Module 11

Lecture – 63

L' Hospital Rule

Okay, so today we will discuss the L' Hospital Rule and Taylor's theorem.

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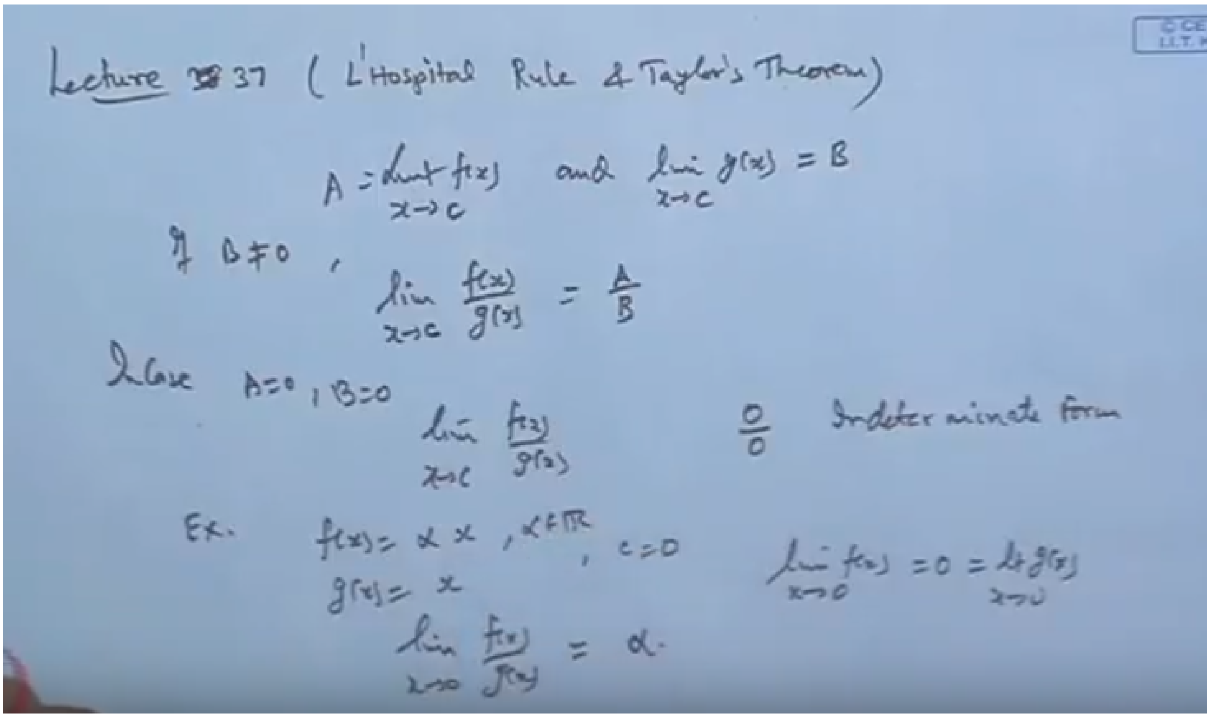

We have seen already in the limit of the function FX,when say X tends, to say suppose a is about X tends to say C is say a and limit GX, when X tends to C, is say suppose B then if B is not equal to zero then one can find the limit of the FX, by GX, when X tends to C and basically this is equal to the a by B. Okay but if suppose in case a is also zero, B is also zero, if b are zero a may be anything, then also we can discuss something and it is shown that if the limit of the derivative exists, then this must be say zero and in case if this b zero and a less than zero negative, minus infinity and B is 0, a is positive then limit will go to the plus infinite. Now the case when a and B both are having the limit value with zero, so when you consider the limit of the function FX over, GX when X tends to C, it comes out to be 0 over 0 form, which is known as the in determinant form, indeterminate form. Why in determine? because you cannot say the value of this is 1 or value is some finite number or value does not exist, in fact 0 over 0, in the well it may exist, value the limit may exist, limit may not exist also and like this. for example if we take the function FX, say alpha times of X and GX say X, and choose the limit c as 0, then obviously limit of FX, as X tends to 0, is 0 which is the limit of GX, when X tends to 0, both are coming to be 0. So when you take the limit of FX, over GX, as X tends to 0, then in fact this is coming to be if you take the limit of FX, over limit of GX, then it is basically coming to build 0, over 0 forms. But if I substitute the value FX and GX, then the limit will come out to be alpha. Because x will get cancelled and x is independent of any X. so limit will come out to be alpha a real number, we are alpha is a real number. so it means the limit of this 0 over FX by GX, when it demands to be the 0, over 0, then it may exist and may have a value a real number or sometimes may not exist also that we will take some example we have the limit does not exist in that case. Ok.

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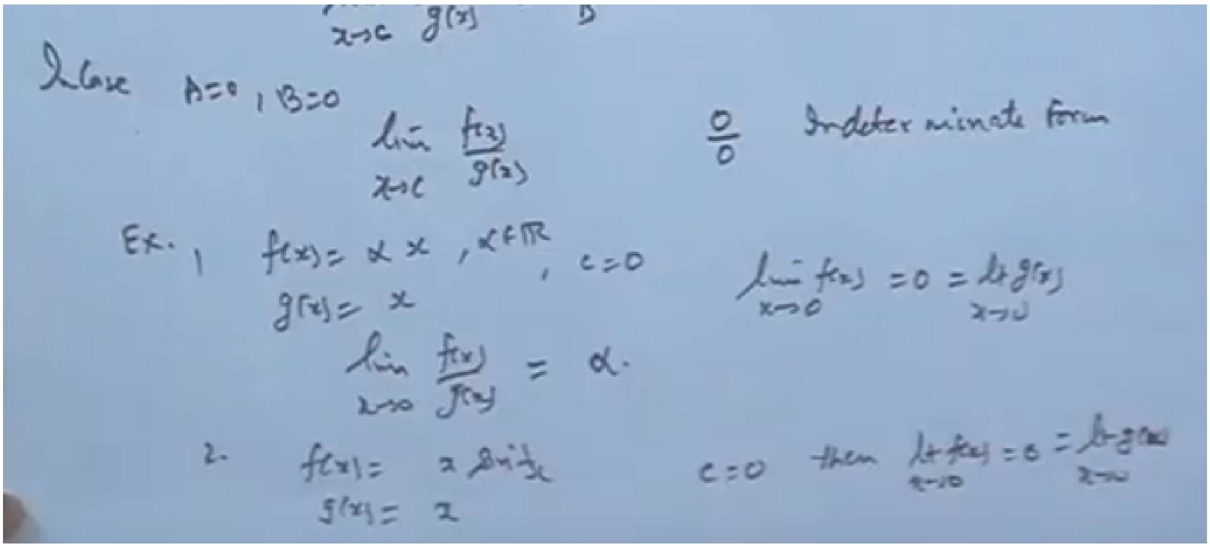

Say for example limit does not exist suppose I take another one, FX equal to X sine, say 1X and GX is equal to X. so if I take C equal to 0, then both the limits, FX, when X tends to 0, is 0 limit GX, when X tends to 0, is also 0.

