Hospital rule again use that, so to find the limit apply the L hospital rule

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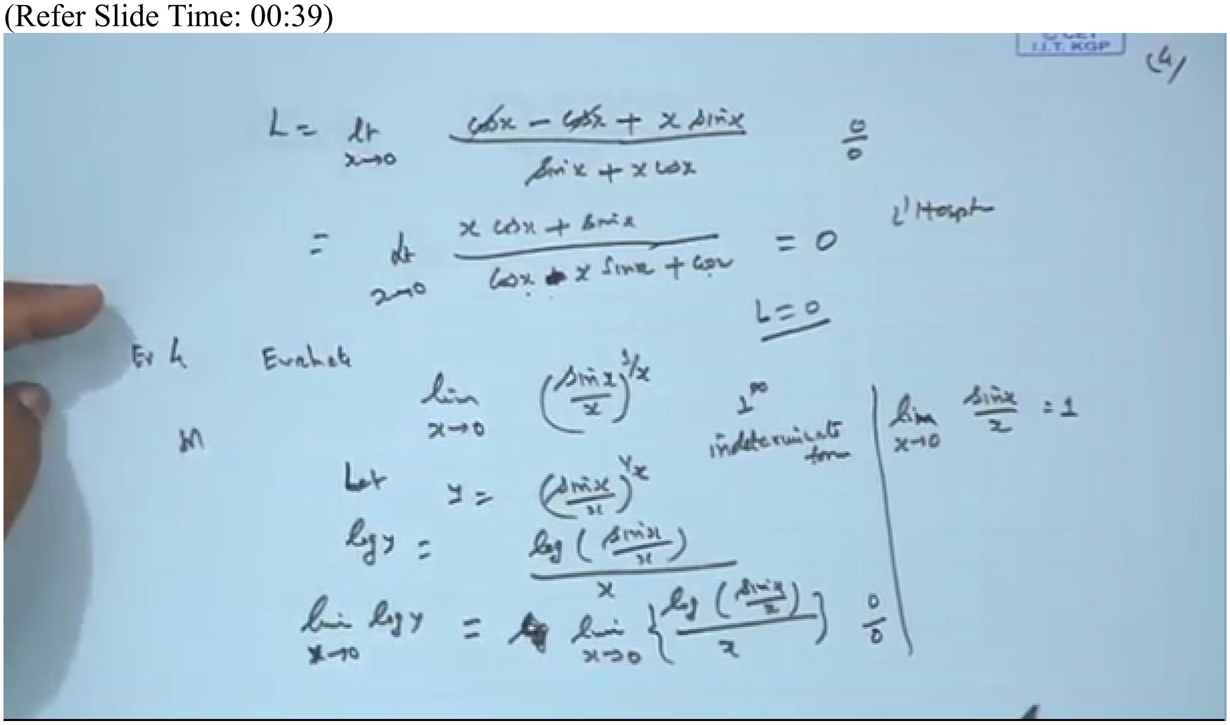

and the L hospital rule says differentiate numerator and denominator separately and then substitute X = 0, so we get limit X tends to 0, 1/X log sine X/X, and this is 0/0, so by L Hospital rule this is equal to log mean 1 upon this, so X/sine X into X cosine X over this, okay. And then derivative of X will be 1, so limit of this X tends to 0, okay.

