Module: 11

Lecture: 64

Taylor's Theorem

Course

On

Introductory Course in Real Analysis

So, today we will discuss few problems on continuity, differentiability, and application of the derivatives like a mean value theorem, monotonically increasing and decreasing functions so, on but prior to start with the problems we will take up the topic which we have not covered in the last lecture that is the Taylor’s Theorem.

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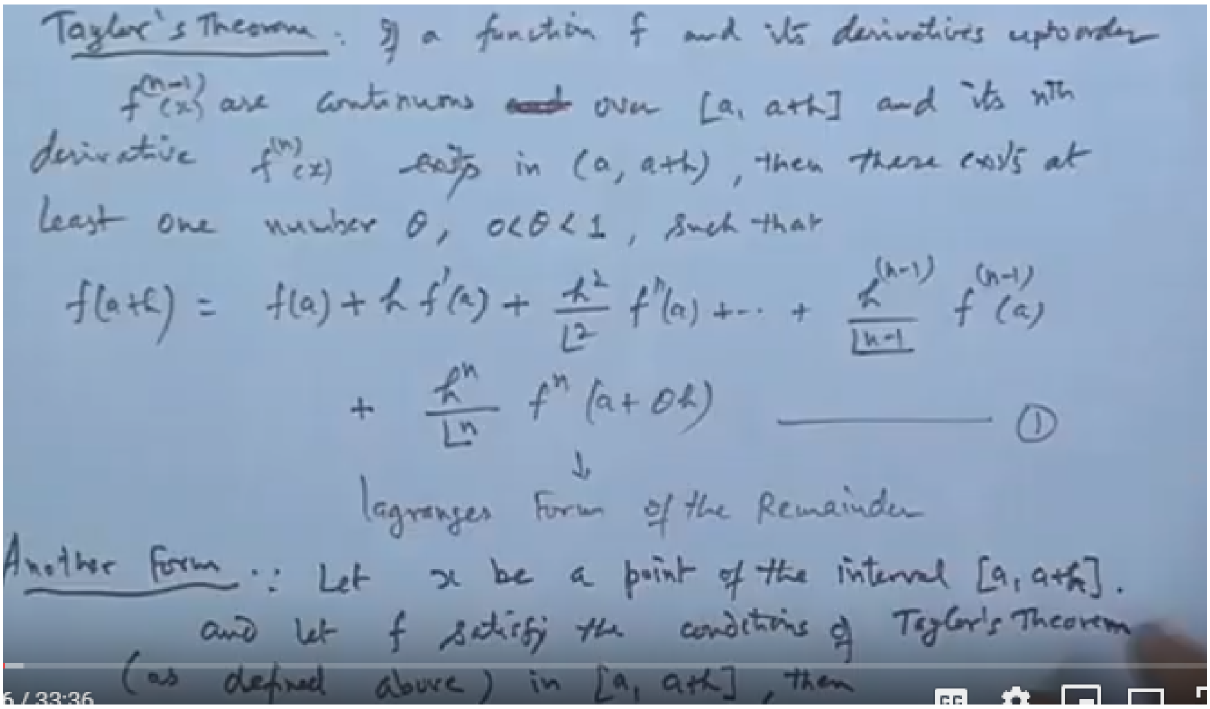

So, before that we first, complete that roof of the Taylor's theorem and statement we have already discussed but still we rewrite again, this Terrace theorem it's also known as the generalized mean value theorem and this says if a function f, if a function f , and its derivatives up to say r dot order. n plus 1 up to n minus 1 are continuous up to order n minus 1, are continuous, up to say order n let us we take up to order n plus 1 all continuous and continuous over the closed interval over the closed interval say a to a plus h. This, closed interval B and differentiable and its nth derivative and its, any derivative and its, an Derivative that is FN, FN x so here FNX and a derivative exist. and a derivative exists, in the open interval A to A plus h. then there exists at least one number, one number, thither lying between, 0 and 1 lying between, 0 & 1such that, such that the expansion of this, function at the point A plus h each can be written in the form of a series, f a plus h F prime a plus h square over factorial 2 F Double Days, a plus h n minus 1 over factorial n minus 1, F of n minus 1 A plus, plus h n over factorial, n factorial FN a plus theta H, a plus theta this expression.

So, f function f can be expanded in the power of H, in the power of H in this form and the last term is called, the remainder basically is called the Lagrange's form of the elemental. This is called the Lagrange’s form of the elemental there are various form we are not interested in the other just log now here, last time behaviours in the last lecture, we have taken in a similar statement. because it's coincide with the previous if I choose another form or we can say of this statement is if suppose I take F, a plus h and let X be a point be, a point of the interval a to a plus h. Okay? Eight minutes and let F satisfies, f satisfy the condition conditions of tellers thorium as defined above, as defined above in the interval a to a plus h, a to a plus h. Okay? So, that it satisfy the condition for a two index also mean okay. Then if I take a plus X then the equation one can be rewritten in this form.