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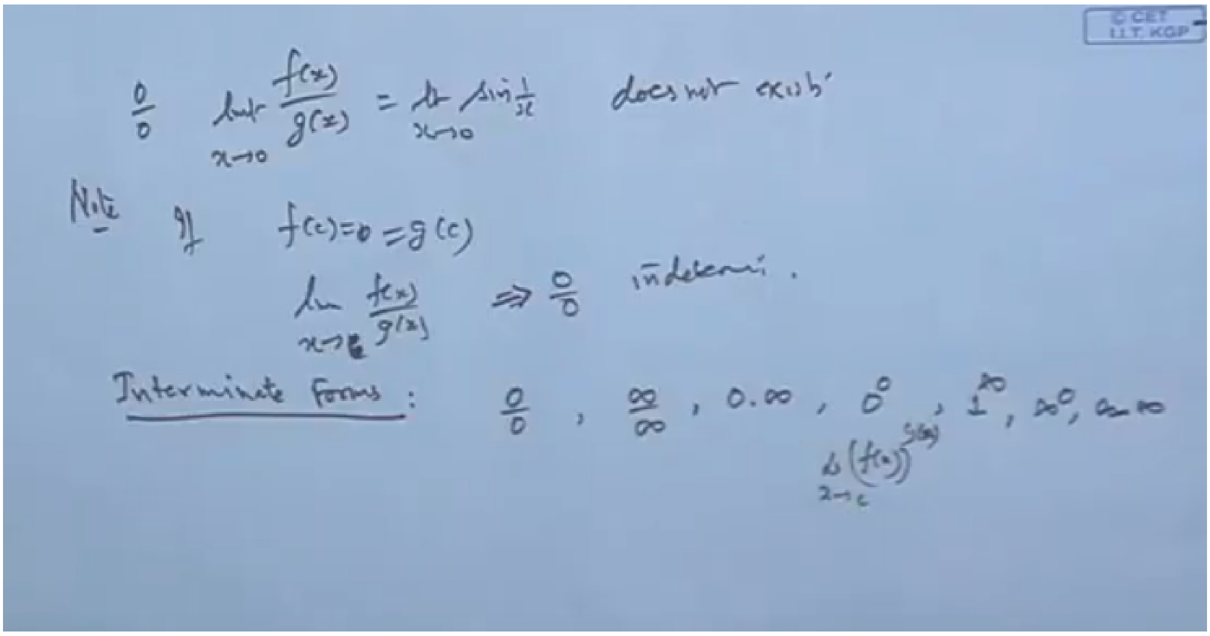

But when you take the limit of FX by GX, as X tends to 0, it is it is coming out to be the limit f sine 1 by X, when X tends to 0, which does not exist. So in the initial form it is 0 over 0 form, but basically the limit does not exist. So that's why to evaluate such case, to find the limit of FX by GX when it emerge to zero over zero, then we require some rules and that is given by the L' Hospital Rule. Now one more thing here we are choosing the limit, even if F of C is zero, or G of C is also zero, then in that case limit of FX by GX when X tends to C, is also in the same form, zero over zero, it takes this form, which is also in determinant form. so if the function is defined at the point C limiting point, and if it is coming to be 0 over 0, then the limit of FX over GX, to calculate the limit becomes a problem, or even the limiting value is also coming 0 by 0, then it's also we cannot use any limit rules for the limits, that limit of the rational function, is the limit of the numerator over limit of the denominator, which we cannot apply here. so in such cases we required that some concepts and that is to how to evaluate, this limit of this ratio, when it is coming to be 0 over 0 form and that is given by the L' Hospital Rule. so before going to L' Hospital Rule there is one we will say, the In determinant forms, is the 0 over 0 is the only determinant form? No, there are many cases which can which are considered as an in determinant form. So let me say the in determinant forms. These forms are first form is 0 over 0, infinity, over infinity, is also considered to be in determinant form, 0 into infinity, that is if f x tends to 0, GX tends to infinity, then FX into GX, this limit will go to 0 into infinity form or FX by GX, if GX also, then 0 to the power 0 if f x tends to 0, GX tends to 0, the limit of FX to the power G X, this ok when X tends to C, if it is coming to be 0 to the power 0,then it is also considered to the lymph in determinant form, 1 to the power infinity will also be taken as in determinant form, and infinity to the power 0, is indeterminate form, infinity minus infinity is also an in determinant form. So they are various types situation, where we can say the limit, the forms, which you are getting is an in determinant form. So evaluate this type, we require certain tricks, ok. because just by substituting the value or taking the limit.

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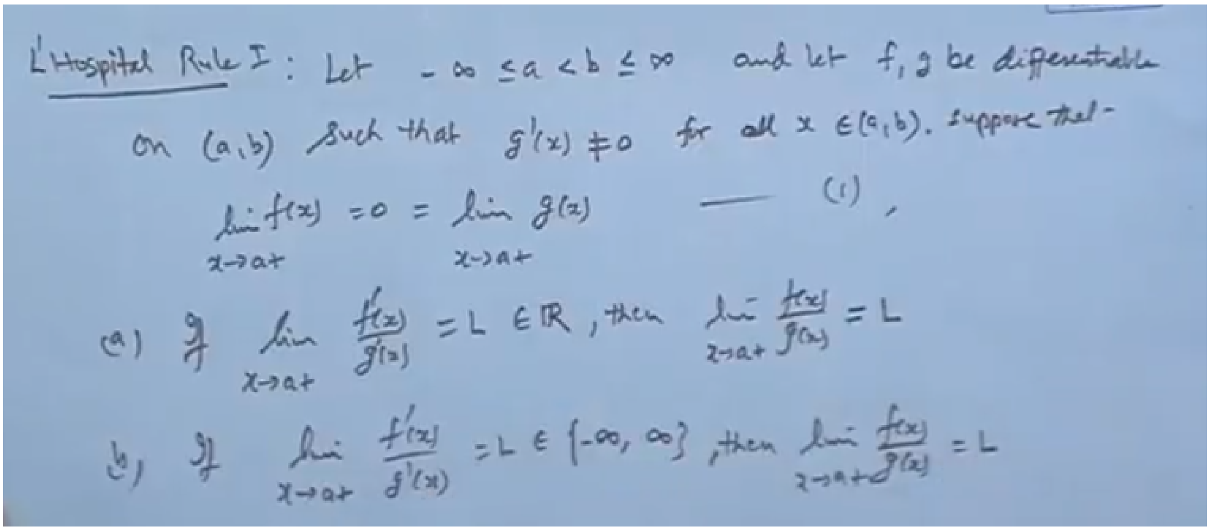

