NPTEL

NPTEL ONLINE CERTIFICATION COURSE

Course

On

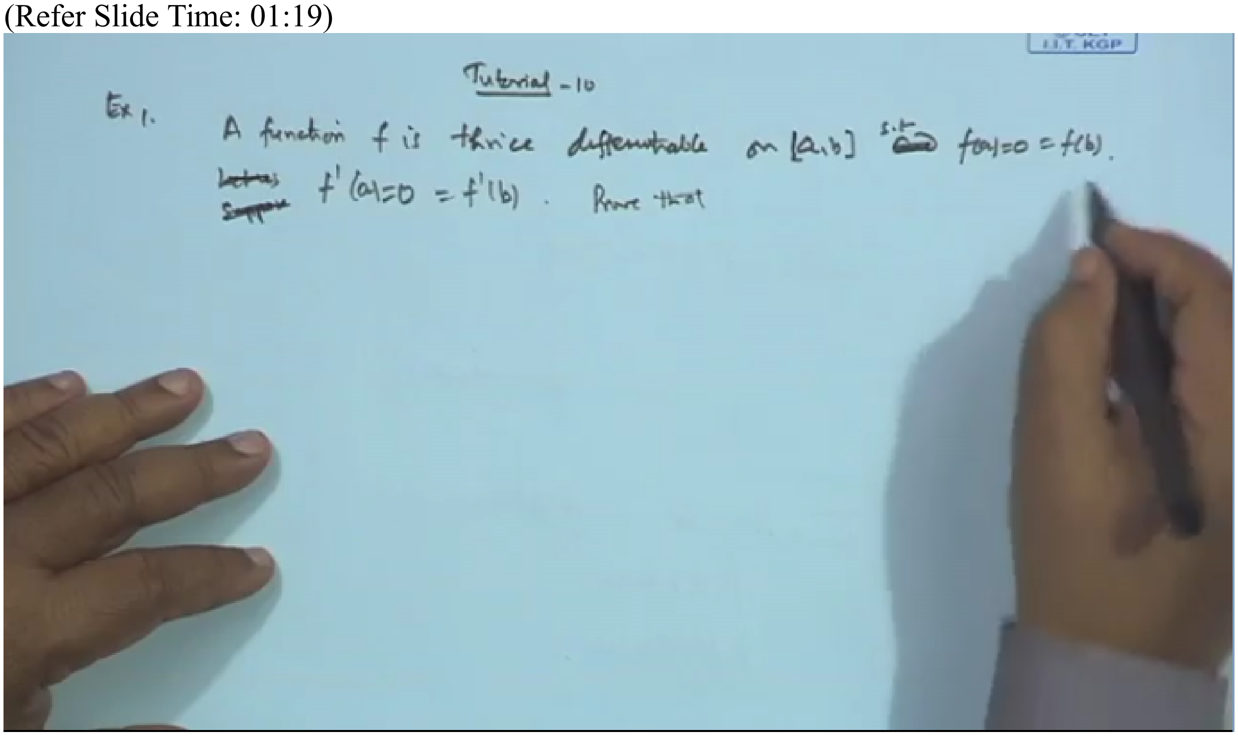
Introductory Course in Real Analysis

By

Prof. P. D. Srivastava

Department of Mathematics
IIT Kharagpur

So this is the next lecture in fact tutorial lecture say 10. We will discuss few problems on based on the mean value theorems, so first problem is a function F is thrice differentiable on a closed interval say A, B, and F(a) is 0, F(b) is also 0, and suppose this. Let the function F prime A is also 0, let us suppose F prime(a) is also 0, F prime(b) is also 0, then prove that F is thrice differentiable function on A, B, we can also say such that this and this happens,
(Refer Slide Time: 01:19)



