Module – 11

Lecture 62

Application of MVT & Darboux's theorem

Okay, sting example, as we discuss it, okay. Now let us come to the, other applications of main value theorem.

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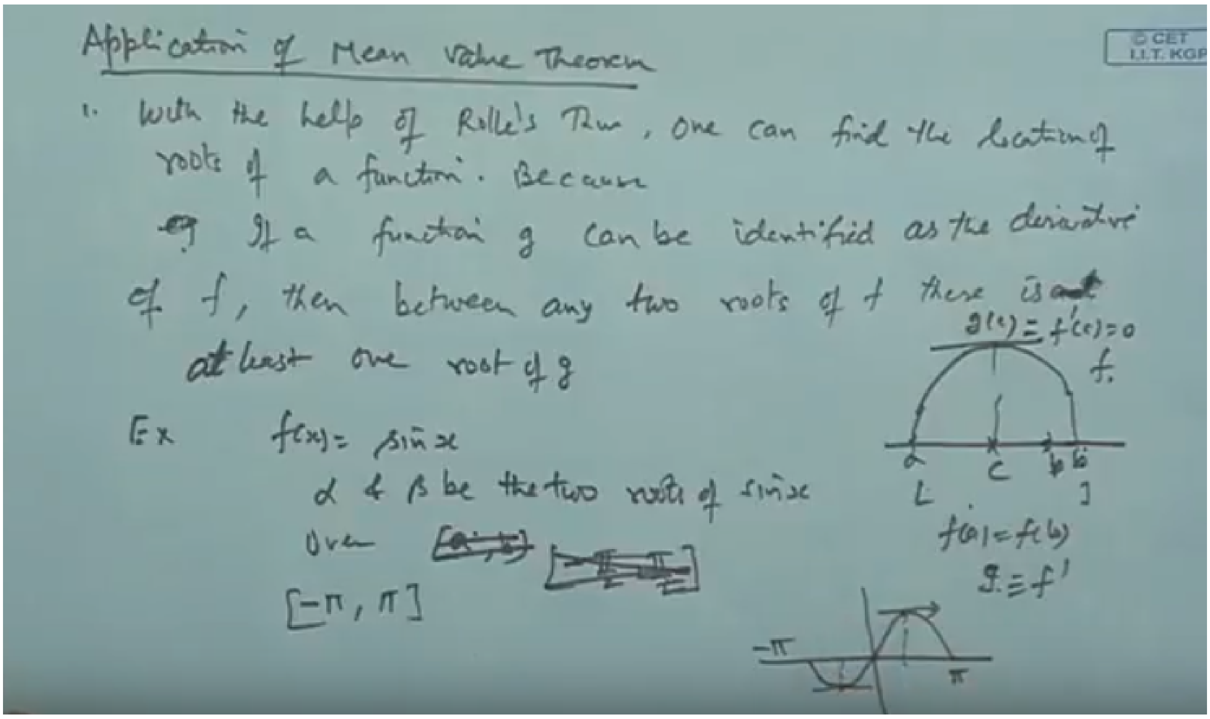

So first, the Rolle’s Theorem, with the help of Rolle’s Theorem, one can, one can find or one can find, the location, of, location of roots, of a function, of a function. For example, for example, for if a function, if a function, the reason is, because, sorry, because, if a function G, can be identified, can be identified, as the derivative of, as a derivative of F, another function f, then between, then between, any two roots, any two roots, of F, there, there is, there is at least, there is, at least, there is at least, there is at least, one root of G. Because, what the Rolle’s Theorem says, if a function, f, G, between any two roots of F J, okay. If a function f such, that f of, this is interval a B, F of a, equal to, say F of B, this is B. Okay? This is B, so this is the function f. If the function f, we say continuous over the closed interval a B and differentiable over the open interval a B and at the point a, both fa and FB is 0?

Then there exist a Point C, in between, where are the derivative of the function, f prime C, is 0. It means, if I say, G is a function, which is equivalent to its derivative, so between any two roots of a b, function f, there is a root of G. Because this is equivalent to G C, so that's what. For example, if we look the function FX, which is say sine X. Okay? Then let alpha and beta be the two roots, be the two roots, of this function sine X. The sine over the interval, say any Interval I take, say between the interval, say, minus say a to b, just a to b. Okay? Or let us take the exact root, because this, sine becomes 0, is 0, sine pi is 0, so between minus, say, over the interval, minus PI by 2, to PI by 2. Let us take this interval. Okay? PI by 2 minus PI by 2, it will not help, because 1 will be 0, other than it. So don't take it. Let us take the interval, say minus PI to PI. Let us take this interval. Okay? So this is the function, sine 0 is 0, sine PI, then minus PI, so pi. So the function f is such, which is continuous and differentiable over this interval.

