Module – 11

Lecture 61: Applications of Derivatives

Okay, so we have already discussed the Main Value Theorem, we will just type, main value theorem. Logran Changement Value Theorem, Rolles Theorem, then generalized main value theorem that is Cauchy main value theorem and also the few applications of the derivatives. And the most interesting application which we have discussed is the monotonic character of the function, just by looking the sign of the derivatives. Okay? And we have seen if over the interval derivative, is greater than 0 positive, greater than equal to 0, then the function will be, greater than equal to 0, then function is monotonic increasing function, if it is less than equal to 0, the function is a monotonic decreasing function. So in this lecture we will give few more applications, of the derivatives, as well as, our mean value theorems.

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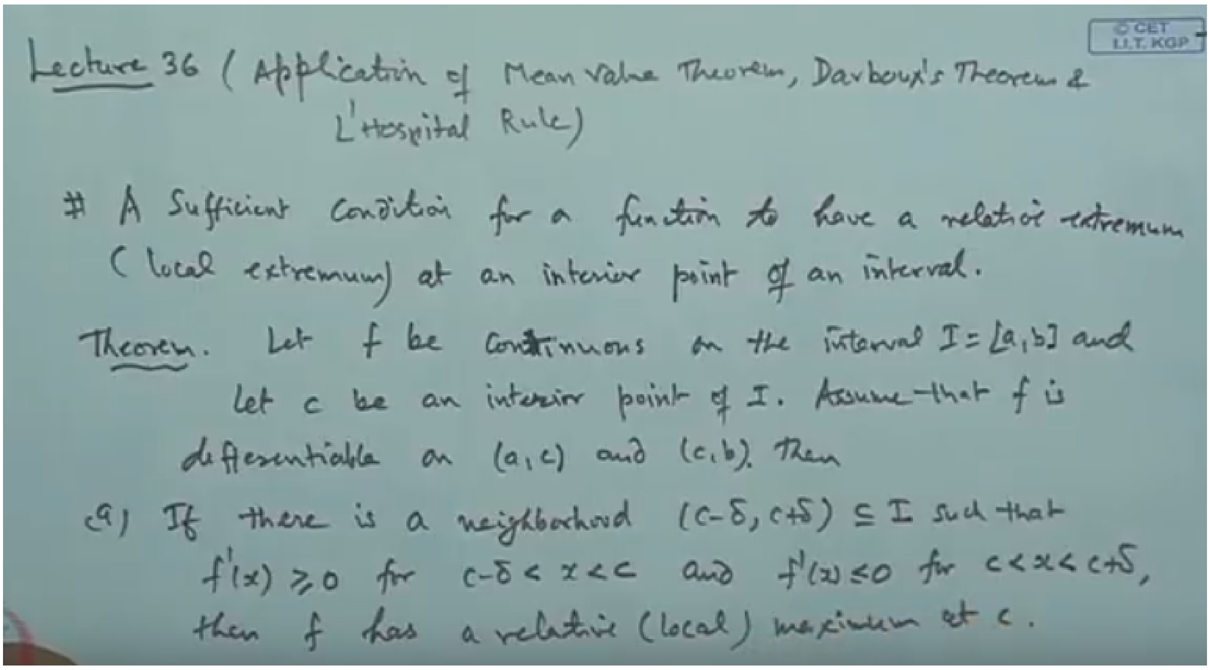

Okay, with the help of the sign of the derivative, near a point X, in some neighbourhood of the X, one can identify, whether the point corresponds to a maximum point, relative maximum point, relative minimum point or maybe none of them. So this is the criteria, of a sufficient, sufficient, in fact the sufficient, condition, condition, for a function, to have, relative, relative, extremum ,that is local, local extremum, extremum, at an interior point, at an interior point, of an interval. These are the sufficient condition. Of course the result says which is also known as the, ‘First derivative test’, for extremum. Let F be, continuous, continuous, continuous, let F be continuous, on an interval, on the Interval, say, I, which is a closed and bounded interval. And let C be an interior point, interior point, of the interval, I. Assume that, that, F is differentiable, differentiable, on AC, open interval AC and open interval C B. Then, the result says, if their is, if there is, a neighbourhood, neighbourhood, say, C minus Delta, to C plus Delta, contained in I, such that, the derivative of the function f, that is F prime, X is greater than equal to zero, non-negative, for C minus delta, less than X, less than C. Means towards the left of this, is great positive, non negative and F prime X, is, less than equal to zero, non positive, for the C, in right-hand side of the interval, that is, C less than X, less than C, plus Delta. Then, then, the function f has a relative or we also use the word local, maximum, at C.