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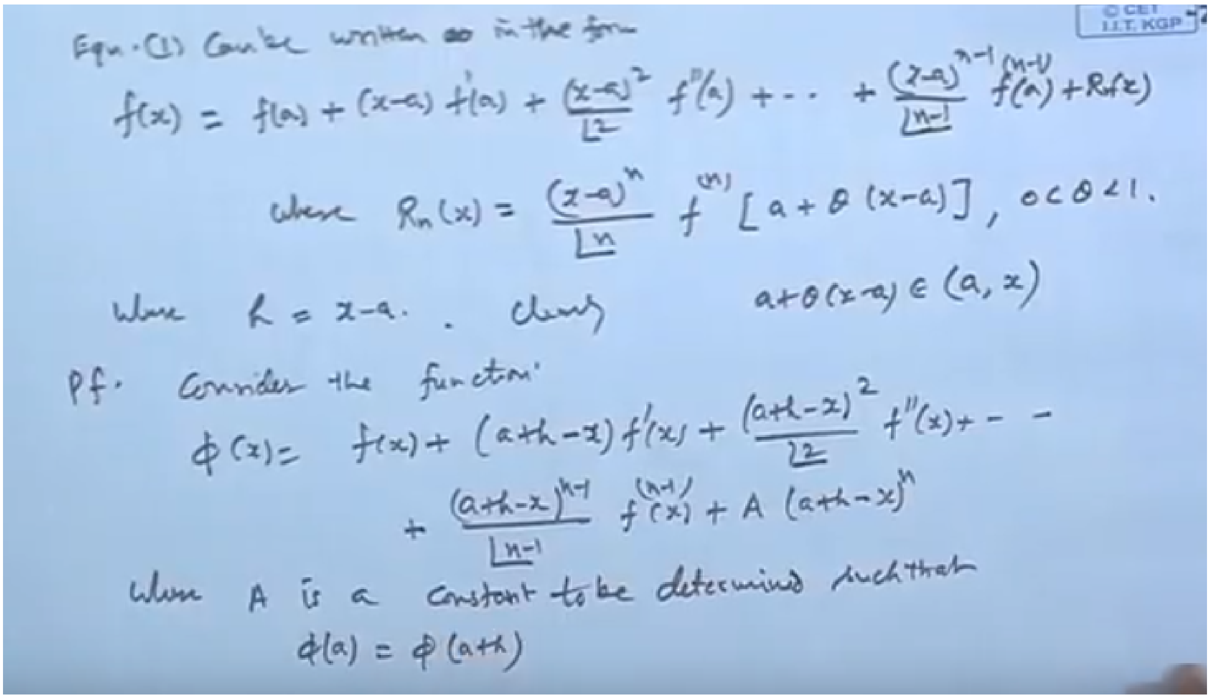

then equation one can be written as, f of X, equal to FN, plus X, minus a, F prime a, plus X, minus a whole square, over factorial 2, F Double Days, a plus, so on, up to say X minus a, to the power n, minus 1, factorial n, minus 1, F n minus n, minus 1, a derivative at the point A plus the remainder term R in X, we are our NX is H X minus a to the power n divided by factorial n, n a derivative at a point a plus theta X minus a ,where the theta lying between 0 & 1 then 1 can be written in the form, in the form this where H is taking to be H X, X minus a.

So, if in this form if I take H to be X minus a, then the function is well-defined, function is thought continuous it's derivative of to order n minus 1 is continuous over the closed interval a B and F n plus 1 FN exists this means, they are also differentiable over the open interval a B when we say FN exist, means the derivative of to order n exist in the open interval a to a plus h, and they are also continuous over the closed interval a to a plus h then what he says is if we picked up any point in the interval a B so I am taking the point like a corner point X plus h in fact this will be a be interval VL I choose a point X in between a B and that point is a plus h then expansion of the function FX in the powers of X minus a ascending powers of X minus a can be expressed in this form we are the first n terms in the series and the last term the Nth plus 1/8 term that is equal to any term is the remainder term and which called the Lagrange's form of the remainder and this point is obviously a, point clearly the point a plus theta X minus a since theta lying between 0 and 1 so obviously this point belongs to the interval a to X a to X, because it cannot be a a because theta is not 0 it cannot be X, as Theta is not one, okay .so it lies in between this so they like some point in between the interval where the remainder term can be expressed. Okay?