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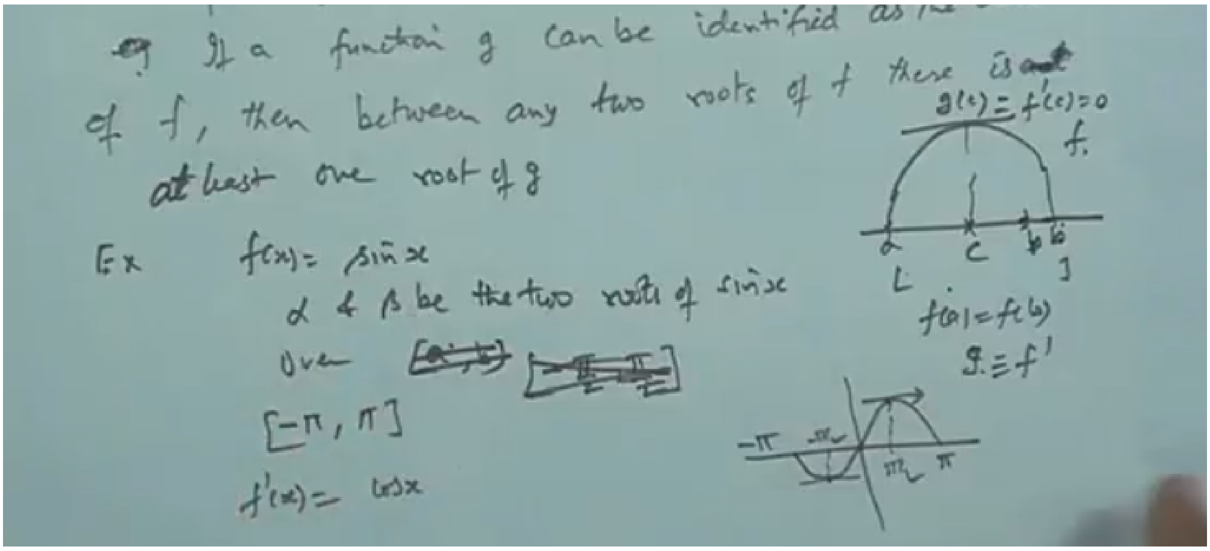

Now there exists at least one root. Obviously this is one of the point, where the derivative Vanishes and you have this another point, where the derivative vanishes. So what is the derivative F prime X? The F prime X is cos x, so cos x vanishes at the PI by 2 and then minus PI by 2. So it shows, that between any two roots of the sine X, there is a root of cos X. And vice versa also, if I start with cos X, then it's derivative is, sine X, minus sine X and forth, then again, between any two roots of cos X, there is a root of sine x, so one can locate, the roots, of the function, with the help of Rolle’s Theorem.

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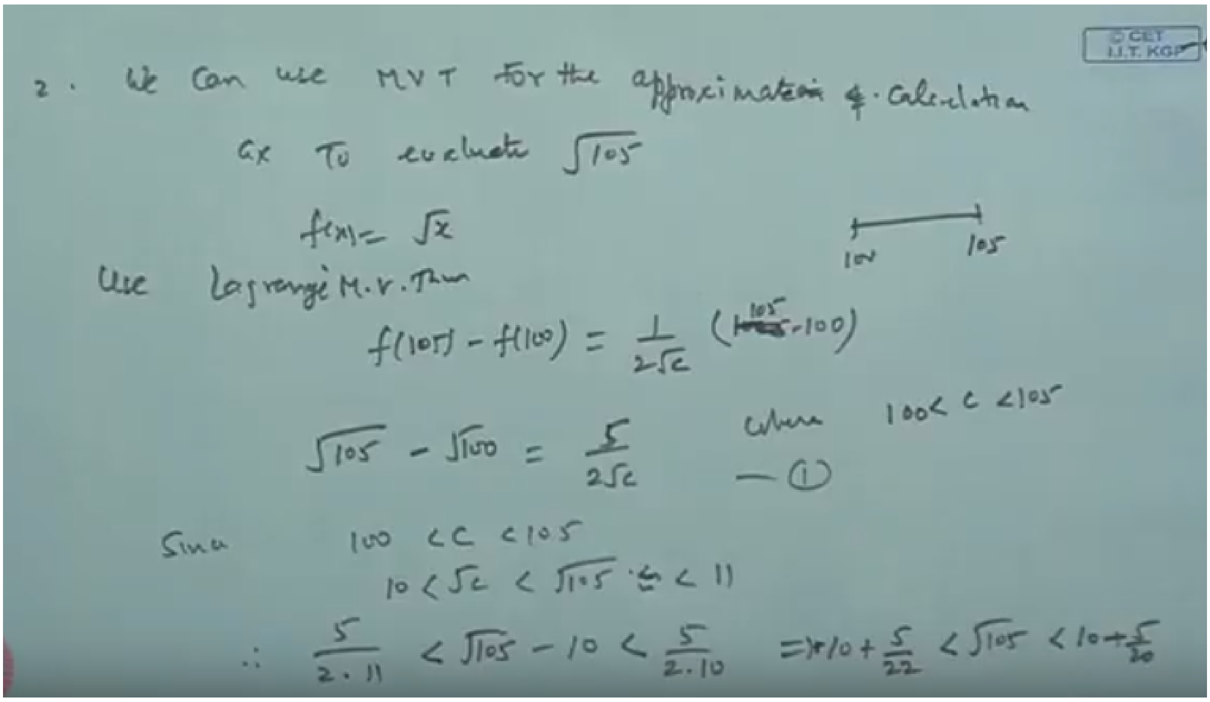

