Module: 11

Lecture: 65

Riemann/Riemann Steiltjes Integral

Course

On

Real Analysis

So today we will discuss, the integration of the real valued functions on an interval basically we will do the integration, entry.

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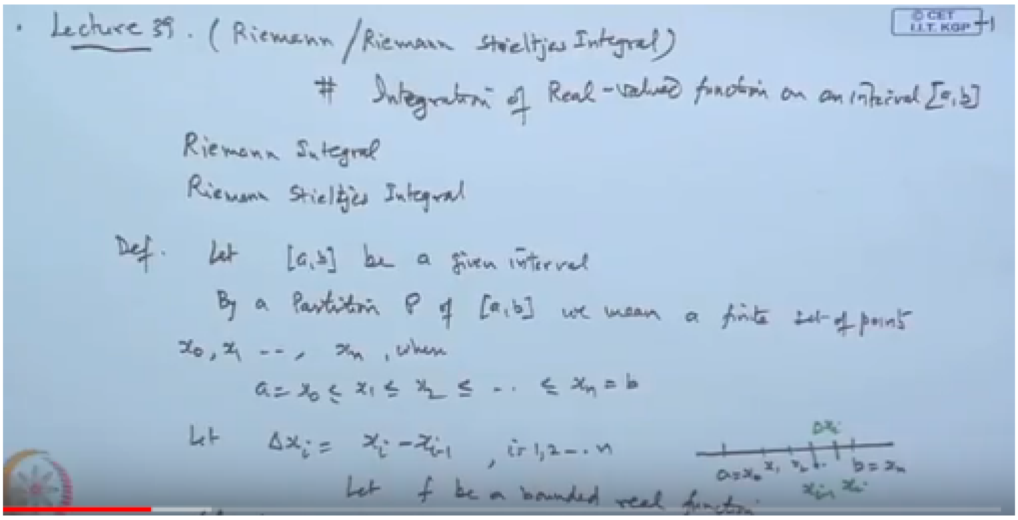

Integration of real valued functions on an interval say a V, so we will discuss today this part. Okay? Now, in this integral of the real world function we have the two type of integral we will discussed one is the Riemann integral, another one is the Riemann integral and then Riemann is tell still chase integral, in fact the remonstrance integral is the generalization of the Riemann integral and in the particular case we can say this. Now this is or it's very definite integral you know, so it is an extension part of the definite integral, when we go for the Riemann interrelation, so let's see before going the Riemann integral, let's see first the definition, how to define the Lehmann integral. Suppose a, B be an interval that a B be a given interval, interval by partition of a, b, partisan P of a B, we mean, we mean a finite set of point set of points say X naught, X 1, X 2, X n, where a is say X naught, which is less than or equal to X 1, less than equal to X 2, less than equal to area and less than equal to X n which is a B, so basically this is our interval a B, what we are doing we are partitioning this interval into a sum interval by choosing the point X naught, X 1, X 2 and x n in between AV we have the X naught is the initial point coinciding with a x n is the terminal point last point coinciding with B, these X 1 X 2 x n are the distinct point and maybe sometimes it may be overlapping that is we can start with the X naught X 1 then go for this X 2 start with the X 1, like that bit are so possible for that. So, let this set of connects which find a set of these points over the interval a B we satisfy this condition is called the partition of the interval a B, ok so let Delta X is stands for Delta X I it stands for X I minus X I minus 1 we are I very ½, suppose we have this point say here we have X I minus 1 and this is say X I, so this interval we are denoting as Delta X I, X I minus X I minus 1 is Delta ok, now let us suppose f be a bounded function let f be a bounded real function, real function bounded real function define defined over the interval a B, over the interval. Okay? Now, since function f we are choosing to be a bounded function and we have divided the interval heavy into a sub intervals, like Delta X 1, Delta X 2, Delta X, it's the length of these sub intervals and each sub intervals, X I mean each flow, so we can choose, for each partition.