We cannot say the limit is correct. ok or we cannot able which may be real number, which may be infinity or minus infinity, also and let F and G, be differentiable, F and G, be Differentiable, on the open interval a, B, such that, such that the derivative of G, does not vanish anywhere inside this interval, a, B, that is nonzero for all X belongs to a, b. Now suppose, that the limit of the function FX, when X tends to a plus, is 0, which is the same as limit GX, when X tends to a plus, both are having the limiting value right hand limit, exist and equal to value 0, Let it be 1. Then the result says if, this condition, if limit of FX, over GX, f prime x, over G prime X, when X tends to a plus if the limit of their ratio of their derivatives exist and is a real number L, then the limit of the FX by GX, when X tends to a plus, from the right-hand side, will also exist and will be having the same value, as the limit of a prime, over G prime is there when X tends to A prime. The second is if suppose L is infinity or minus infinity. so if limit of this derivative F prime, X over G prime X, when a tends to a plus, is suppose L, which is either in minus infinity or plus infinity only, then the limit of this FX over GX, when X tends to a plus, will also be minus infinity or plus infinity, depending on L, okay? so this is what is known as the Law. So what this L' Hospital Rule says is That, if we assume the differentiability of the function, differentiability of the function, but not necessarily at the point a, then the behaviour of the limit of FX by GX and the behaviour of the limit of f prime, x over g prime x, will be the same.

That is the limiting behaviour of a prime, over G prime, will be the same as the limiting behaviour of FX by GX, whether L is finite or infinite or minus. Now here second point which we also want to discuss, though I have taken the limit, a plus, means the interval is this a, B and we are taking the limiting value from the right hand side, but there is same result continue to hold good, when we take the limit b minus or a minus, suppose it is completely defined or limit exists, if the function has a limit at the point X equal to a, then we can replace a plus by a, like this so if the limit of f prime over G Prime, at X tends to a is L, then the limit of FX, over GX, over X tends to a, will also be L. so in case limit FX, as X tends to a exists and 0 and equal to the limit of GX, when X tends to a, when F and G both are having the limit, I am not taking only the right-hand limit let us limit exists. then if limit of this ratio F prime, over G prime X, when X tends to a, is equal to a exists, then limit of their ratio will also exist and will be the same as L. it means the proof will remain the same, whether we replace a plus by a, or a plus by a minus, provided the limits are there if the left hand limit exists we use this, if the limit exists we use this if the right left hand limit exists, sorry, right and then this left hand limit exists, then a minus and like this, but the result continued to hold. so we will estimate this proof when the right hand limit exists. Okay and for others cases the proof is the same.

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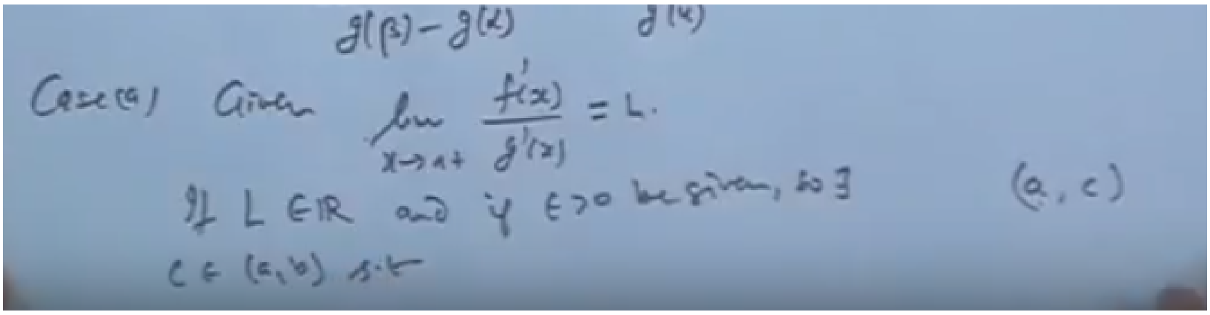

So let us see the proof. Now let us take an alpha beta. lying between a and B, okay now since the function G, since G, which is given to be differentiable, over a B since G is given differentiable over a, B, over a, B and G prime X, is not equal to 0. so this automatically implies the value of G beta, will be different from G alpha. Why the reason is otherwise, otherwise by Roll's theorem, if a function G, which is continuous over the closed interval a alpha beta, differentiable over the open in alpha beta, G is given a differentiable over alpha beta and if the endpoint alpha and beta the values are same then there will exist a Point C, lying between alpha and beta, such that that is a and B, such that derivative of G, at the point C, must be 0, which Contradicts, our assumption that G prime X, never vanishes for any X, belong to the interval a, B. so always this result will be true, that G alpha fine. Ok? now take the interval alpha beta, apply Cauchy mean value theorem, for the function f and G, over the interval alpha beta. so what the mean value theorem says. if F and G both are continuous, over the closed interval alpha beta differentiable on the open interval alpha beta, then F of B minus, F alpha, divided by G beta, minus G alpha is the value of the F Prime's set C, or F prime u divide by G prime u for some u in lying between alpha and beta. So by mean value theorem there exists n u belongs to alpha beta such that F beta minus F alpha, G beta minus G alpha is the derivative this thing for some u. okay let it be clear this is my D now let us take the case one, say a what a, it is given limit of this is given limit of F prime over G prime, when X tends to a from the right-hand side is L, this is given so given limit X tends to a plus F prime, over G prime derivatives, is L. okay so let us take, so if L is given that is if L is known which is in R and let epsilon and if epsilon greater than 0 be given. as some positive number now limiting value of this is L. it means when the point a close to some interval say AC, AC I take so whatever the point X in between this the limiting value of this prime is L, mean difference is very very small so this lies between L minus epsilon and L plus Epsilon. so we can say there exists a C belonging to the interval a,