The second, we can also use the main value theorem, to approximate or for the approximation, for the approximation, of the roots, for the approximation, of some number, say approximate calculation, you can say, for approximate calculation. Let us see how? suppose we wanted to find, to evaluate this, evaluate, under root, say, hundred five. Okay? Now if I look this, take the intervals suppose 100 to 105. Okay? And the function FX if I take to be root X, then apply the Lagrange's main value theorem, Lagrange's main value theorem. What we get is f of 105, minus f of 100, is equal to the derivative of this function, at some point C, into the length of the interval, 105 minus 100, 105 minus hundred, where C lies between, 100 and 105, where C is this. So what is this is? This is equal to hundred five, minus hundred, this is equal to 1 by 2, root C, into 5, so 5 by root 2.

We can approximate roots 2 now. So since our C, lies between 100 and 105, so root C will lies between 10 and under root 105, this is almost approximately or less than, say 11, less than 11. Okay? So it means, this value, so what we get. Therefore, under root 5, 105, minus 10, will lie between, these two bonds; the bond will be 5 over 2, into 11 and 5 over 2, into 10. Okay? Therefore, the root 1 0 5, will lie between, this 10 plus, 5 by 22, and less than, less than, 10, this is 10, so minus 10, minus 10 here, because this will go from here and this will come, so 105 with this and this, minus, say 10 plus both sides, so this is 10 only and then, it will be 5 over, 10 plus, 5 over 20. Now this is an approximate value for this, so one can get the value accordingly. Okay? So this is another replication of this.

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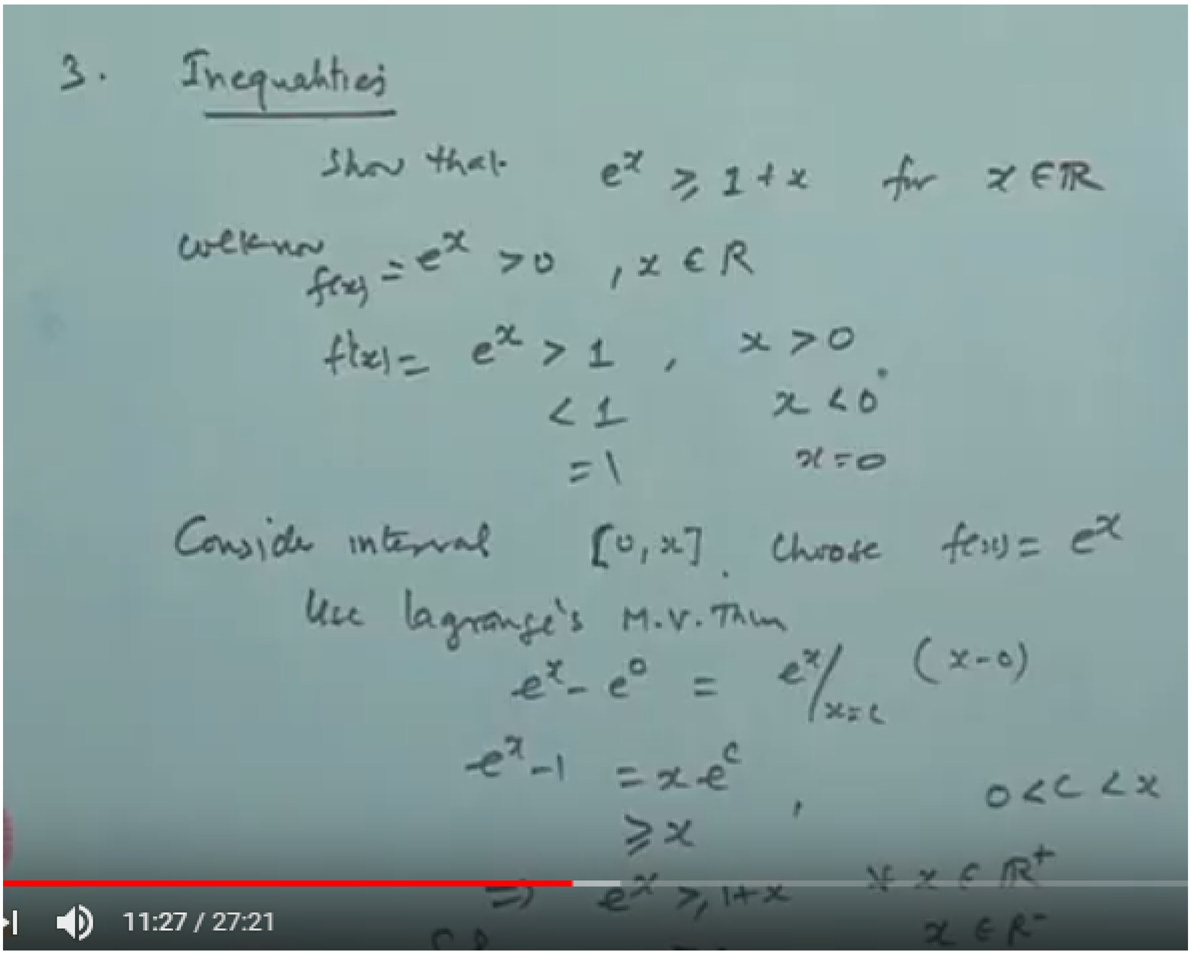