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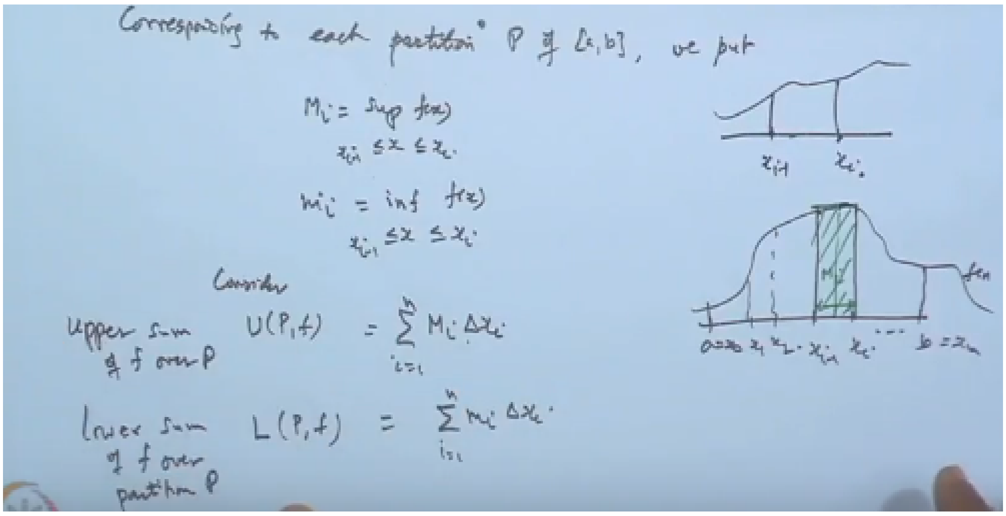

So corresponding to each partition P of a B, we take, or we take, or we assumed ,or we put mi is the supremum value of the function FX when X varies from X I minus 1 to X I, mean this is our interval X I minus 1, this is X I, function f is a bounded function, need not be continuous but it's a bounded function, so once it is a bounded function, over the interval closed interval X I minus 1 to X I, then obviously it will attend is much supremum and minimum value and infimum value over this interval, because it's the bounded function. Okay? So supremum will exist, infimum will also exist ,so let the infamous denoted by small m I ,this is the infimum value of the function FX over this interval x I minus 1to x I, is it Okay? Now, let us take the sum consider, this sum mi Delta X I and I is ½ means, over this interval, this is our interval a B, we have partitioned this thing as X naught. X 1, X 2, X I minus 1, X I and so on, this is X and here is something like this function FX ok. So, over this interval, over this interval, function have attends is supremum value, say at this point and infimum value suppose this point, so we multiply the soup the supremum value of the function that is this is our capital m I, by the length of this interval, so when you multiply this by the length of the interval you are taking basically this rectangle area of this rectangle, is it not? So this we are doing for each sub intervals over each sub intervals we are calculating this and taking the sum. Okay?
This sum be denoted by is, by you PF, because this sum depends on P, as well as the function f, because supremum is taken for the of the function f over this sub interval. So over each interval am I may change depending on the function, as well as the partition, delta x I depends on the length of the partition so if the change the partition Delta X I will also change ,so this sum we call it as a upper sum, upper sum of the function f over this part is corresponding to the partition P, similarly when we write mi Delta X I, I is equal to 1 to N, this term we denote by L, PF n is called the lower sum of the function f over the partition p, over parties b, so upper some of F over P and lower some of f over p. Okay? Now if we change the partition the upper sum and lower sum will keep on changing, so, but what is the but this upper sum and lower sum will always be a bounded function ,bounded thing by since function f is bounded over the interval a B. So it means, so there exist the two numbers, small m and capital M, such that the value of the function FX will always fall between these two range, because F is bounded, so bounded means it will have the least number and the largest number so m and capital M realize for all X lying between X and B and B so this is true? Is it not, so if we take any partition P.