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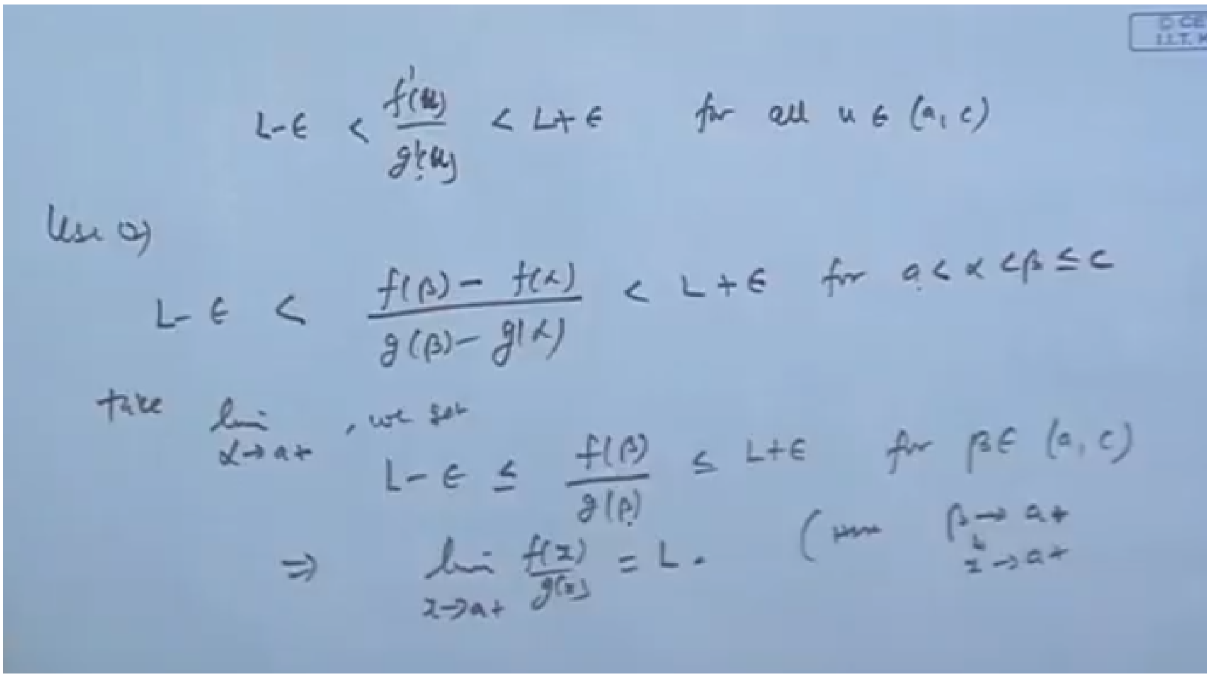

B such that f prime x, over G prime X, lies between L minus epsilon and L plus epsilon, for all u let us be this u, so let us take the all u, okay or you for all u, belongs to the interval a, C, ok. We can find the neighbourhood, of right neighbourhood of a, where if we take any point, in that right neighbourhood of a, the this condition will be satisfied, okay. But a prime u, over G prime u, we have already find out from there is it not? So from two, we have discussed this, from two, F prime over G prime is this, so substitute this value, so use two. Hence from two, we can say L minus epsilon, is less than F beta, minus F alpha, over G beta, minus G alpha, which is less than L plus Epsilon, for a less than alpha, less than beta, less than equal to C. now take the limit take limit, limit as alpha tends to a plus, because a plus this limit will be 0, is it not so this limit will be 0, this will be 0, so what we get is, so we get an L minus epsilon, is less than equal to when you take the limitingly equal to sign may also come, so this is less than equal to epsilon and this is true for all beta belonging to the interval a, C. therefore this shows the limit of F be FX by GX, when X tends to be because this limit is it not, when this one is there so limit beta tends to a, so when you take X, in place of beta I am taking X, so X tends to a plus, is also because here let beta approach to a, from the positive side, that is equivalent to say that X approach to a plus we are X belongs to this interval so this sort okay.

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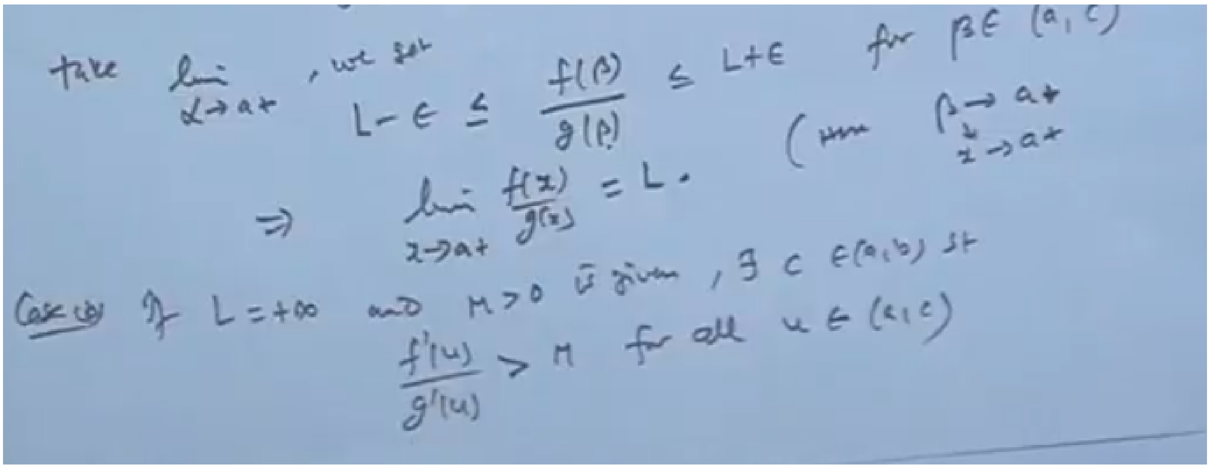

The second case, when L is plus infinity, L is plus infinity and if M we choose greater than zero, it's given because the limit is given to be L, which is infinity. So what do you mean by this it means that there exists a C, belongs to a, B, because this is given this is given, this is suppose infinity it means when point is closer to a, in the right in the neighbourhood right neighbourhood of a, then this limit can this value can exceed any positive number, so I am taking m to be greater than zero then a number you can be obtained, they'll exist C such that for all u this condition is satisfied, is greater than M for all u belongs to a, C, okay and but since epsilon is arbitrary number.