The proof of this which we have not done it last time let's see the proof its proof is a simple in fact we what we do is we construct a functions such as it so that we can apply the Rolle’s Theorem and once you apply the rows from a point C can be obtained where the derivative becomes 0 and from here the remainder term will come okay so that's the our idea of the proof so let's consider the function Phi X H ,F X plus a consider function Phi FX plus a plus h minus X F, prime X plus a plus X minus X, whole square by factorial 2 F Double Days X .and so ,on plus a plus h minus X to the power n minus 1 factorial n minus 1 f n minus 1 X plus a, a plus h minus X power n. we r is a constant where a is a constant, constant to be determined such that the value at the endpoint of Phi that is Phi is equal to Phi a process we put the restriction on Phi in such a way so that it can be computed.

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Pf. Consider the function

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{\lfloor 2} f''(x) + \cdots + \frac{(a+h-x)^{n-1}}{\lfloor n-1} f^{(n-1)}(x) + A(a+h-x)^n \quad \text{— (2)}$$

when A is a constant to be determined such that

$$\phi(a) = \phi(a+h)$$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + A h^n$$

So ,suppose I put this restriction and then from here let it be - in the - if I replace X equal to a and X equal to a plus h but we get as soon ,as you substitute X equal to a plus h all the terms gets cancelled except you are getting FA plus h. So, we get F of a plus h and when you take X equal to a then what you get in the right hand side we get FN plus h, F prime a plus h square by factorial 2 f double dash a and so on, H to the power n minus 1 factorial n minus 1 FN minus 1 a plus a into H to the power n, ok. So, this function, now the function Phi is a continuous function.

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I.I.T. KGP

Since ϕ is continuous over $[a, a+h]$ and diff. in $(a, a+h)$ and $\phi(a) = \phi(a+h)$, So By Rolle's Thm, $\exists\ \theta,\ 0<\theta<1$ st

$$\phi'(a+\theta h) = 0$$

But

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{\lfloor n-1} f^{(n)}(x) - nA(a+h-x)^{n-1}$$

At $x = a+\theta h$,

$$0 = \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{\lfloor n-1} f^{(n)}(a+\theta h) - nA(1-\theta)^{n-1} h^{n-1}$$

$$\Rightarrow A = \frac{1}{\lfloor n} f^{(n)}(a+\theta h), \quad 0<\theta<1,\ 1-\theta \neq 0\ \&\ h \neq 0$$

put in (2) we get proof.

When $a=0$: Taylor Thm reduces to

Since the function Phi is continuous over the interval a to a plus h and differentiable in the open interval a to a plus h and also satisfy the condition Phi of a equal to Phi of A plus h. Why, if you look the function Phi the function Phi is basically is a linear expression is an expression which involved the function FX and the term a like L minus X or L minus x squared, so on. So it's a, of the 1 X to the power n and the day then since function f X n it's derivative of two orders say n minus 1 they are continuous and this X to the power n type Alpha X, plus beta X, to the power n type, they are continuous function so prorate of each term will be continuous so Phi will be continuous. Similarly Phi is also differentiable and at the endpoint of the interval, if an, A plus h, it attains the value same value. So we can apply the Rolle’s Theorem. So by Rolle’s theorem, there exist a Point C, so there exist sum theta, there exist data lying between 0 & 1, such that, the derivative of the function, at some point A plus theta H, is 0. So a plus theta is a point lying between to a to a + H. Okay? But what is the five prime X? What five prime X if we just go and differentiate it, you will get many term get cancelled and only you are getting differentiation of this, one term, when it's derivative here and this. And rest is, telegraphically, it gets cancelled. So we get finally, the value as, a plus X, minus X, a plus h, minus H to the power n minus 1. Factorial n minus 1, FN X, minus P, in minus n, a, a plus h, minus X, to the power n, minus 1, this you get. Now this quantity at X equal to, a plus theta H, the 5 days of this number is zero. So from here, what we get? We get X to the power n minus 1, 1 minus theta, n minus 1, over factorial, n minus 1, just substitute this value, X equal to this number, into F n, a plus theta H, minus P, n a, 1 minus theta, n minus 1, H to the power, n minus 1. So if it solve it, we get the value of a, will come out to be, the thing and that is 1 by factorial n, 1 by factorial n, FN, a plus a plus theta H, a plus theta H, where the theta, lying between 0 and 1 and 1 minus theta, is not equal to 0, and H is not equal to 0, and H is also not equal to 0. Okay? So we get the remainder that a. and then substituted this value in one. So put in, in 2, we get the result proof, with the remainder term. Okay?