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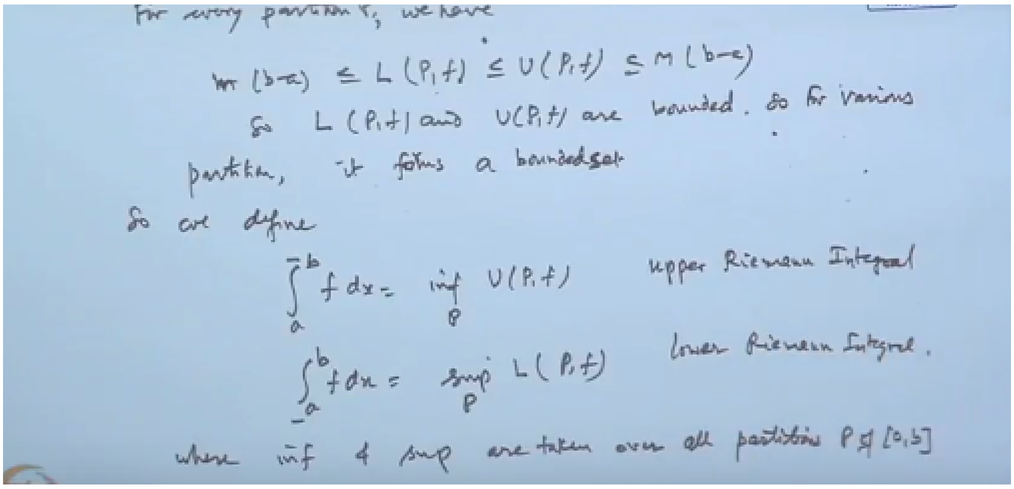

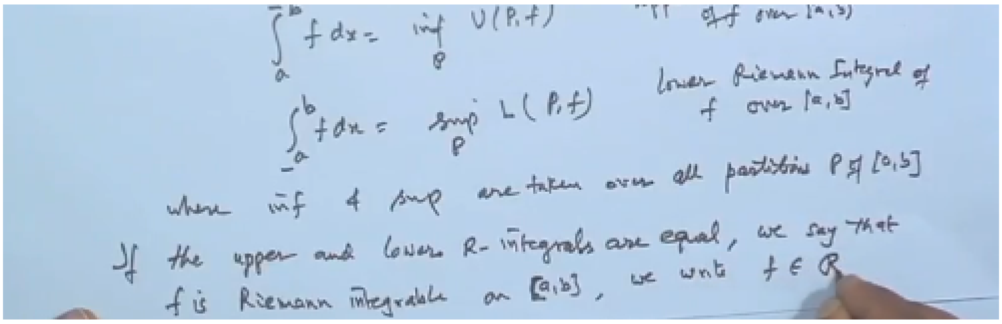

So, for any partition P, or for every partition P, we have M times, B minus a, will always be less than equal to lower sum of this which is less than equal to upper sum, of function f over the partition P which is less than equal to M into B minus a. Why? Because this lower sum and upper sum over this interval X I minus one lower sum will always be less than equal to upper sum over each subinterval, because mi is the infimum value, Capital m is the supremum value, so because of this it will be less than equal to a person, then take the summation over all such sub intervals, so obviously the lower sum will all total lower sum will always be less than the upper sum, this is one thing, second one is when we take the function f, F is bounded by M in Capital m, so this is the length of the interval so if we multiply this by the B - a ,the total length of the interval then this will be the minimum area bounded by a curve, whose lower values M and the length over the length B - a and this is the upper bound for the function f ,so M it was so M into B minus a will lie between these two one, clear, therefore our lower sum and upper sum is a bounded function, so it's a boundary set, so lower sum and upper sum are bound it's alpha all bounded, or they form the bounded, set are bounded, so for the partition for various partition, various for any various partition, it forms, it forms a bounded set, keep on changing the partition the lower and upper sum will change, but it will remain bounded between these two limits, so it is bounded. So, once it is bounded, it means, we can take the infimum value and supremum value of this, so this is upper bounded by this, so the infimum value of this will also exist supremum value of this will also exist, because this is lower bounded by this, infimum value will be at the most equal to this term and supremum value at the most equal to this, so what we see here that if we take from this, from this one, so we define so, so we define the in a to b bar upper ,FDX is the infimum value of the parties of the upper sum PFF, we are infimum is taken over all the partition, infimum is taken over all such partition P, similarly a to b lower bar F DX, is the supremum value of the lower sum P F, where the supreme is taken over self pardon, we are so we are infimum and supremum are taking over, over all partisans, all partisans P of a B is it Okay? And since these are bounded, so supreme and film will exist, hence about this integral will exist, this is called the upper Riemann integral. This is known as the upper Riemann integral and this one is called the lower Riemann integral this is called the lower Riemann integral okay, so, we get this role Riemann integral of F over that interval a B, so lower integral of F over a b, this upper Riemann integral and lower Riemann integral that's what we say, yes f is Riemann okay. Now, if lower Riemann integral and upper Riemann integral con sides, that is they have it same value and independent of course it's the partition, then we say the f is Riemann enterable okay, so we say if, if the upper and lower upper and lower Riemann integral are integrals, Riemann integrals are equal are equal, then we say, that f is Riemann enterable, f is Riemann integral, f is Riemann enterable on the closed interval a B, on the closed interval a B and we denote this and we write it as, f belongs to our ,we are all denotes the class of we are.