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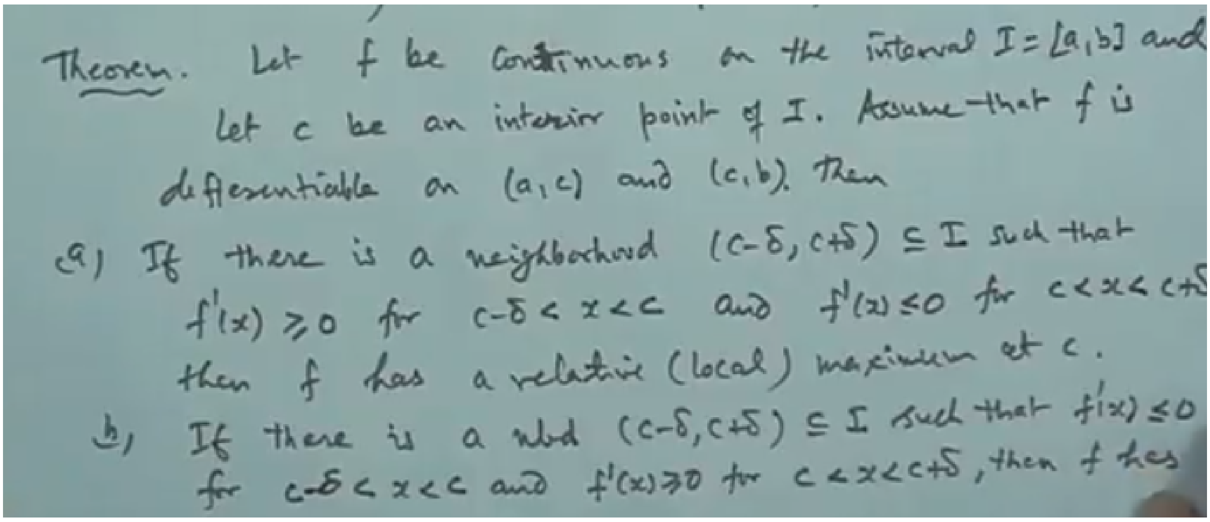

The second part is also, says, if there is if there is our neighbourhood, there is a neighbourhood C minus Delta, C plus Delta, in I, such that, the derivative of the function f is non positive, for the, left-hand side interval, C minus Delta, less than C and non negative, in the right-hand side, of interval of C, that is C my, less than X, less than C plus Delta. Then f has, a relative,

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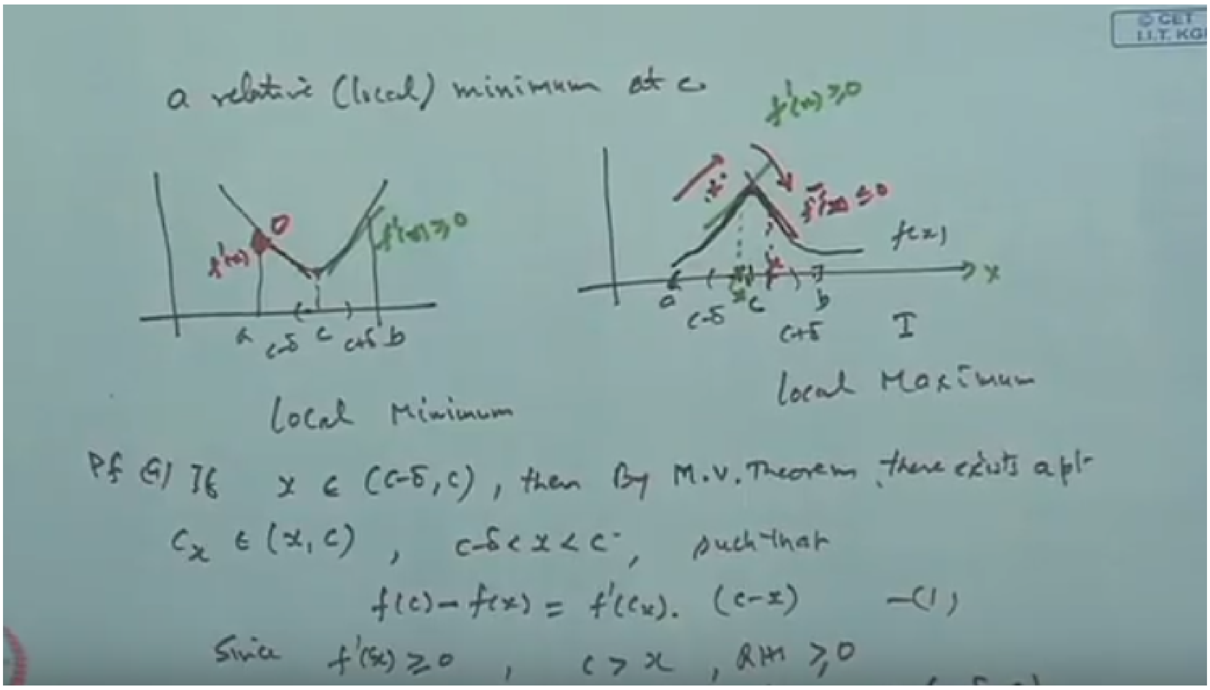