Then inequalities can also be established, with the help of this. The inequality is suppose, I want to establish, show that, e to the power say X, is always be greater than equal to 1 plus X, for X belongs to R. Okay? Now, we know, that e to the power x, is always be greater than, is positive quantity, is always be positive quantity, when X belongs to R. Is it not? 1 plus X, ll fx, in negative 1 by X is. And its derivatives, and its derivative, F prime X, is e to the power x, itself. So F prime X will always be greater than 1, when X is greater than 0 and is less than 1, if X is less than 0, because it is 1 upon e to the power and for the X equal to 0, it is 1. Okay? So 1 occurs if one. Clear? So this will be the solution for. Now to estimate this result, consider the interval, consider the interval, say, we take the interval, 0 and X, this closed interval and choose the FX, as, e to the power x. Now this function is continuous, differentiable, in the interval inside the interval 0 x.

So by the main value theorem, Lagrange's main value theorem, the value of the function at the endpoint, is equal to the, derivative of the function, X, at the point X equal to C, into the length of the interval, X C. So we get what? E to the power X minus 1, is equal to, e to the power C, into X, where C lies between what? C lies between 0 and X, C is greater than 0. So e to the power C, will be greater than 1, so it is greater than or equal to X. At the most 0, when C is equal to 0, it is 1, so it is there. So this shows. E to the power X is greater than equal to 1 plus X for every X belongs to R Positive, now similarly we can so for X, belongs to are negative it is 2, greater than equal to this hence for all R it is 2. So this can be shown. So this also inequality we can use it in main value theorem.

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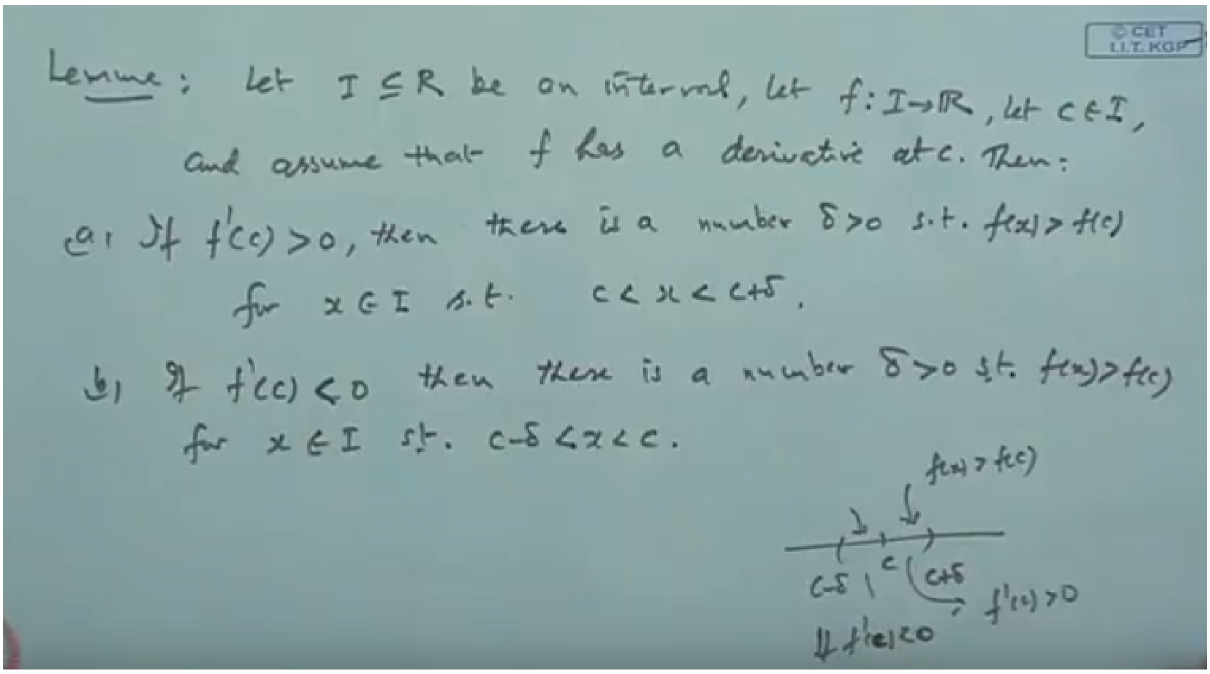