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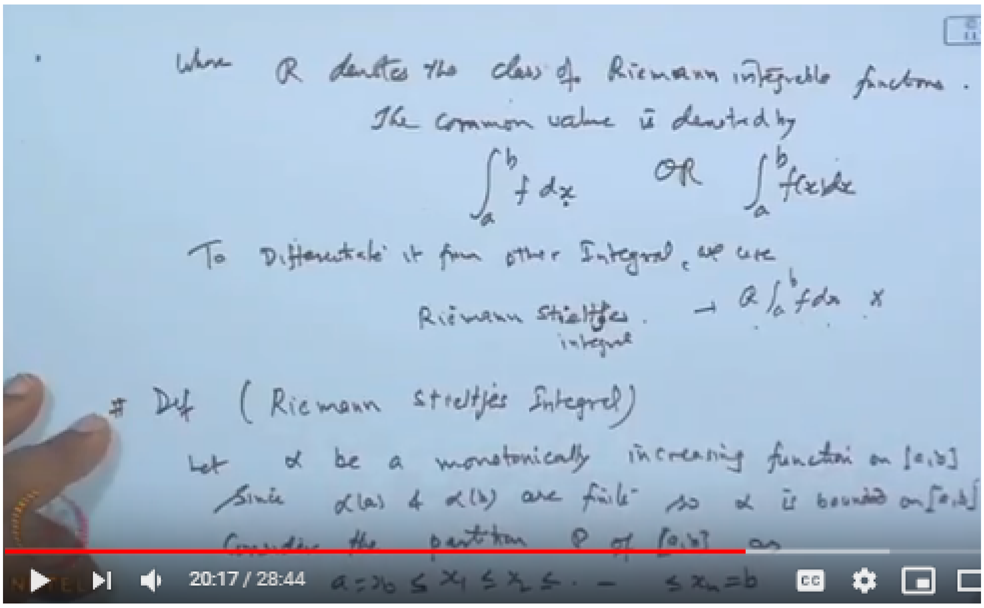

R denotes the class of all, or cassette classes of all Riemann, Riemann, Riemann integrable functions, integral functions Riemann integral functions okay, the common value we do not by the comma value is denoted by integral A to B, F DX or integral a to b, f x, d x or sometimes to differentiate between, to differentiate it from other integrals, like lemon states integral, we use this say we use the Riemann integral over r okay, so here we will take the Riemann integral A to B F DX as usual, otherwise some authorized like are a B also to so the Riemann integral, but here this notation we will use for the Riemann is tetras integral. Okay, so some author use, but here we won't write, we will take up only for the Riemann stasis integral, Riemann still it still J's integer okay, this we will take up later on what is this limit is, we denote this by r, so that it will differentiate between these two okay, so this is what we are, so what this so that, if f is a bounded function, f is a bounded function, then a person in lower sum will definitely exist and upper integral L all integral exist, not the question of whether they are equal or not? If they are equal, then we say the existence of the lemma integrals there, if they are not equal then we say the Riemann integral does not exist, okay, so existence part we will take later on, first let us see the other integral which is known as the Riemann States is integral, a generalization of our Riemann integral and then we will study the Riemann States integral in detail. So as a particular case, we can get all the results for Riemann integral also; okay so let's see the next definition for Riemann still this integral. Okay, now what we do here in this is, before going at b, reminisce t okay, so we will take let alpha be a monotonic, be a monotonically, monotonically increasing function, on the interval a B, on the interval a B, okay, now alpha is monotonically increasing function. So alpha and alpha B are finite, assuming that since alpha a and alpha B, these are their number alpha and alpha B are finite, so we can say alpha is a bounded function ,so alpha is bounded on the interval a B. Because monotonic increasing, so alpha a and alpha b, these are real numbers, they are finite, hence it has a finite when all the values of alpha lying between alpha and alpha B, because it is monotonically increasing function, hence it's will be a bounded function on a B, so once it is bounded, then let us consider the same partition, consider the partition P of a b, H, a is x naught less than x1, less than equal to X 2, less than equal to X n, which is say B, okay.