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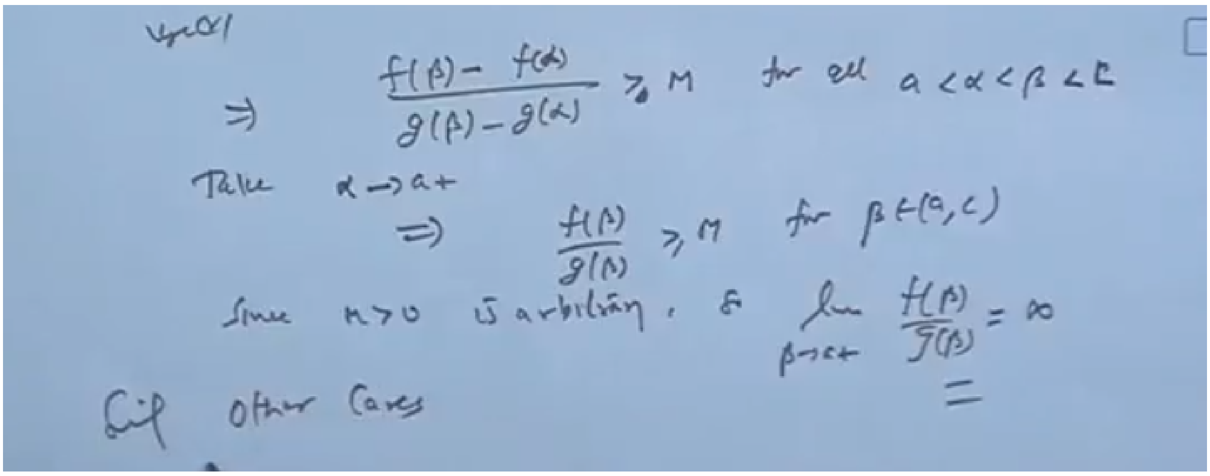

this sorry then from here this implies f of B this implies again substitute to so f of beta, minus F of alpha, divided by G of beta, minus G of alpha, is greater than equal to what M, is greater than m strictly greater than M, for all alpha and beta, lying between this bond. Okay in that right neighbourhood of a. so take the limit as alpha, tends to a plus and immediately we get the F beta, over G beta, is greater than equal to M, for all betas, belonging to A,C and this shows since M is greater than 0, is arbitrary, arbitrary large number. so limit of this F beta, over G beta, when beta tends to a

Plus, will be Infinity, infinity and that's what. Other cases follows in a single similarly we can so for others cases results here okay.

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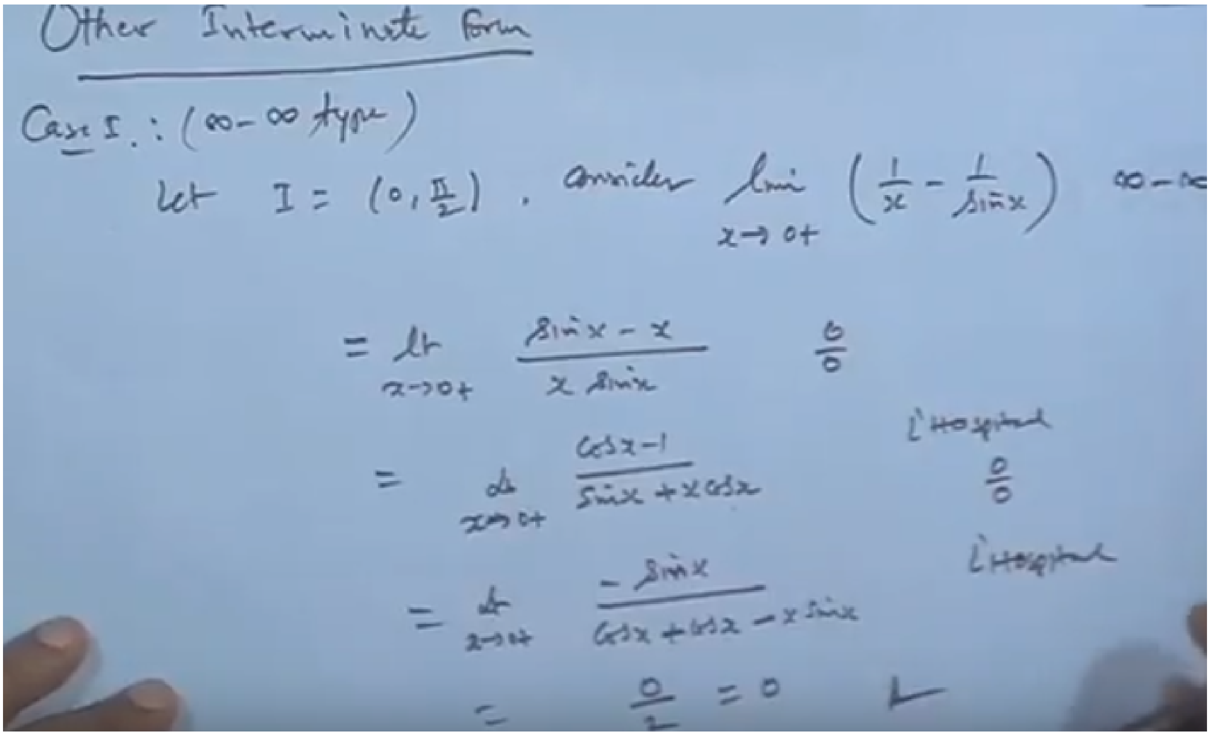

So let us take the other determinant forms other in determinant, determinant forms. so case by case, let us take say first cases is, infinity minus infinity, I will give by example seeing that in case of this we are infinity minus infinity type. okay so say for example and let I, is the interval 0, PI by 2 and consider the limit of this as, X tends to 0 plus, 1 by X, minus 1 by sine X, so when you take X tends to 0 it is of the form infinity minus infinity, so how to solve find out the value, what we do is we take the LCM so limit X tends to 0 plus, this is the same as X sine X, minus sine X, minus X, now it reduces this is infinity minus infinity now it is 0 by 0, so apply the L'hospital, so when we apply the L'hospital, differentiate the numerator and then differentiate the denominator, so we get sine X, plus X cos x and take the limit as X tends to 0 plus, if this limit exists then limit of this will be the same, is this now X tends to 0 it is 0 again 0, so it is again 0 over 0, so on so further apply the L'hospital rule. So when you take there is because all the FX and GX satisfy the conditions of L'hospital rule. So we get X tends to 0 plus, cosine X, is minus sine X, divided by say cosine X and then X, so again cosine X, then minus X sine X, so when you take the limit of this what you get is X tends to zero, so that comes out to be now zero over two, so this is zero, answer is this. Okay.