Now there are certain Darboux’s theorem we required the Darboux’s theorem, okay. So in order to prove the Darboux’s theorem, we first need this lemma, prove we can skip because let R, I, which is subset of R, be an Interval and let f is a mapping from I to R, let's C is a point in I and assume, that F has derivative, has a derivative, derivative, at C. then this lemma says if derivative of a function at the point c is strictly greater than zero, then there is a number delta greater than 0, such that, such that, f of X, is greater than f of C, for X belonging to I, such that it lies in the right hand side of the interval, of the neighbourhood of C and if, if F prime C is negative then there is a number, Delta greater than 0, such that f of X, f of X, will be greater than FC, for X, X belongs to I, such that C minus Delta, less than X, less than C. so what this is says is, suppose F is a function and C, I is a interval, C is a point, belonging to this and function has a derivative at this point.

Now if the derivative is strictly greater than zero, then what it says is, we can always find a right hand side neighbourhood of this C, where the function will have a positive value, where the function FX will be always be greater than the value at the point C, so here in this interval the function FX, will always exceed the value at the point C, it mains that F of C, will behave at the lowest point in this interval, if derivative is positive and if this f of X is odd and if derivative is negative, then in the left hand side of this neighbourhood, the value of the function will always be positive, if here. So here if the derivative is negative at the point C, if the derivative at a point C is negative, then we can get this in neighbourhood left-hand where the function is exceeding the value of C, if derivative comes out to be positive, at the Point C, then we get the right hand side interval we are the function value exceeds the value at the point C. So using this lemma we can prove the Darboux's result. In fact the proof of this is very simple,

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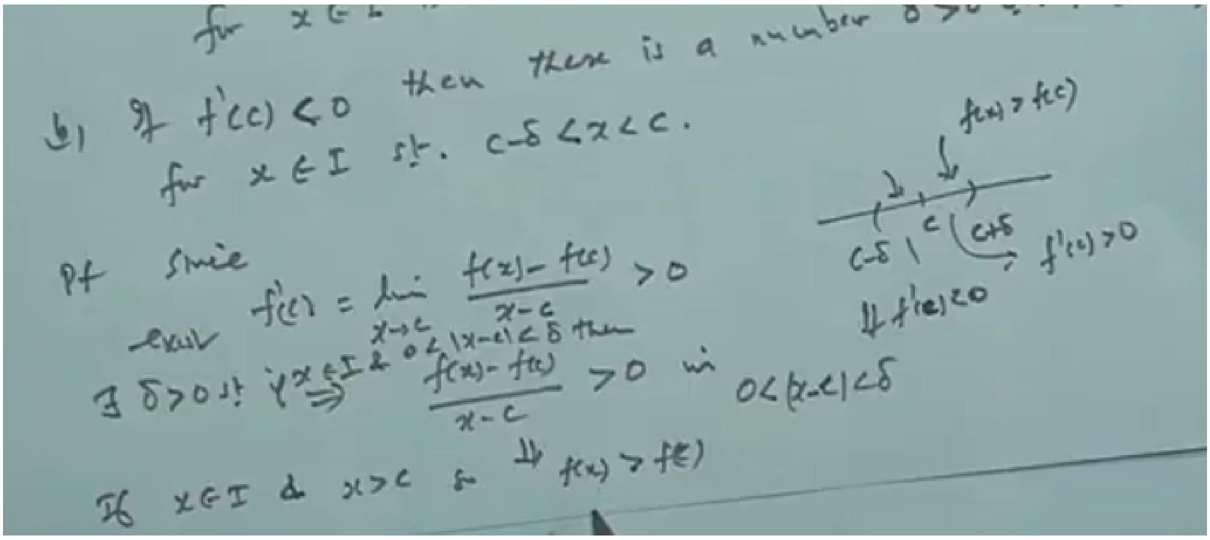