a relative or local we can say, local minimum, minimum at C. So before going to the proof, let us see, that geometrically what it is. Suppose we have, two cases, one is this, say C, is this interval. Okay? Now a function is supposed is this, suppose this is the function? Okay? FX. Now what the property says, f be a continuous function, on the interval a B, suppose this is the interval a B, where the function is continuous, continuous, on the Interval. And let us C be an interior point, in, of the interval a B, and assume that function f is also differentiable, on this, means, this interval a, a C and C B, the function is differentiable. About C we are not talking right now. Okay? So in this, the function is differentiable, in this function is differentiable, the function is throughout, over the closed interval, is continuous. And then what he says is, if there is a neighbourhood if there is a neighbourhood around the point, around the point, say C, C minus Delta, C plus Delta, this is the C minus Delta, it is C plus Delta. So in this neighbourhood, if the function behaves like this, if the derivative of the function, at this derivative of the function, at this point, is non-negative, while the derivative of this function, at this point, is non positive, then the point c, will correspond to the maximum point. Let us see what. So if the curve is like this, this is our curve. Okay? Now if we look this, suppose I take a point, here, X point. Okay? This is our X, now if we look this here, the derivative means, slow, F prime X, this is the denote the slope of the function, at the point X, on this. So if F prime X, is greater than equal to zero, so slope will be positive, it means, it makes an acute angle, with the axis of X. Now, when in this interval, if I choose, in this interval, if I take a point, here, then correspondingly, the tangent, if I draw at this point, the tangent will be something like this, the slope becomes obtuse. That is the derivative F prime X, is less than equal to zero. So it means, if, nearby X C, if nearby C, the tangents changes it's behaviour, from positive, to negative. That is, it is keep on coming here and then, as soon as it reaches to a point, where it has a local maximum it has a change, and the direction is changed. Is it not? And this change point, the point where it changes its direction, will correspond to the point of maximum.

The same case happens, in second part. If we have, say suppose, we have this curve and here is this a B and we have this Point C. So let us take a neighbourhood C minus Delta, C plus Delta. When we picked up the point below, in the left hand side of this and draw the tangent ,that F prime X, here, is negative, here this was positive and here it was negative, F prime X, is negative, less than equal to 0, means non positive, here this is non positive. And then, when it goes to here, it is positive. So here the F prime X, is greater than zero. It means, the tangent again changes time, from negative, slope changes, the direction, as soon as it is c, the point on the curve, corresponding to C, and this point, will correspond to the minimum point, so it is a local minimum for this. So geometrically we can explain these things. And let's see the proof. Analytical also we can prove it, that in case of one, it is a local maximum, while in case of the two, it is a local minimum. Okay? So let's see the proof of it. If first we prove as per, let if X belongs to the interval, say C minus Delta C, in the left hand side, so this is the figure 1, figure 1. Okay? In this, then, by main value theorem, by main value theorem, there exists, by main value theorem, there exists a point, exist a point, say X, in the interval X C, where X lies between, the, where X belongs to this, means X lies between, C minus Delta, less than this, less than, so I am choosing the interval C a, and X C, and applying the main value theorem. So there exist, a Point C X, in this interval, such that, such that, the F of C, minus f of X, is equal to the derivative of the function, at a point C X, into C minus X. Okay, this, one. Now since Our, let it be 1. Since it is given, over this side, f prime, X is positive, f prime X as CX, this is C X, is greater than or equal to zero, it's given. Now C is already greater, C is greater than X, because of this Interval. So the right hand side, right hand side is positive, greater than or equal to zero. Therefore, left hand side has to be positive.