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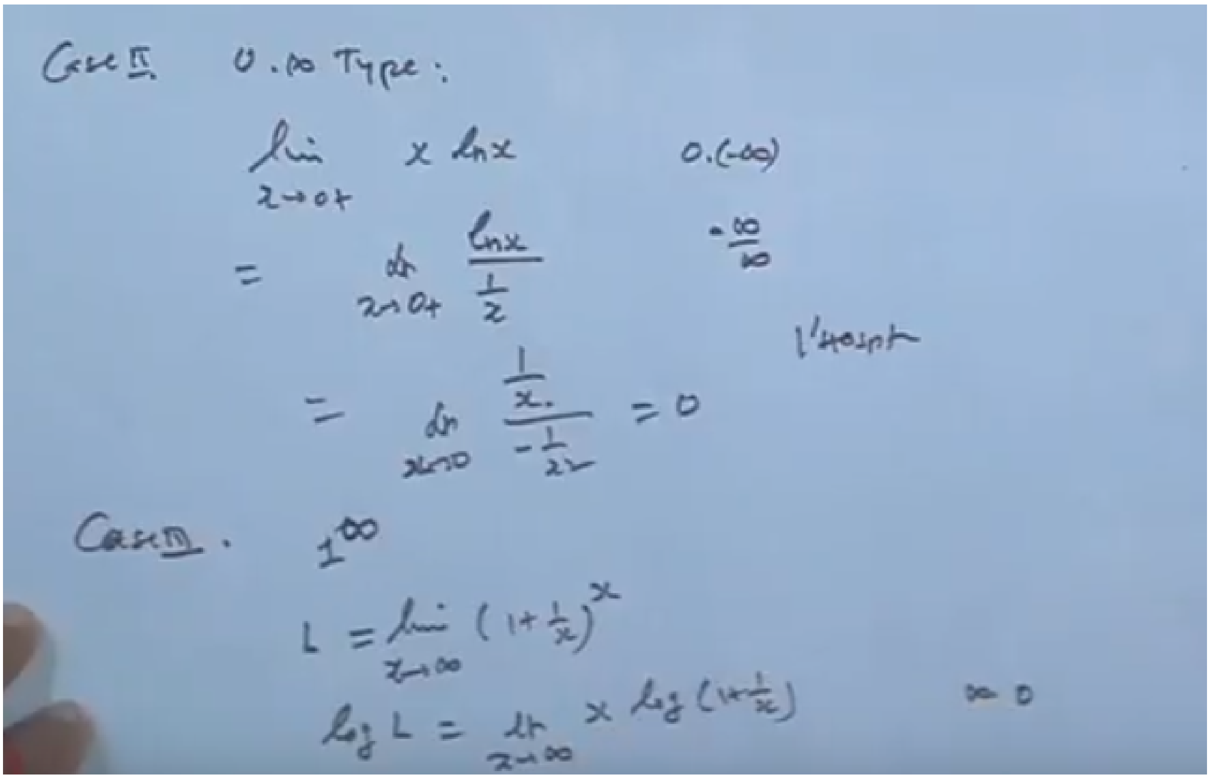

second case is, say 0 into infinity type, suppose I want to take the limit of this, X tends to 0 plus, X of Ln X, so when it's X tends to 0, it is 0 into minus infinity that is 0 into infinity type, so how to get it this one is, what we do is we put it this in the form of 0 over 0, or infinity over infinity, so if I take the Ln X divided, by 1 by X as X tends to 0 Plus, then it is infinity by infinity Type. Ok of course minus infinity makes notice a difference, then apply the L'hospital rule now. So once you applied L'hospital rule 1 by X, and here denominator we get the minus 1 by X square, as X tends to 0 and then when it comes up you get the value 0. so it is coming to be zero. Third case, when it is of the form, say 1 to the power, 1 to the power 0, 1 to the power 0, for example suppose I take the limit of this 1 plus 1 by X and X tends to infinity power X 1 to the power infinity sorry X tends to infinity, ok 1 to the power infinity, so it is when X tends to infinity, this is 1 this infinity, so infinity to the power infinity, so what we do is we consider this limit L, take the log, so when you take the log it is comes, because limit is a continuous function log will go inside and we get log 1 plus 1 by X, and X tends to infinity. so basically when X tends to infinity is infinity into zero type.

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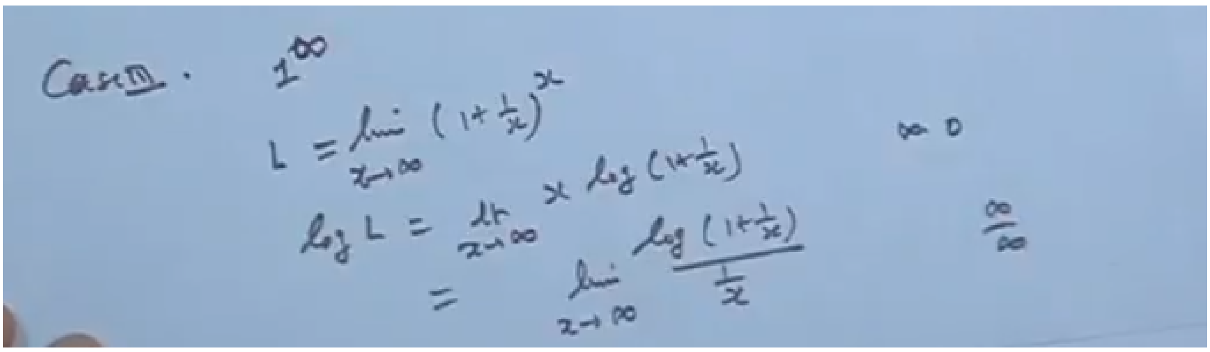

So this can be put it again either zero over zero,or infinity over infinity type. so we can put it this as 1 plus 1 by X, divided by 1 by X, limit X tends to infinity, that is infinity over infinity type and then apply now L hospital rule.