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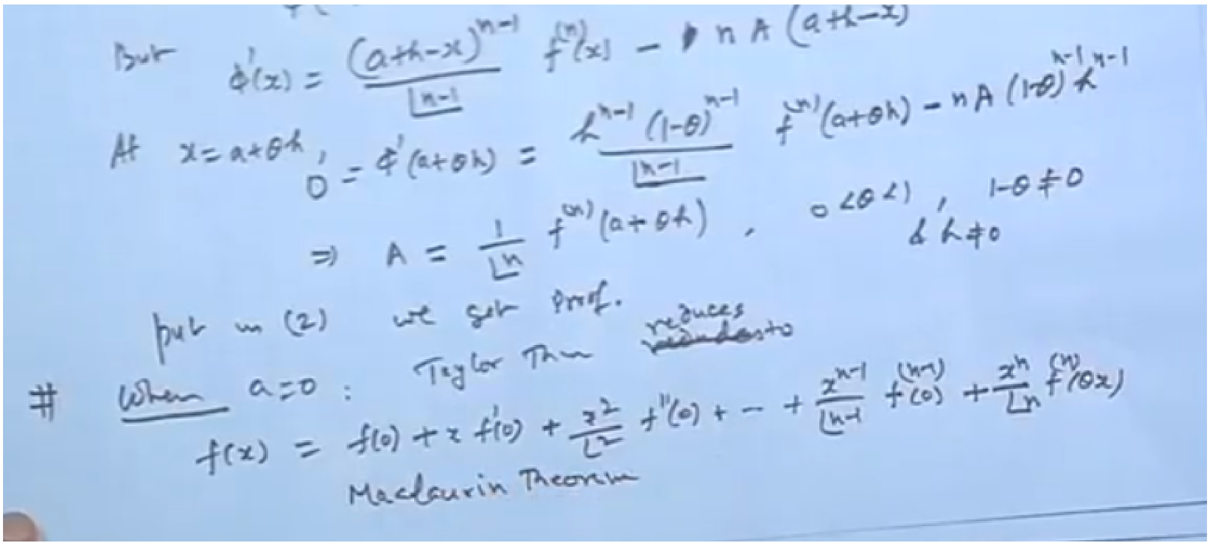

Now as a particular case, we can say, when a is equal to zero, when a equal to 0, then the Taylor series, Taylor's expense or Taylor's theorem, reduces to, reduces, reduce, reduced to, reduces to, the expression, which is FX equal to F 0, plus X, F prime 0, plus X square, over factorial 2, f double days, 0 and so on, X to the power n, minus 1,, factorial n minus 1, FN minus 1 0, plus the remainder term, that is X to the power n, over factorial n, F n theta X and this form is known as the Maclaurin’s Theorem, Maclaurin’s Theorem. Okay?

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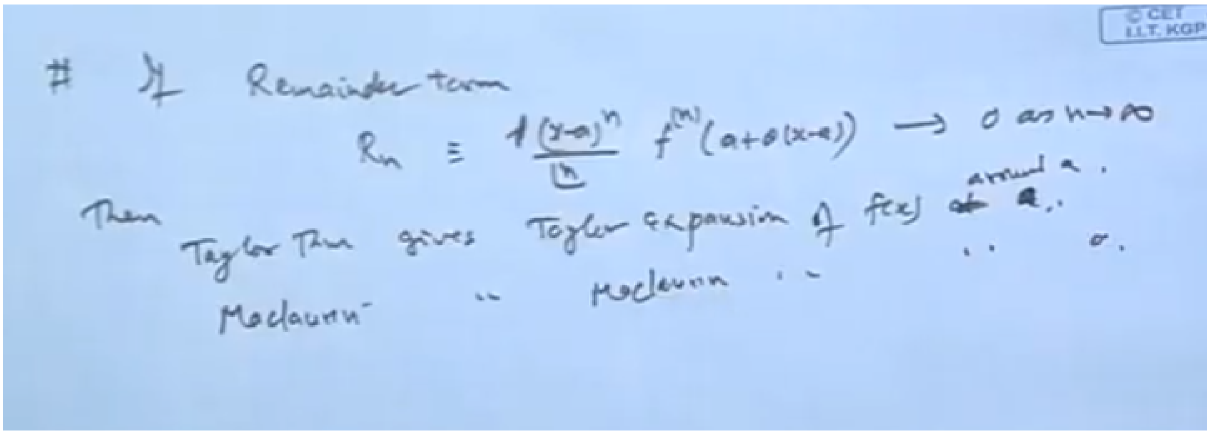

Now further, if our, if the remainder term, that is RN, RN, which is basically equal to, H to the power N, or X minus a, to the power n, over factorial n, X to the power n, over factorial N, FN a plus theta, X minus a, a plus theta X minus a. If this remainder term goes to 0, as, n tends to infinity, then Taylor's theorem, gives the Taylor expansion, of the function, FX, of the function FX, at the point X at X, at the point X, or around the point zero, around the point zero,, of course, around the point zero, at zero, taylor expansion, no, at, at the point X, minus a, at the point around the point a, around a, in the power of X minus and, the Maclaurin's theorem, give the corresponding, Maclaurin expansion, of the function, around the point, zero. That is what you get. Okay?