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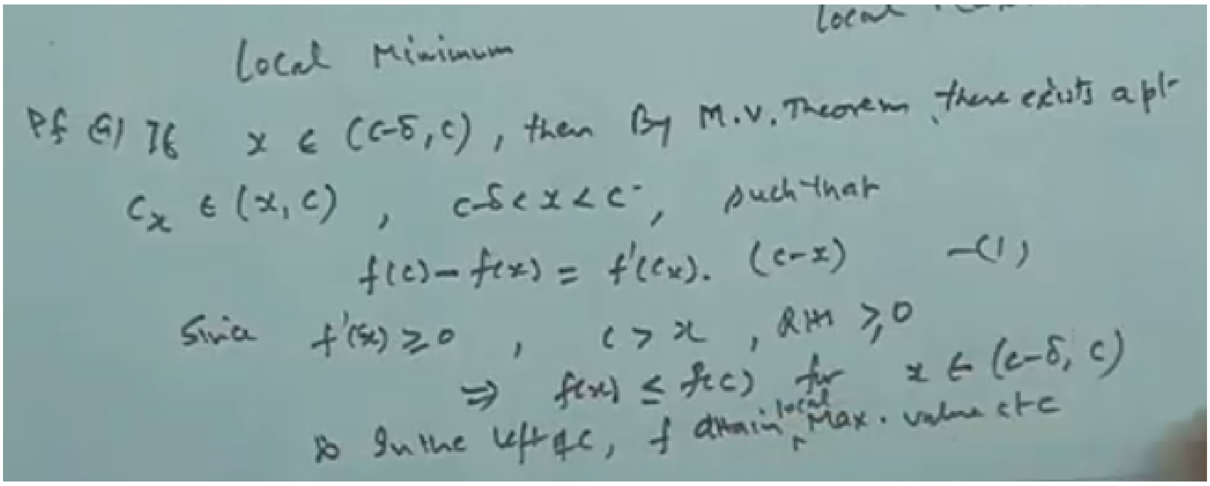

So this implies this is only possible, when f of X, f of X, is less than equal to, FC. Okay? For all X, belongs to the, interval C minus Delta to C. Clear? It means, in this neighbourhood, the function; f attains the maximum value at C. So in the left side, in the left of the C, in the left of C, the function f, attains maximum value, at C, at C, local maximum value, you can say local, because we are choosing only the neighbourhood around this. Now on the other hand, if I take this,