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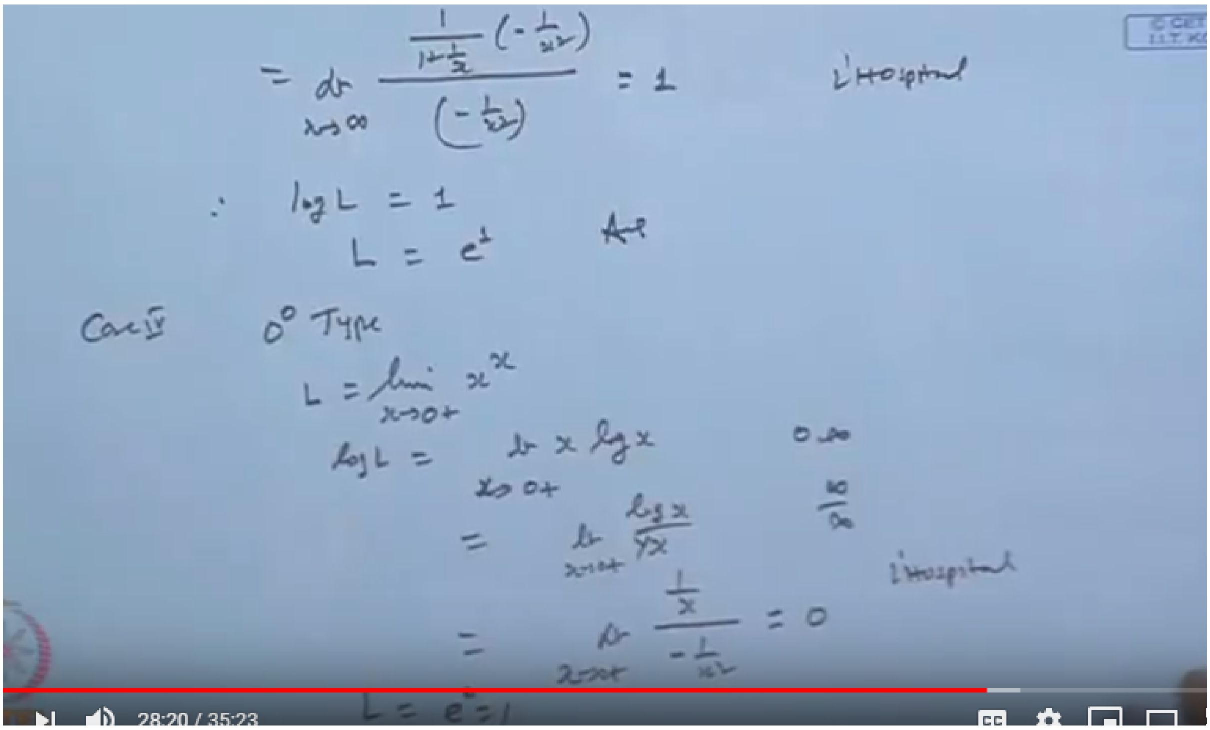

So if we apply the L hospital rule, what we get differentiate the numerator, numerator differentiation will give 1 over, 1 plus 1 by X and then derivative of this will be minus 1 by X square and then minus 1 by X square, so and then take the limit as X tends to infinity. So basically this limit comes out to be one. so since log of L is ,1 is it not log of L is 1, therefore so log of L is 1, therefore antilog of this will e to the power 1, that is e so answer is basically L, e to the power one. Then next case is for when it is of the form say 0 into infinity, 0 into 0 type, 0 power 0, type, so for example, if we take the limit X to the power X, when X tends to 0 plus, so it is 0 to the power 0, again in this case take the L log and when you take log, this problem is reduced to the case when it is 0 into infinity type. so just like a previous one we can take it log X, by X as X tends to 0 plus, then apply it is infinity, sorry 1 by X, so infinity over infinity, now apply the L'hospital rule. So when you apply the L'hospital rule differentiate the numerator and denominator, separately and take the limit within X tends to 0 plus and this limit will come out to be what 0, so the L will come out to be e to the power 0 that is one answer.

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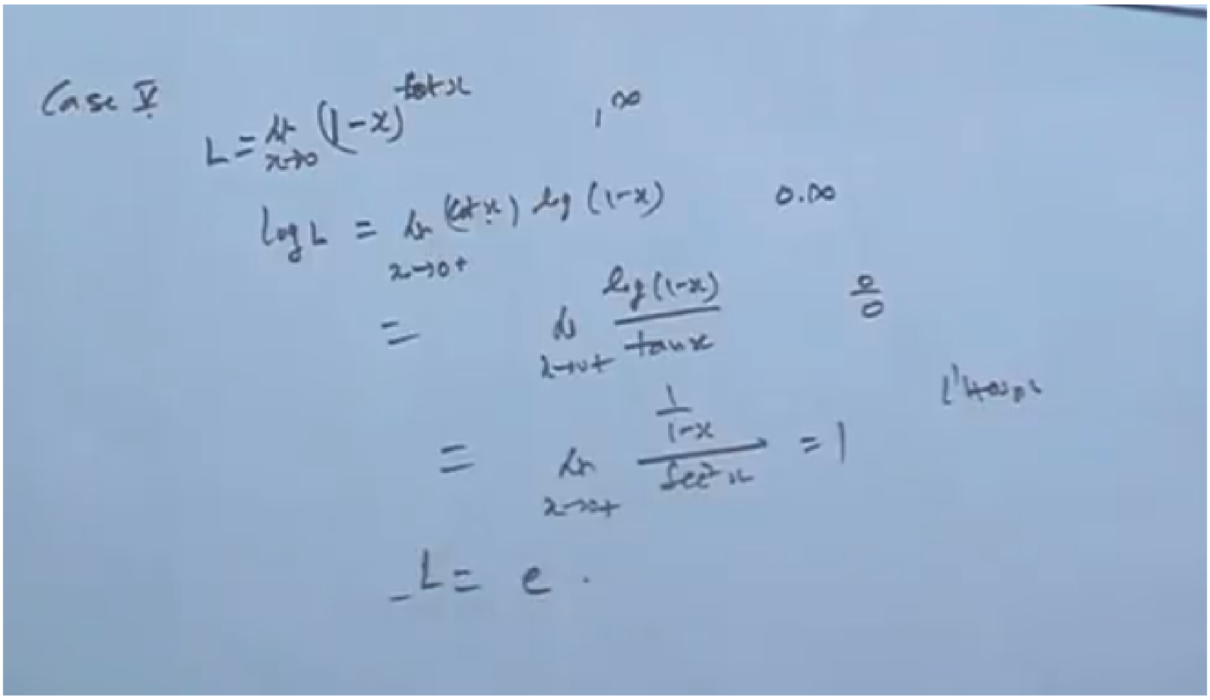

Then another case fifth, which is of the form say, 1 to the power, say again what form left now, infinity minus infinity we have taken, and then let's take another case is, limit of this X tends to 0, 0 in to minus infinity, X log X, that we have already consider, so I think this is all forms we have discussed you know let's see, okay. So we can say another form infinity and let's take one more example, thence what we want suppose I take 0, say 1 minus X, X tends to 0, okay, into say 1 to the power infinity, so let us take the tan x, no, cot x and X tends to 0. So what this is the 1 to the power infinity, okay. So what how will you do it? Take L and then log L, so Log L, will be equal to what? Cot x into log 1 minus X and X tends to 0 plus, so cot 0 is infinity, this is 1, 0 so 0 into infinity, then again you can take log 1 minus X, divided by tan X and X tends to 0 plus, so it is 0 over 0, apply L'hospital rule and we get from here is, differentiate numerator, differentiate denominator, take the L limit, when this goes to 0. so when you take the 0 upper limit comes out to 1 and this is fake is also 1, the limit will be 1. So L will be equal to e to the power one, answer will be this. So that's what is so. So it's almost we have completed this theorem,