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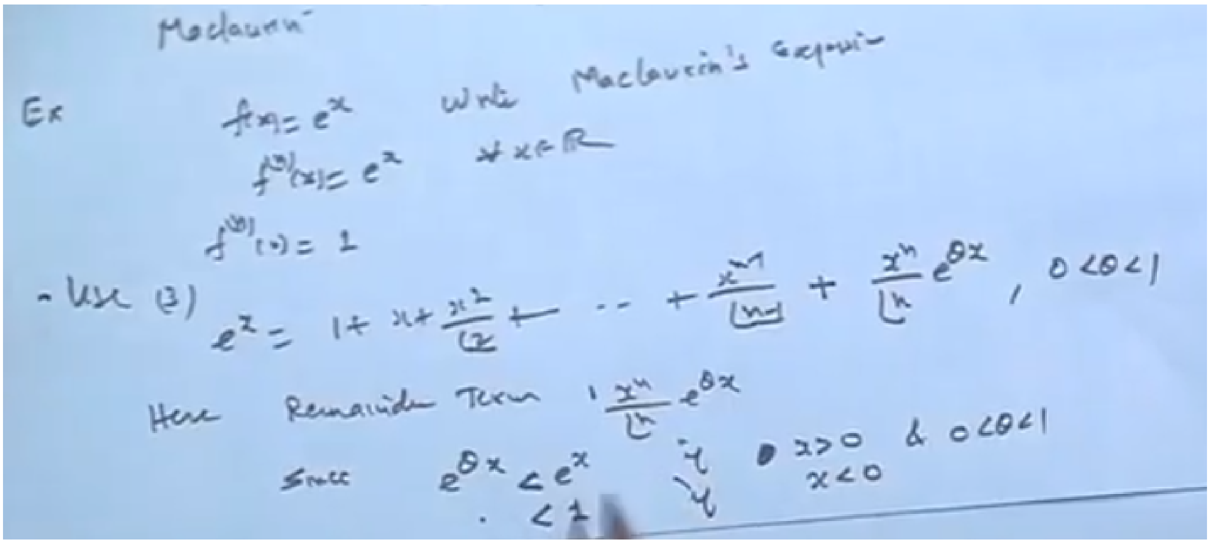

So for example, suppose I take the function FX, equal to e to the power X. Write down the Maclaurin's expansion, of this function, Maclaurin's expansion. So what is our derivative? If I differentiate n times of this, is nothing but, the e to the power x itself, so this is true for every X belongs to R. So when you take the zero, it is one. That is used on now Maclaurin's expansion, the Maclaurin series expansion, let it be three, so, when you use the three, so, use three, you will get, the expansion is 1 plus X, plus X square, by factorial 2, and so on. X to the power n minus 1, factorial n minus 1 and then you will get the remainder term and the remainder term will be what? X to the power n, over factorial n, e to the power theta X, we have the theta lying between 0 and 1. Okay? But if the remainder term goes to 0, then we get the Maclaurin series or expansion of the function e to the power X so here the remainder term is X to the power n over factorial e to the power theta X now if we look the e to the power theta X. Now e to the power theta X ,if I take this since e to the power theta X, the value of this, if it is theta X is positive, then the value will be, that value is less than e to the power x, if theta is positive, sorry, if X is positive. Because if X is positive and theta lying between 0 and 1. So e to the power X, the e to the or theta s will be less than e to the power X and when X is negative when X is negative then it is coming below so it is dominated by one always if this. It means the e to the power theta X, will always be a, bounded function, by e to the power X. Okay? So it is bound.

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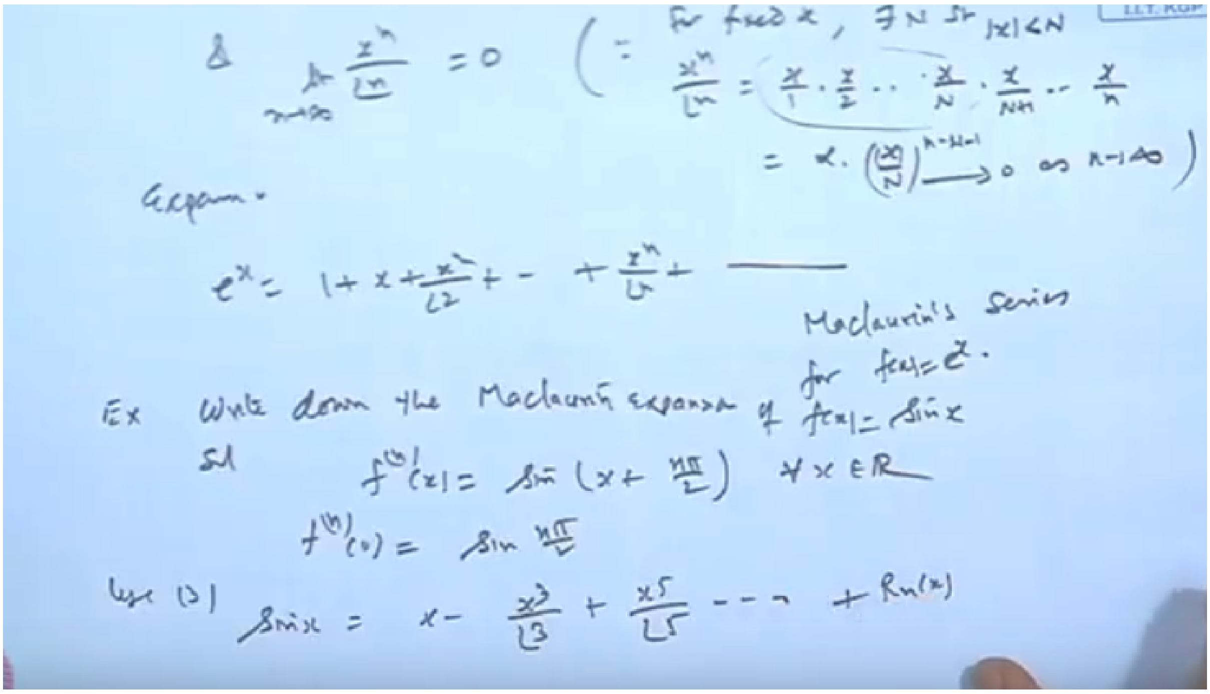