Since the derivative exists since this derivative F prime C exists, which is the limit of this FX, minus FC, over X minus C, this exists and it is given to be say greater than zero. Now when the limit greater than zero X tends to C. then obviously this entire thing must be positive. so this is only possible when limit is strictly greater than zero, it's only possible when the this quotient should be greater than zero, otherwise if it is negative it cannot get the limit to be strictly greater than 0, but X is, if I choose X, in the interval, for so there exist in place this is greater than zero, in some neighbourhood 0, less than X, less than mod X minus C less than Delta, ok so mains there exists, if it is greater than zero, then there exists or Delta greater than 0, such that if X belongs to I interval and 0 less than mod X minus C, less than Delta, then this horse, okay so if X satisfy the condition, if X belongs to I and satisfy the condition, X is greater than C, then from here we get implies, so F X must be greater than F of C, otherwise this will be negative. So that proves the result for the first part, similarly for the other. Ok, so this we can see, now Darboux's theorem which is,

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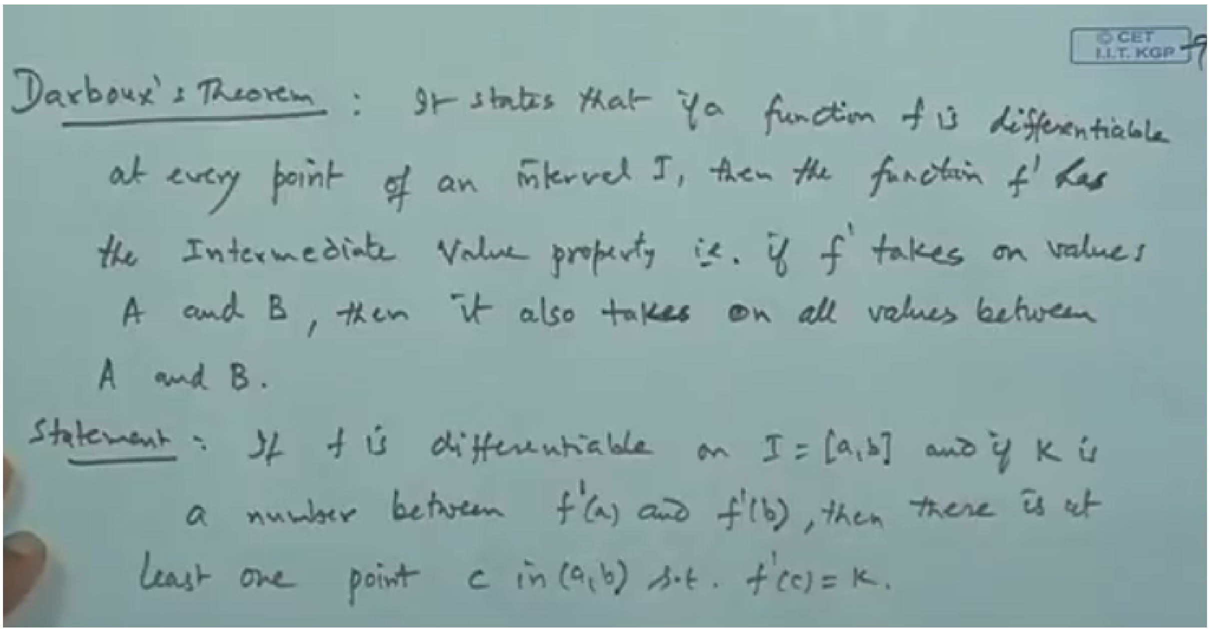

Darboux’s theorem, basically what is the Darboux’s theorem is? it states it states that if a function, if a function f is differentiable, function f is differentiable, at every at every point of an interval, say I, then, the function f prime, that is the derivative of this function f prime, has the intermediate value, as the intermediate mediate value property, it mains that is the meaning of this is that is if F dash takes, if F that takes on values, a and B, a and B, then if it then it also takes it also takes on, takes on, all values between a and B, okay, so this is our, so exact statement is, we can see exact statement, a if F is differentiable, on the interval I, is close and bounded interval and if K is a number, K the number between, F prime a and F prime B, then there is at least one point C, in the interval a, B, such that the derivative of the function at the C scale. it mains if F is differentiable throughout this, mains derivative of the function exists, including the derivative at the endpoints, then if I picked up any arbitrary number, any number which lies between F Dash A and F dash B, then there will be at some point, where the derivative of the function, at that point will coincide the numbers. Okay so it can attain all the values in between a and B which is taken like this. Okay so missing maximum and minimum value okay.