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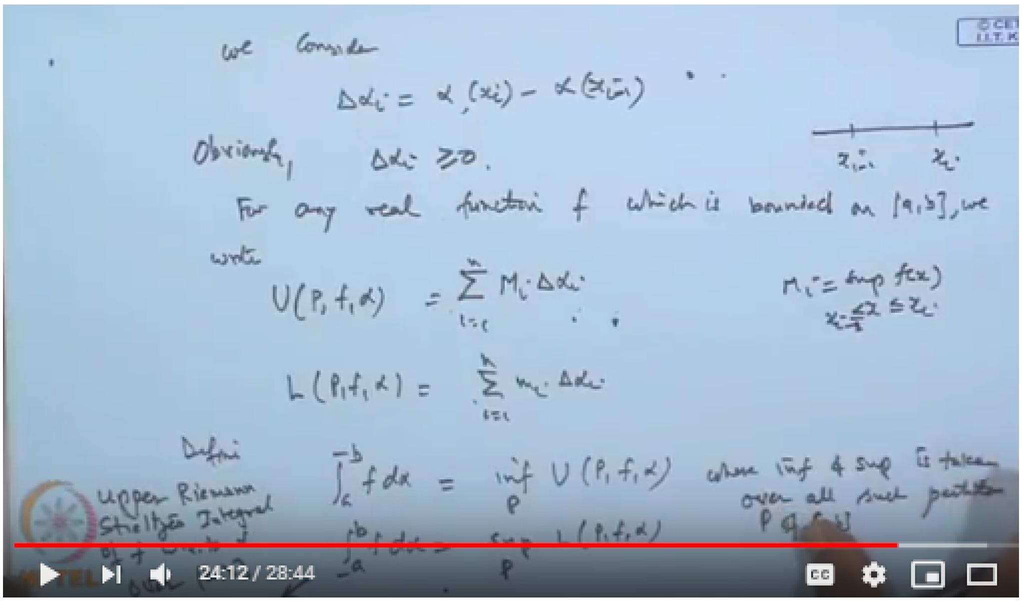

Then, consider instead of this Delta X I, now I consider ,the Delta Alpha i, we consider Delta alpha I H the value of alpha at a point X I, minus the value of the Alpha at a point X I minus 1. Because this is the interval X I minus 1, this is X I ,so earlier what you are doing we were taking the X I minus, X I minus 1, is the Delta X I, now alpha is taken as a monotonic function so considering the value of the Alpha X I, minus Alpha X I and denoted by Delta alpha I, so obviously, this Delta alpha I, will be greater than or equal to 0, because alpha is a monotonic function, increasing function, so alpha xi, will be greater than equal to alpha xi -1, therefore this will be non-negative quantity, now for any real function, for any real function, which f which is bounded, which is bounded, on the closed interval a B, on the closed interval a B right, we write the Sigma mi Delta alpha I, is 1 to N as u of P F and alpha, we have that mi means, the supremum of FX over X lying between X I minus 1 to X I, same thing ok and L P F alpha we are writing as, I is 1 to N small mi Delta alpha I, now this is again we call it the upper sum and the lower sum of the function f with respect to the alpha, alpha it depends on Alpha, is it not, so now this upper sum and lower sum, they will be defined in terms of the function f that is mi n is small a bye I alpha I, where alpha is this one Delta alpha is this one, now in a similar way to choose now, so define the upper integral A to B F, D alpha H infimum of u p f alpha, we are infimum is taken over all such partition p and lower sum is denoted by this supremum of l p f alpha, we are again supreme is taken over all partition, we are in fermium and supremum is taken over all such partition, all such partition, all such partition P of a B, or post-partisan of a b supreme, now if this further because again this is bounded function, so supremum will adjust, so this will, these two will adjust, this is called the upper Riemann still treats integral of the function f of the function f ,with respect to alpha ,with respect to alpha, over the interval a B and this will call it as a lower and this okay this we call it as a lower Lehman integral, in a similar way we write rural women stays integral of f with respect to alpha over a B, okay, now if both these values coincide then we say f is riemann States integral, so if let it be one and two, it's better we write it okay let it be one two.