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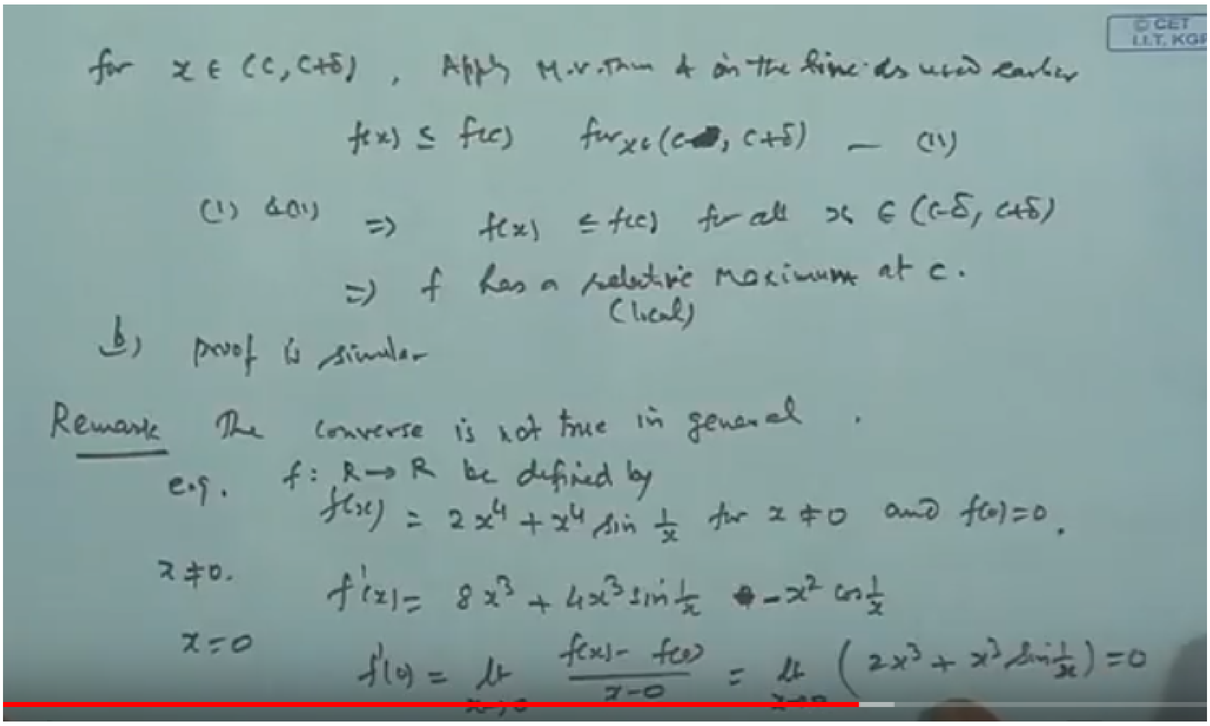

say X, for X, belonging to C, to C plus Delta and again apply the main value theorem. Then what we get? We get the X, there exists some x, main value theorem, over this, same so by, mean value theorem, we can say. Then with f of X, is less than equal to FC, for by main value theorem and in a similar way and on the same lines, lines, of as used earlier, proof. Means, there exist a C X, lying between the, say C minus, comma X, where this will lie, so for or we can say, this less than equal to this. For X belonging to C minus Delta, to C plus Delta, follow. Okay? Follow and this is to, sorry, this is C to, C plus Delta, for all X, C to, C plus Delta. Therefore, one and two will gives, that implies, the FX, is less than equal to FC, for all X, belonging to C minus Delta, to, C plus Delta. And that is sufficient to prove, F has a relative Maxima or local, we can say, maximum at the point C, at C, this proof. The proof, for the part follow, similar, proof is similar. So we just. Okay? So that's gives the result. Now, the converse of this is not true remark. Because what we have seen is, we are assuming the derivative is nonnegative, in the left hand side, and non positive in the right hand side, then there is a Maximum. Here in this case, non negative here and positive, non positive here, here it is a non positive and non negative, then minimum. But if I think, say the converse part of it, Darboux Theorem. Suppose f has a maximum at the point C, then, this or minimum at the point C, then we can, can be dry, so that the function will change its behaviour, from positive, to negative. So that converse is in general true, the converse is not true in general. Okay? For example; Say, suppose I take the function FX, which is equal to, say, let F is a function, f is a function, from R to R, be defined, be defined, by FX is equal to, 2 X to the power 4, plus, X to the power 4, sine, 1 by X, for X, not equal to 0 and an F of 0, equal to 0. Okay? Then clearly this function, then this function, clearly what, when X is different from 0, the derivative of the function will be, 8 xq, + 4 X q, sine, 1 by X and then, minus X square, cos 1 by, sine cos, plus and then minus 1 by, so it is minus, I'm sorry, so it will be minus of X, because sine is cos 1 by X and 1 by X is, minus 1 by X square, so that's why minus sign, is there.