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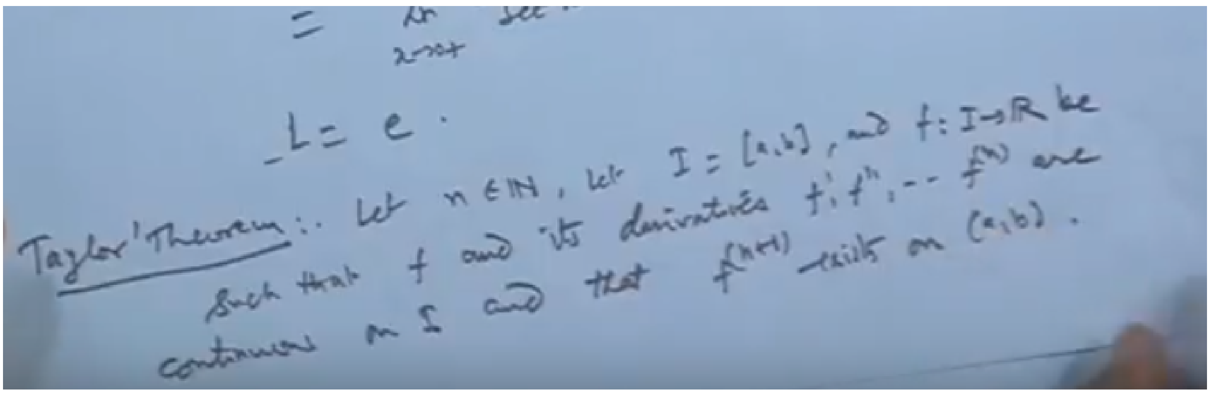

Now let's come to the Taylor’s theorem. The Taylor’s theorem it's an extension of the mean value theorem, mean mean value theorem we give the reason between the functional value and its first derivative whether it is a question La Grange is mineral rah FB - FA or B - a derivative of the function or maybe the Cauchy mean value theorem, we are the two functions F and G, are there and then we get the relation between F, G and their derivatives. Taylor's expansion Taylor's theorem is an extension of the mean value theorem we are behaving rationed between F and it's higher order derivatives and we basically used to approximate the functions with the help of the first few terms of the Taylor series, or Taylor's theorem. What this theorem says is let n belongs to capital n, okay and let I is the closed interval a, B, and let F is a mapping from I to R, be such that, be such that, F and it's higher order derivatives, derivatives, say up to order n, F prime, F double prime, up to FN the Nth order derivatives up to are continuous, are continuous on the interval I and that the Nth plus 1, at derivative Exists. On the in open interval a, B,

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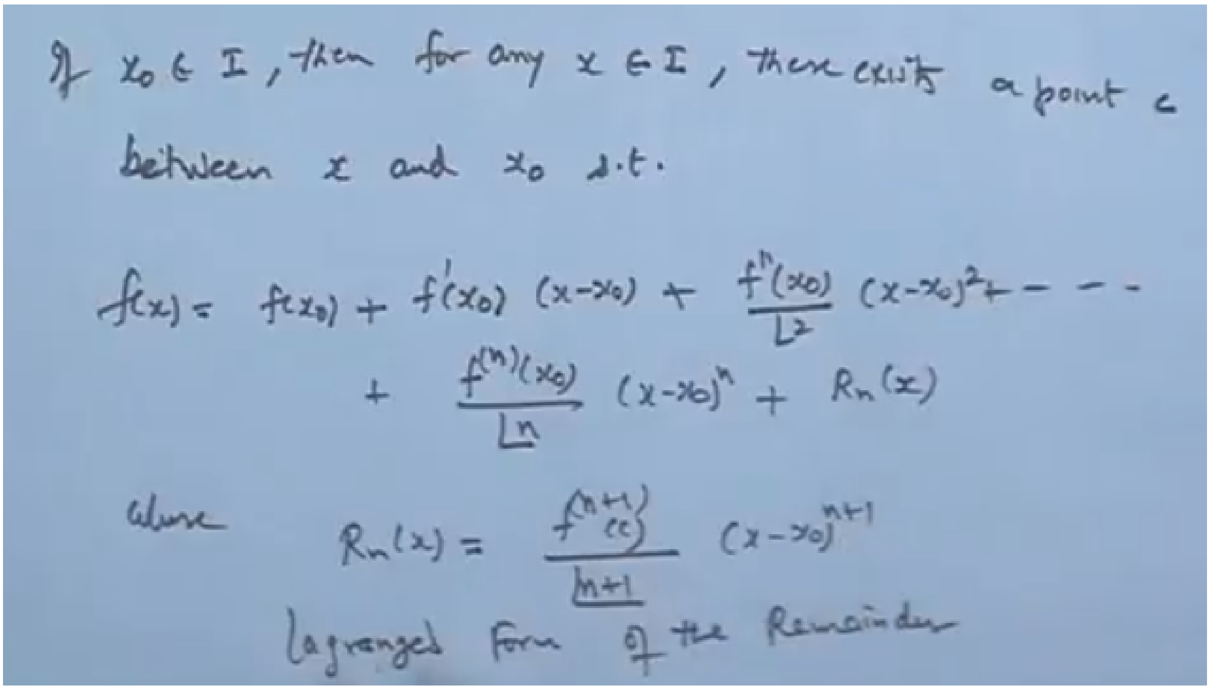

Now if, if X naught, is a point in the interval I, then for any x, belongs to I, I there exist a Point C, between X and X naught, such that F X can be expressed as FX naught plus, f prime X naught, into X minus X naught, plus F double prime X naught, over factorial 2 X minus X naught whole square, and so on and then plus F n, at the point X naught, divided by factorial n, X minus X naught to the power n, plus the remainder terms Rn X, RN X where R n X, is given by FN plus 1 C, over FN plus 1 C, over factorial n plus 1, 1 into X minus X naught, to the power n plus 1, and this is known as the LaGrange form of the remainder. so thus function f is such which is differentiable and continuous, up to say n plus 1th order of time and n, n plus oneth order derivative exists, in the neighbourhood of the point X naught, in the neighbourhood of the point X naught, then there exists some exists in the open interval a, B of course and the X naught, then there exist a Point C, where the expansion of the function FX, can be written in the form of the series, where the first n terms, of the series is the polynomials of degree N and the this term is called the remainder term and it is known is the

Lagrange's form of the Denominator. Now in case if the remainder are n goes to 0, then this same expansion is known as the Taylor series, expansion for the function FX. Okay so we will discuss it next time the proof of this.
Thank you much.