And, and X to the power n, by factorial n, limit of this, as n tends to infinity, will be 0. Why it is zero? Because the reason is, when it for X is fixed for fixed X we can identify there exists an N such that mode of X is less than capital n so we can find X, to the power n factorial and x over 1, x over to, say x over n and then you are getting x over n, plus 1 up to x over n. So what happened? These are the finite values and this value, so sum alpha into x over say this number, every term will be less than n, so we can write x by n, rest to the power n, minus n, minus 1, so this is less than 1 mode of this so this will be tending to 0 as n tends to infinity so this will go to 0 so therefore the remainder term in this will go to 0 so expansion of H to the power X will be 1 plus X plus X square factorial 2 plus X to the power n factor and so on and this will give the Maclaurin series for this, this is the macro let's see is for the function FX you fold the function FX which is e to the power X, Okay? similarly if we take the write down the, write down the Maclaurin's expansion, of the function FX equal to say sine X. Okay? Now we know the derivative of the sine X the derivative of sine X is comes out to be sine X plus, n pi by 2 and this is true for every X belongs to R. Okay, so take the X is 0 so what you're getting is sine n PI by 2 now when it is n PI by 2, it depends on n. Okay? So forth, n is even when you are taking this becomes PI for 2 pi etcetera. So we are getting the sine X, is it not and then the x + theta, so this tip, the value of this will be 0 and well in other odd numbers, the value will be some x plus 1 or minus 1, so if you use the 3, we get the expansion sine X, is X minus, s cube, by factorial 3 plus X 5 by factorial 5 and so on, like this. And what is the remainder term rnx?

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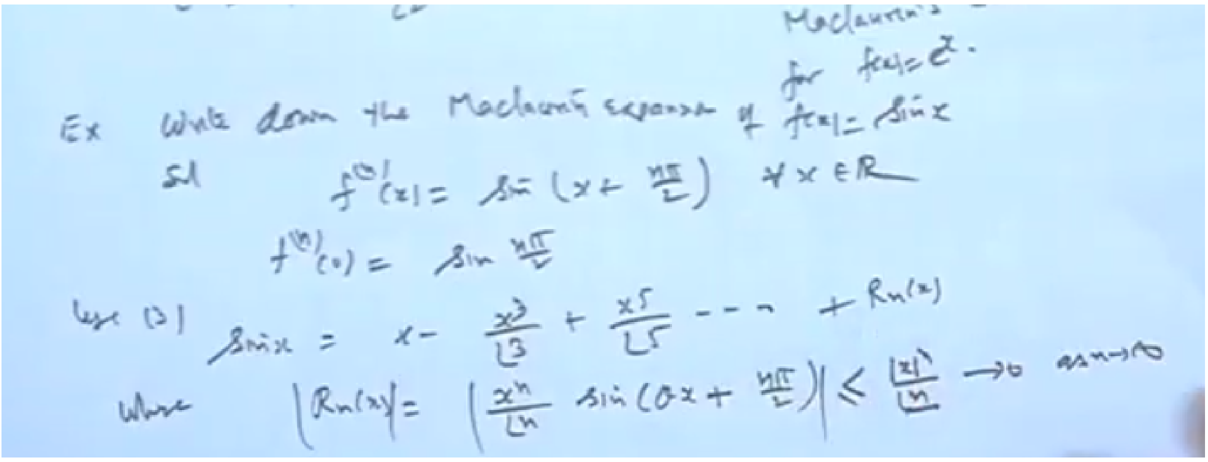

The remainder term r n will be X, x to the power n, y factorial n, sine of theta X ,plus n, PI by 2, this is the remainder, so mode of remainder mode of this, but this limit and where remainder term is this, which is dominated by mode X to the power n, by factorial n, less than or equal to this, because this is always less than equal to 1, so as n tends to infinity this will go to 0 agents. So again we can expand it and get the series expansion for the function, so that is what 1 minus X, is.