Now this will be the same as limit of this X tends to X/sine X, when X tends to 0 into limit of this X tends to 0, X cosine X - sine X divided by X square, now this limit is 1,
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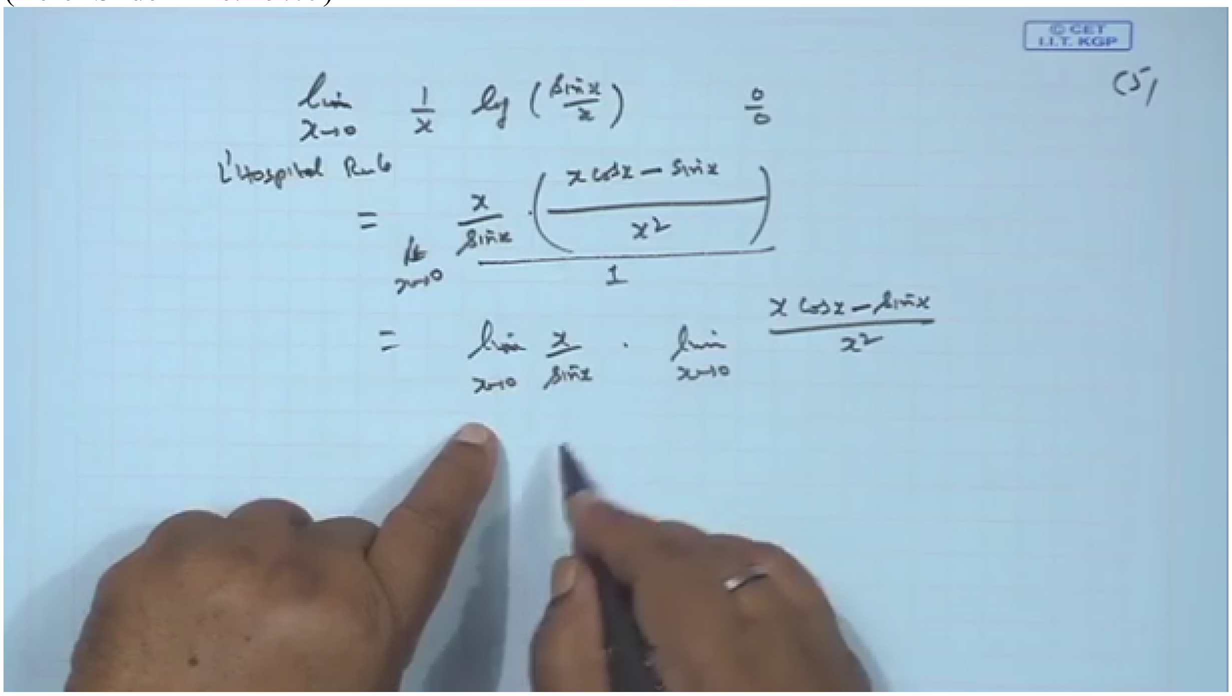

so we are not bothering about it, the second limit is limit X tends to 0, X cosine X - sine X/X square again it is 0/0, so apply again L Hospital rule and when you use the L Hospital rule differentiate numerator and denominator, so X of sine X with – sign, + cosine X - cosine X divided by 2X, so again this is of the form limit X tends to 0 - sine X/2, and that will be equal to 0, so this is what?
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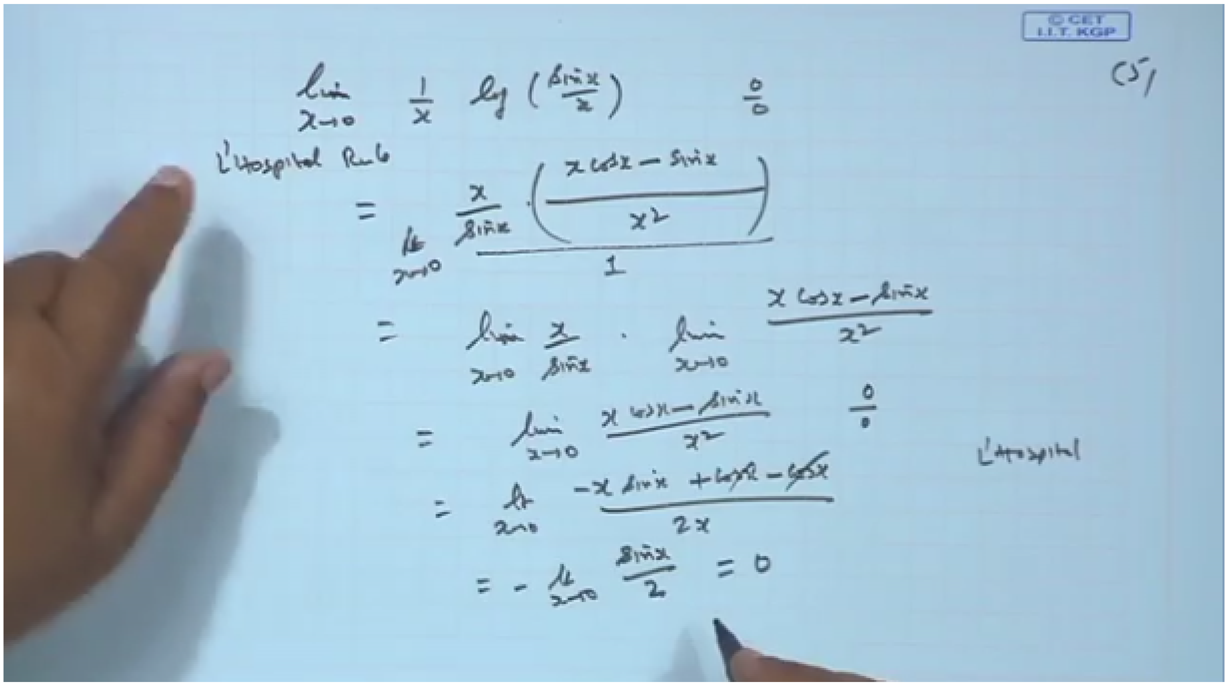