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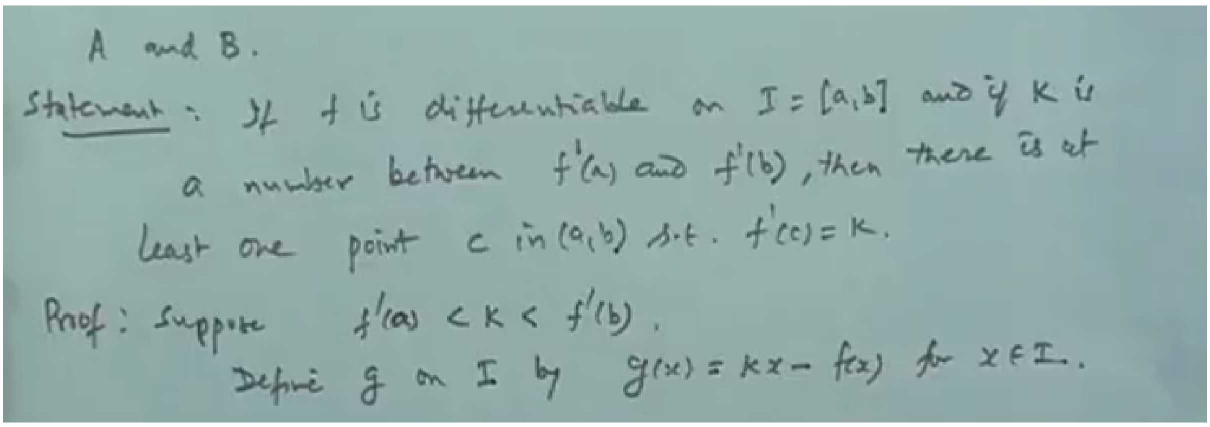

so proof is, like this suppose that, K lies between suppose K lies between F Prime A and F prime B, ok now we define the function G, on I, by G X equal to KX, minus FX, for X, belonging to I. Let us define this now since function f is continuous and differentiable, so G is also continuous and differentiable.

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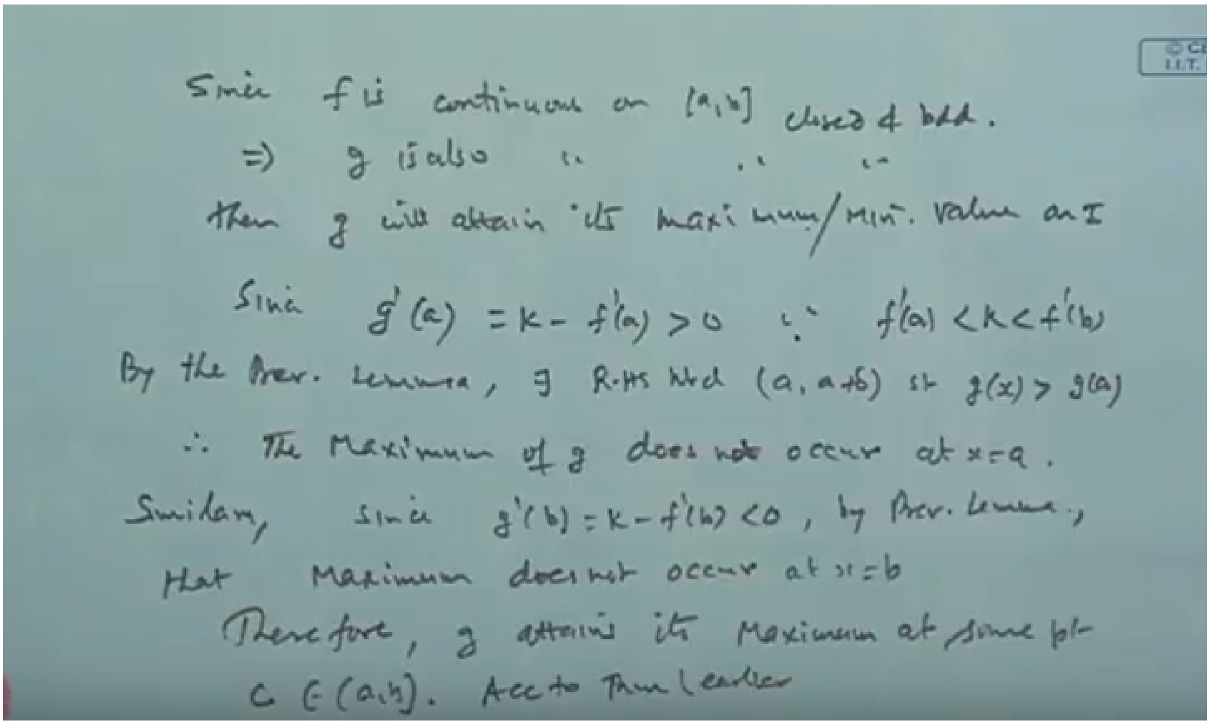