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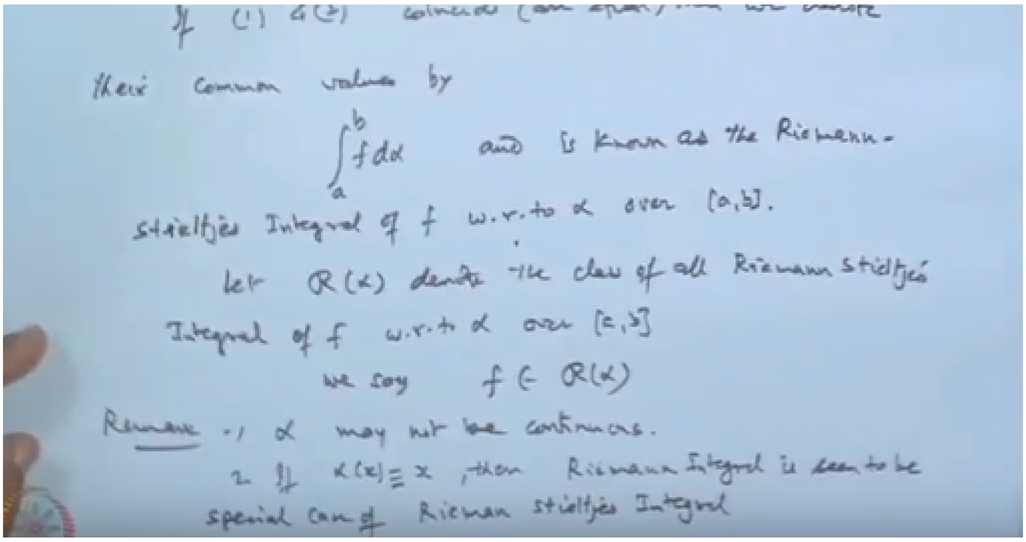

So, if one and two coincide, if one and two coincides have the same value ,that is are equal or equal then we denote, we denote their common values, their common values, common value by integral a to BFD alpha and is known as, is known as the Riemann still this integral Riemann integral of f with respect to alpha, over closed interval a B, that's what is ,okay so the class of all, the class let our alpha denotes the class of all Riemann still just integrals of f ,with respect to alpha over a b, so in this case we say so we say, that F belongs to R alpha. Okay, there we are denoting simply by r, here we are denoting alpha remonstrance is integral of the function f over ,in both the case I am considering F to be a bounded function ,need not be a continuous here also, not alpha need not be a continuous function, remark alpha may not be continuous function, continuous it's simply an monotonic increasing function, still this second one is if we take alpha X equal to X then the Riemann stretch integrals convert then we say f is integral the Riemann integral, is then Riemann integral is the is seen to be a special case of Riemann ,of Riemann is still just integrals, thus simply alpha X equal to X when it reduced the Riemann instance in thing I will give the Riemann instance integral and from there we can get the Riemann integral e, okay so this is an extension Riemann States integral is an extension of over in monetarily, hence whatever the property we will drive for the Riemann is States integral as a particular case many choose alpha X equal to X ,then you get the corresponding property of the Riemann integral. And remember we will always do not art by a Riemann integral and Riemann states integral by alpha, second part is. that earlier we have used the A to be f x, d x is it not, where X is the variable of integration, now here also one can write it FX, D alpha but this is not a very common notation, so normally the common notation is a to BFD alpha but it means that a to b f x, d alpha X. Okay so that's the meaning of this.