This limit is the limit of X tends to 0, log of Y, therefore the Y = E to the power 0 is 1, that will be the sub-coordinate, okay, clear?

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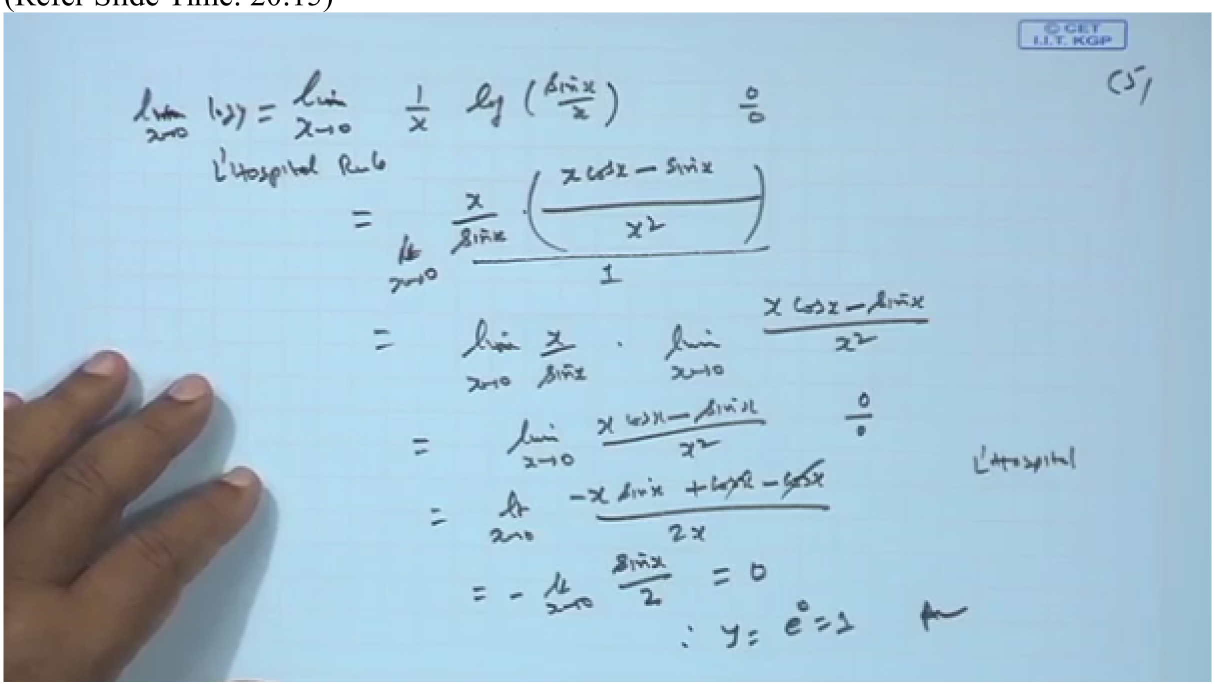