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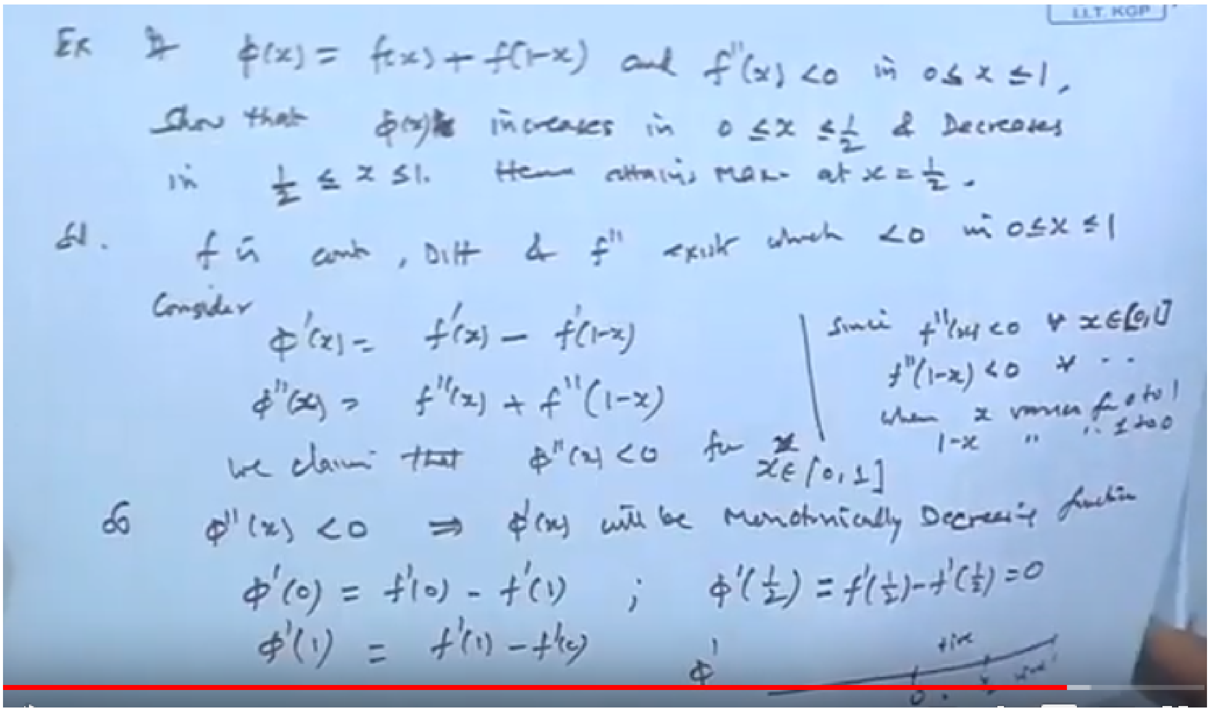

And the second derivative of the function Phi prime is negative, negative, in the interval 0, less than X, less than equal to X ,less than equal to 1, then so that, so that, Phi is an increasing function, Phi increases, increases, of Phi X increases, in the interval 0, less than equal to X, less than equal to ½, and decreases, in the interval, 1/2 less than equal to X, less than equal to 1, hence attends maximum at X equal to ½. Okay? Let me see this is what is given that function f is well defined function over the interval 0 & 1 and second derivative, is negative. So function is giving to be, function f is continuous differentiable and the second derivative exists, which is less negative, in the interval 0, less than X, less than equal to 1. So this much information is known. Now let us consider the Phi days X, Phi days X. If we take the Phi days X, this is the f prime X and the derivative of this is f prime 1, minus X and derivative minus X, minus 1, but we cannot get any information whether f prime is greater than this or not. Okay? So go for the another one, Phi double days 6, so Phi double days X, comes out to be F Double days X and this comes out to plus, F 1 double days minus. Now from here we claim, we claim, that Phi Double Days X, is negative, for X negative. Why it is negative? For X belonging to the interval, say yes, now here we say, here this is, okay. Why f double days X, is negative, in the interval this, when X varies from 0 to 1, so since our f double days, X is negative for all X, belonging to the interval 0 1, closing, so f double days 1 minus X, will also be negative. Why? Because when X is from for every X because this is when X varies from 0 to 1 then 1 minus X, will vary from 1 to 0. So there always lies in the interval 0 1 and for all point which lies in that 0 1 the second derivative is negative so Phi double days in negative for all X in the interval 0 1. Okay? So this is information we have got it that's and now we can come now X is, okay, so what we get is.