So since F is continuous, on the closed and bounded interval, this is closed and bounded interval, so this shows G is also continuous on the closed and bounded interval, so if it is continuous on a closed and bounded interval, then it will attain its maximum value on the interval. Then G will attain, its maximum value, similarly minimum value, also maximum or Minimum value, maximum and minimum value, of course on I, so here we recall the maximum and minimum, when they say less than. Okay so we get this I. now since our Derivative G prime a, that is by definition is coming to be what? K minus f prime A, this is by just differentiated and this G prime a, is positive. Why? because our chosen K, lies between F prime a and F prime B. so this is positive, this is positive. now it follows from the previous lemma, so by the previous lemma, what he says, if the derivative is positive, then there will be a one neighbourhood, is it not? so if the derivative is positive, then there is a number Delta, such that in the right hand side of this neighbourhood the function FX will be greater than this. So we get from by the from the previous lemma, we can say that the maximum of G, well there exist a right hand side neighbourhood, say a to a we can say a to a plus delta, such that the derivative the functional value G of X, G of X, is greater than G of a. therefore the function G, cannot attain the maximum value, at the point a.

So therefore the maximum value, maximum of G, does not occur, does not occur, at the point X equal to a. because it already violet if it is measure it must be GX must be less than equal to G a, which is not true, not occur. Similarly since G dash of B, that is equal to K minus a prime B, is negative. So again by previous lemma, again there is a left hand side where the value will be greater than the less than this, so again this source that G by the previous lemma, that maximum does not occur, that minimum that's not occur, maximum does not occur, at X equal to B. so neither it occurs at X equal to a, nor at X equal to B. Therefore, but maximum should be attained because it is continuous over a closed and bounded interval, therefore G will attain, G attains, its maximum value maximum at some interval, at some point, some point C, which lies in the interval a, B. Okay, now again a function G, is such which has a derivative and maximum is attained at some point, so according to the theorem according to previous we have discusses, according to the theorems, earlier, earlier we have seen that

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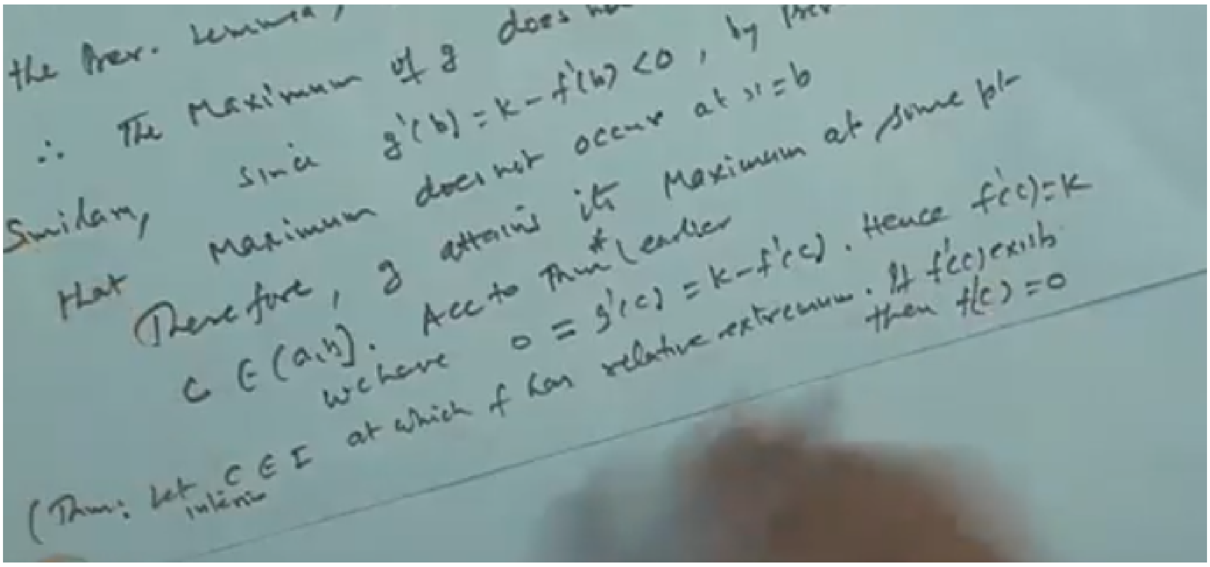

There exist a point, we have zero, which is less than equal to G dash C, which is K minus F prime C, and hence F prime C, is K. According to the theorem, this theorem I am putting star and this star, it's we can say this theorem, is there the theorem says which is interior, theorem that let's see be an interior point of the interval i a interior point and then at which the function has a relative extreme at which F has relative extreme extremum then if the derivative if the derivative of the function at this point C exists then the value of this must be 0 this was the result which we have already proved so using this result we get this therefore this come okay so thank you very much we have not covered the hospital's next lecture we will do it.
Thank you very much.