So these are few examples based on the L Hospital rule, and increasing decreasing function, now let's see the some examples where Taylor series and Maclaurin series, so find the Maclaurin series expansion of the function log 3+4X about the point X = 0, we know the Maclaurin series expansion, if the function F(x) is continuous then differentiable, define in the neighborhood of the 0, then one can expand this function F(x) as F(0) F prime(0) over factorial 1 +F double prime 0 over factorial 2X square, F prime(0) into X, okay, X and like this, so last term will be FN(0)/factorial N X to the power N so and so on, so this will go there, and up to here when you truncate this is corollary element terms and there, so this is it.

So basically the function F(x) is here log of 3+4X, so F(0) becomes log 3, what is the F prime (x)? F prime (x) becomes 3+4X and then multiply by 4, so F prime(0) becomes 4/3, and then F double dash(x) is -3+4X whole square and then 4X, so 4 square, so F double prime(0) becomes -4 square/3 square and continue, so if we see the behavior of the term we can write FN(x) in a similar way we can write -1 to the power N+1, -1 to the power N+1, 4 to the power N into factorial N-1 into 3+4X power –N, N is 1, 2, 3, so when it substitute N = 1 the derivative this F prime is coming to be 4/3+4X, N = 2 this value will come in and continue,

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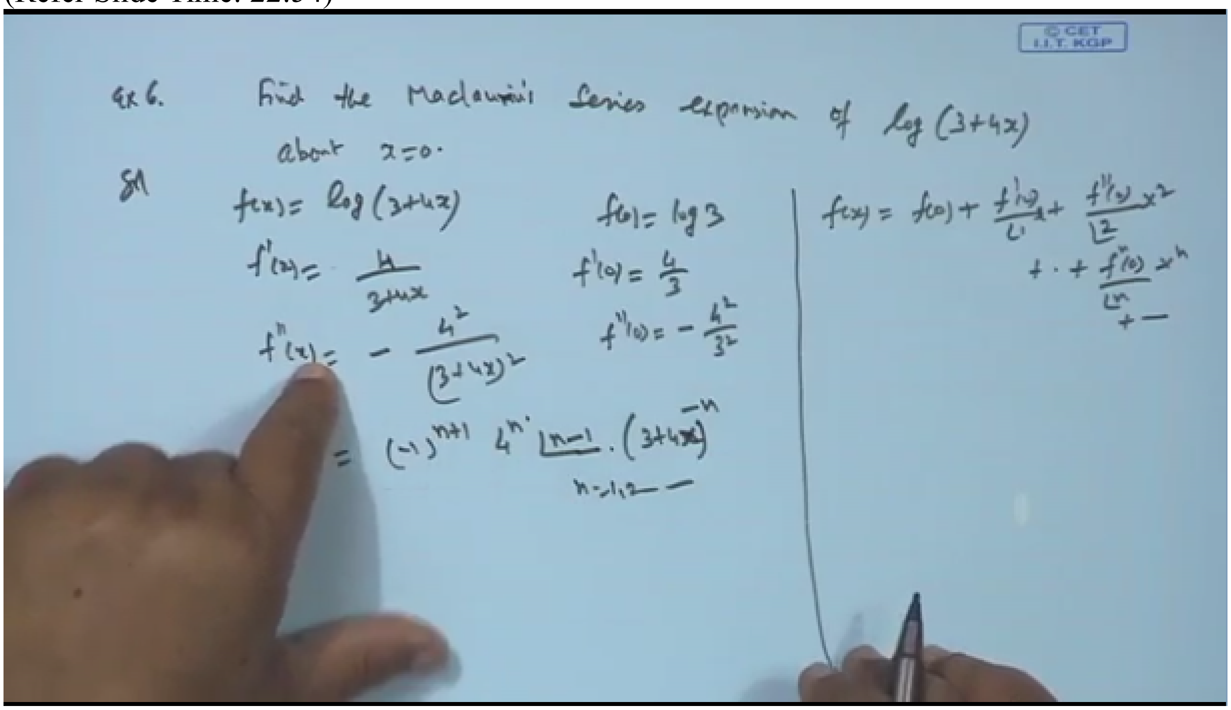