The phi days X, so over 1, so phi double days X, is negative, is negative, therefore 5 days X, will be a decreasing function, monotonically, decreasing function, monotonically decreasing function. So once it is decreasing, over the interval 0 to 1. Okay? Now let, me see. What is that decreasing functions? So the value of the function at the point 0m so what is the value of the function at the point 0? The value of the function at the point 0 is f, 0 minus f prime, the value of the function at a point 1/2 a phi days, say, at the point ½, is nothing but, what? Phi days half, this becomes what? When you take a prime 1, this is 0, a prime half, minus F prime half, which is 0 and then value of the Phi days, at the point, at the points say 1, what is this a? F prime 1, minus F prime 0. So what is this? This is the function, Phi days, phi days is this function, which attends the value, sorry, this is one interval, this is half, so phi days attends the value 0 at this point and then these values, at the point 0, is this value, at the point 1, this value now phi days and phi days said they are differing by minus sign only. Because minus times of this number, is the second. So if it has a positive value here this will have any get a value here and if it is negative value then this has a positive value but phi days is a decreasing function, so phi days a decreasing means, it cannot go from negative to positive. So what we get a, get it, that since it is monotonically decreasing, so we can assume, we can consider from here is, that in between 0 to 1, phi days must be positive. So here it should be positive, here it should be negative, and this. So that it decreases and cross the x-axis and going down.

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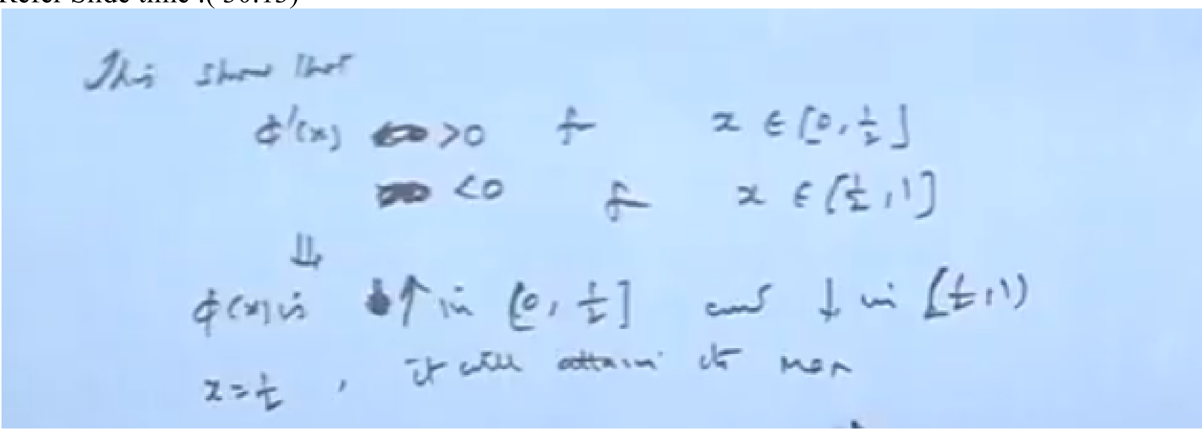

So this shows, this shows, this shows, that the function phi dash X, is negative, for X belonging to the interval 0 to half and positive, for all X, belonging to the half and 1. It means this implies the function Phi, what is the function? Therefore our function Phi is a Phi function, is decreasing, in the interval 0 to half and increasing, no, no, it's wrong. It's the other way round, I'm sorry; it is the other way round, this is positive, this is greater than zero, in this interval and this is less than 0. So this is increasing function, in the interval positive, so it is increasing, this is what, it's coming, no, no, no, it's, I'm sorry.

What is this is, this function Phi days, if it is positive, it is increasing, oh yes and then it is decreasing, so function will be something like this and then at the, sorry, at the point half, it is coming to be, at the point half, what is this is, yes, yes, at the point half, so it is increases, is it not so it will go like this so maximum value will come at the point half, yes, so we get this.

So this is our increasing in this and decreasing in the interval half to one. So our function f Phi, Phi, which is this function Phi, this function FX, so what we get is, the Phi is increase in this decreasing this and attains the maximum value, at X equal to phi. So Phi, at X equal to half, it will attain its, maximum value. Is it not? Because then only, we are getting the function Phi days, this is, this zero and this one is, one so here it is f days 0 and then f days 0, is greater than this, F days 0 is greater than this and this is negative, that's why it, is coming, so it's coming, to be just like that. Okay? So we get this one, the value at the point, this is basically, at the point X equal to ½, this function is this. So what we get is, and that increases in the interval, so and why is increasing in the interval this, phi days X is greater than zero and this is less than zero. Okay? So we get this one and the phi days X. Okay? So this shows our result is good. Okay? So that.