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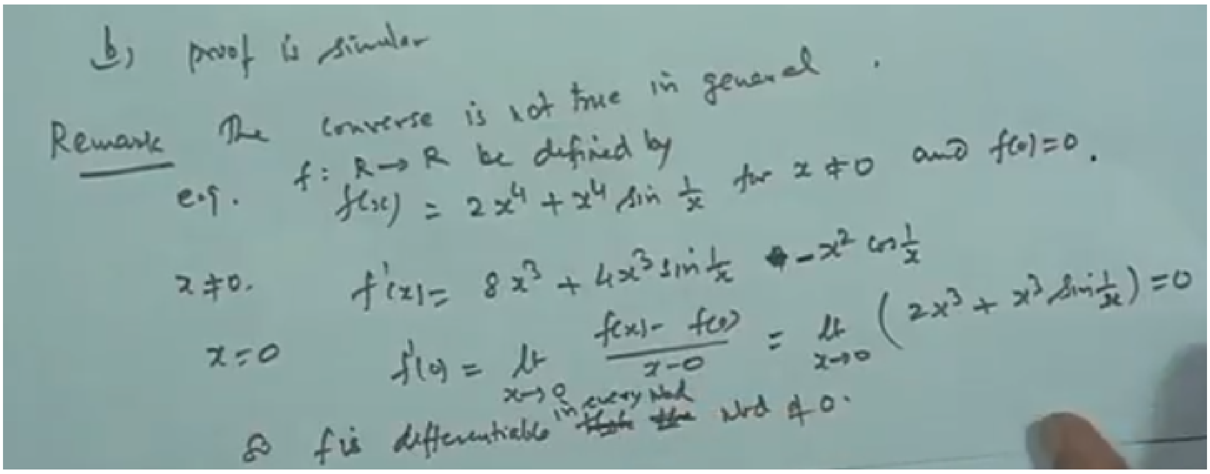

And when X is equal to 0, the derivative of this function, f prime 0, is, limit FX minus f0, over X minus 0, when X tends to 0. And this comes out to be what? When you divide by this, you are getting limit, X tends to 0, 2 X cube, plus, X cube, sine, 1 by X. And this is dominated by X cube and this is also, the total limit comes out to be 0. It means, the function f, so f is, f is, differentiable, f is differentiable, Differentiable, throughout the neighbourhood of zero, in some neighbourhood of 0, in any neighbourhood, in every neighbourhood of, differentiable, in every neighbourhood of zero, in every neighbourhood of zero.

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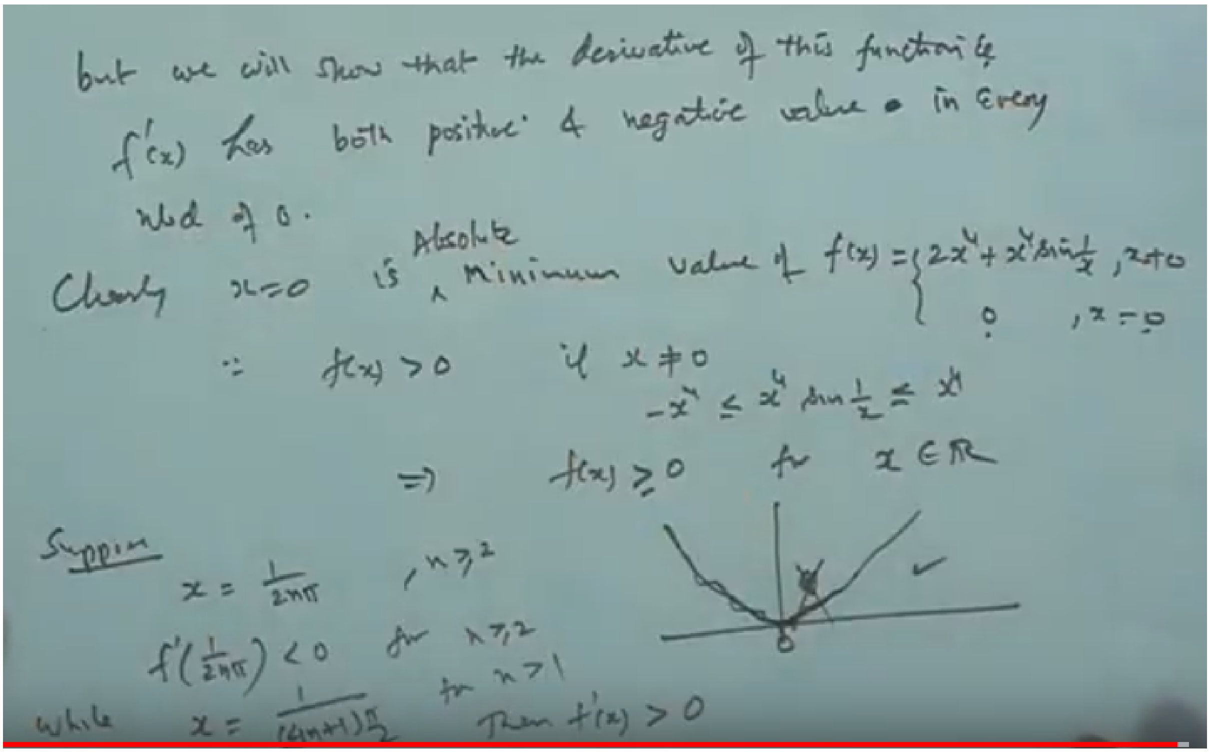