so what will be the FN(0)? The FN(0) and that is the Nth derivative, so this will come out to be -1 to the power N+1, 4 to the power N and then 3, when you take 3 outside, so 4/3 to the power N into factorial N-1, and then substitute therefore Maclaurin expansion of this 3+4X will be is nothing but the sigma, N is 0 to infinity, FN(0)/factorial N into X to the power N, you substitute it we get this one, okay, so we get from here is log 3 + sigma, N is 1 to infinity -1 to the power N+1 4/3 to the power N, divided by N into X to the power N, after simplification we will get this, so this is the Maclaurin series expansion of this function.

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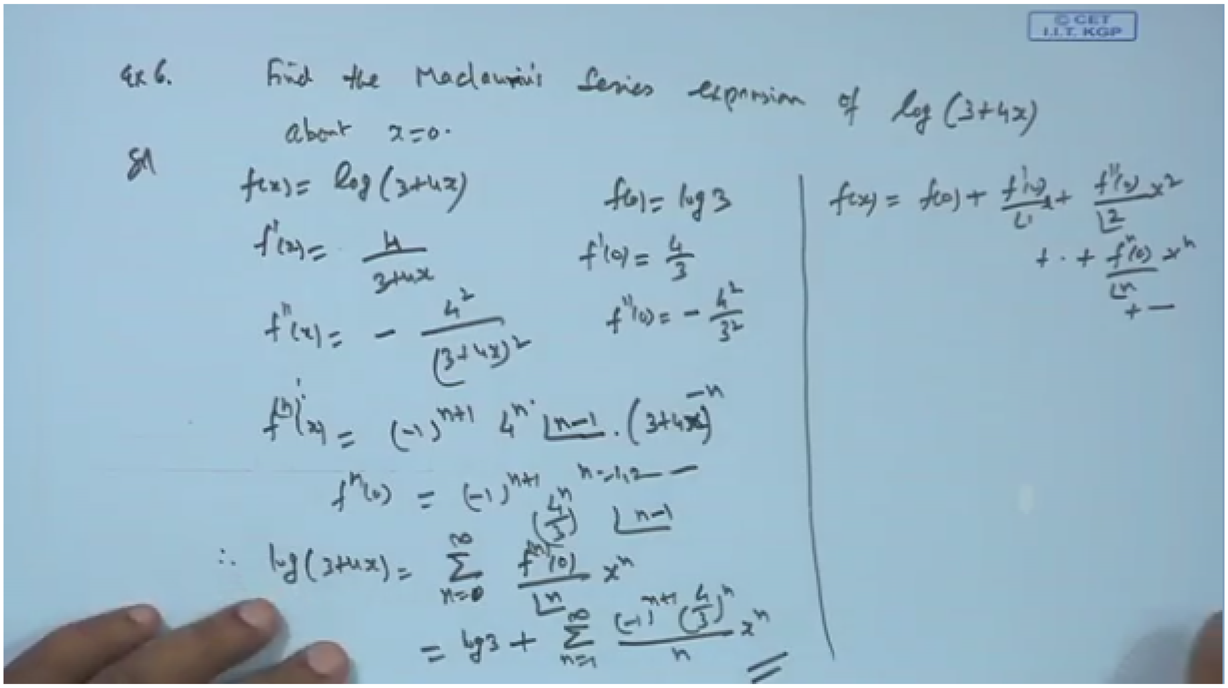

Then see the one problem based on this, find the Taylor series expansion, see this expansion of the function F(x) = 7 by, of the function F(x) = 7/X to the power 4 about X = -3, so we have this result F(x) is there, we can write is X-H by H like this so we can expand this function around the power of X, and this expansion will be equal to F(x-1) and so on, or we can say X naught this we can also write X naught + X naught –X, so we can say X-X naught, so in the power of X-X naught we can say F(x) naught + X - X naught F prime X naught + X-X naught whole square by factorial 2, F double dash (X naught) and continue, so this is the Taylor series expansion of the function, and the last term will, say nth term will be factorial 2N FN (X naught) and so on.

So here our X naught is 3, X naught is sorry -3, and X-X = X+3, so when you expand it in the form of this we are getting the derivative F(x) is 7X4, so F(-3) this is coming to be F(-3), so 7/3 to the power 4 then F prime(x) is coming to be -7 into 4X to the power -5, F double dash(x) is coming to be 7, 4, 5, X to the power -6 and so on, and so nth derivative of this will be -1 to the power N 7, 4, 5, 6, up to N+3 into X to the power –N+4, and that from here we can get the term and get the function F(x) = F(X+3-3) and that will be equal to N =0 to infinity, F(-3)/factorial 3, factorial N sorry, factorial N, and then X+3 FN, X+3 power N,
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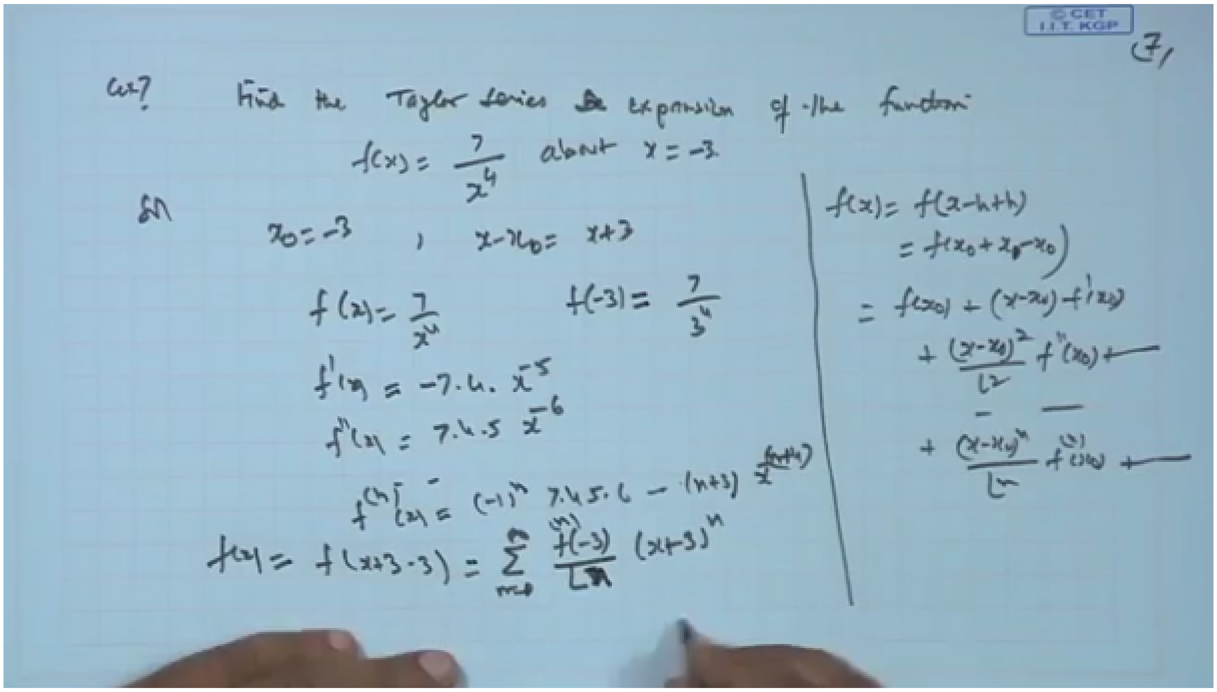

so find the value and substitute it, so that will be the answer. Thank you very much.