But we will show that, but this function, but we will show, that the derivative, of this function, that the derivative of this function, that is f prime X, F prime X, has both, positive and negative values and negative values, negative values, of, in every neighbourhood of 0. One more thing, let me just clear it. The function is defined by this. The function clearly, X equal to 0, is the minimum value, of the function, FX, which is 2 X 4 plus X 4, sine 1 by X, for X is not equal to 0 and 0 for X equal to 0. Whatever the X you choose, this will be always a positive quantity. Why? Because, so it is X equal to 0, is basically, an absolute minimum value, absolute minimum. The reason is, because f of X, will always be than zero, if X is different from zero. The reason is, the sine x to the power 4 sine, 1 by, X to the power 4 sine, 1 by X, this is bounded by minus X, to the power 4 and plus X to the power 4, this sign will be there. Now 2 X 4, when we added both side, then X 4, is a positive quantity.

Therefore the function FX, will always be, greater than or equal to zero. Zero at the point 0. So it is 0, for all X, belongs to R. So 0 is the point which is the minimum point for this function and it. So what we wanted to show, this function is something, at the point 0 it is 0. And rest of this, I don't know what type of this function, is. It will go to something like this. Okay? Maybe, X 4 dominated something, so a curve will be there. Okay? A curve, maybe, like this, something, now this point zero, is the point, where it has tends the, F should be minimum value. But what it says is that this curve, this is not correct. What we wanted to show, that, its derivatives, okay, sorry, it is okay. Derivative, will have both positive and negative value, in this inter. It means, though it has a minimum value at the point 0, but according to the previous result, the result was, which we have shown, is this. That if, the derivative is nonnegative, then there, left hand side interval, it is non negative and right hand side interval, it is non positive. If it has a, maximum and when it is a minimum point, then left hand side derivative, it is, all the values will be greater than equal to 0. And for the left hand side, right hand Side, it will be, less than, say less than 0 and right hand side, it will be greater than equal to 0. But here in this case, we are unable to get, any interval, whether in which, the left hand side, all the functional values, less than equal to zero and right inside the derivative of this, is greater than equal to zero. The reason is like this. So this follows from here. If we take the say, point. Suppose I did, suppose I take, X which is equal to, one over, two n PI, n is greater than 2. Okay?

Then what happens this? The function, f derivative of this function, the derivative is defined like this. This derivative, F prime X, at this point, at this point, so this is positive, now this will be, since it is 2 n PI, this will be 0, this will be 0 and here it will be, something negative. Is it not? So when we take n, greater than 2, so it becomes what? 4, 4 into PI, becomes more than 8, in fact. So we get this value negative. So here we are getting to be negative. And infact this is, we can show this thing is, that this is negative for N, greater than equal to 2 and positive. So it is negative, for N, greater than equal to 2. While if I take, X equal to, 1 upon 4 n, plus 1, PI by 2, for N, greater than 1. Then what happens? The derivative F prime X, derivative F prime X, this is the derivative F prime X, so when you take this term, this of course it will not affect, this value will be 0 or multiplier, so it is positive always, so it is always be greater than 0. So in the same, this sequence belongs to the same intervals, 2 n PI n, then the right hand side of this. So we are getting the right hand interval, where the derivative is having both positive and negative values. Therefore, it contradicts our, that converse is not true, in fact is. So this shows. Next lecture we will do it.

Thank you very much.