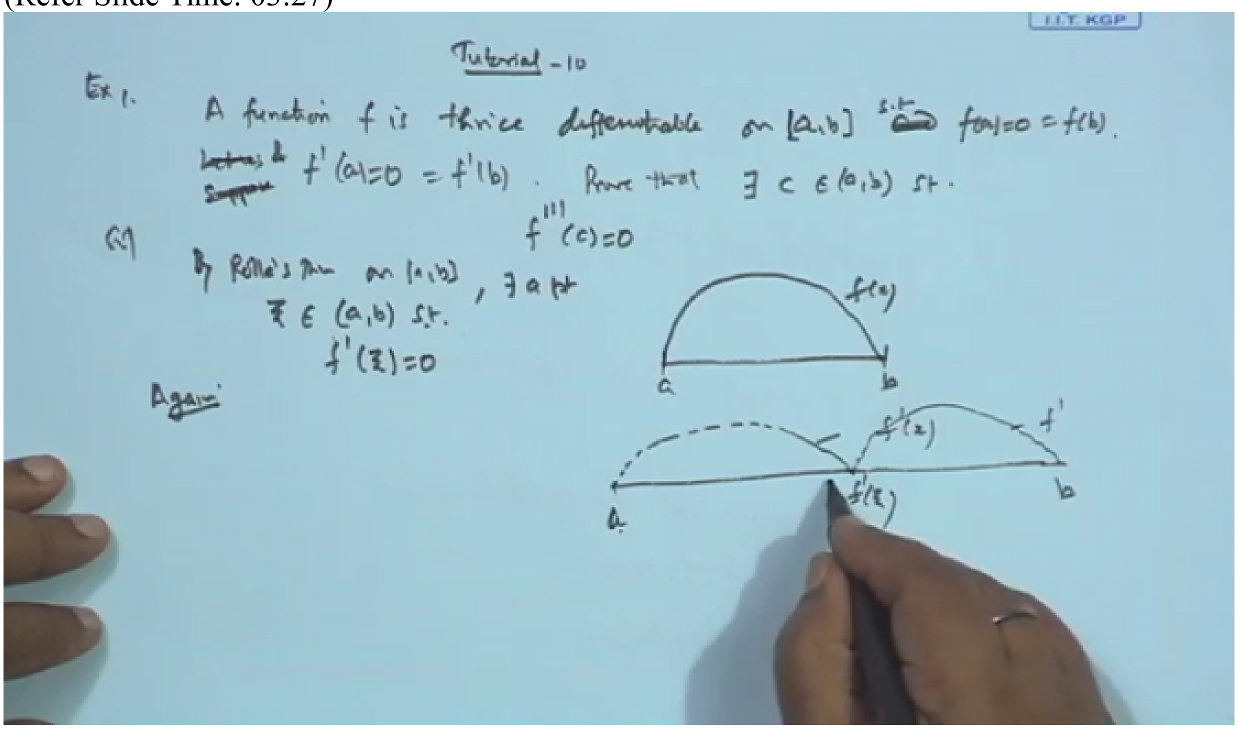
this and this happens, okay, that is better.

A function F is thrice differentiable on A, B such that F(a) is 0, F(b) is 0, F prime(a) is 0, F prime(b) is 0, then prove that there exists a C in the interval A, B such that this third derivative of the function at the point C is 0, okay. So what is given is the interval A, B is given, the function F is thrice difference, so it is continuous function differentiable and at the point A the function is 0, at the point B the function is 0, so this is the function F(x).

So by Rolle's theorem if the function on the interval A, B, if the function is continuous over the closed interval A, B differentiable at the open interval A, B and at the end point the function attains the value 0, then there exists a point say X xi in between A, B such that the derivative of the function vanishes, such that F prime X xi vanish, this is the mean value theorem Rolle's theorem.

Now again it is given that the functional value F dash A is 0, this is F dash(x) function, F dash (a) is 0, F dash(xi) is 0, and then F dash(b) is also 0, this is our F dash function. So a function is vanishing at the point A, at the point X xi, and at the point B,

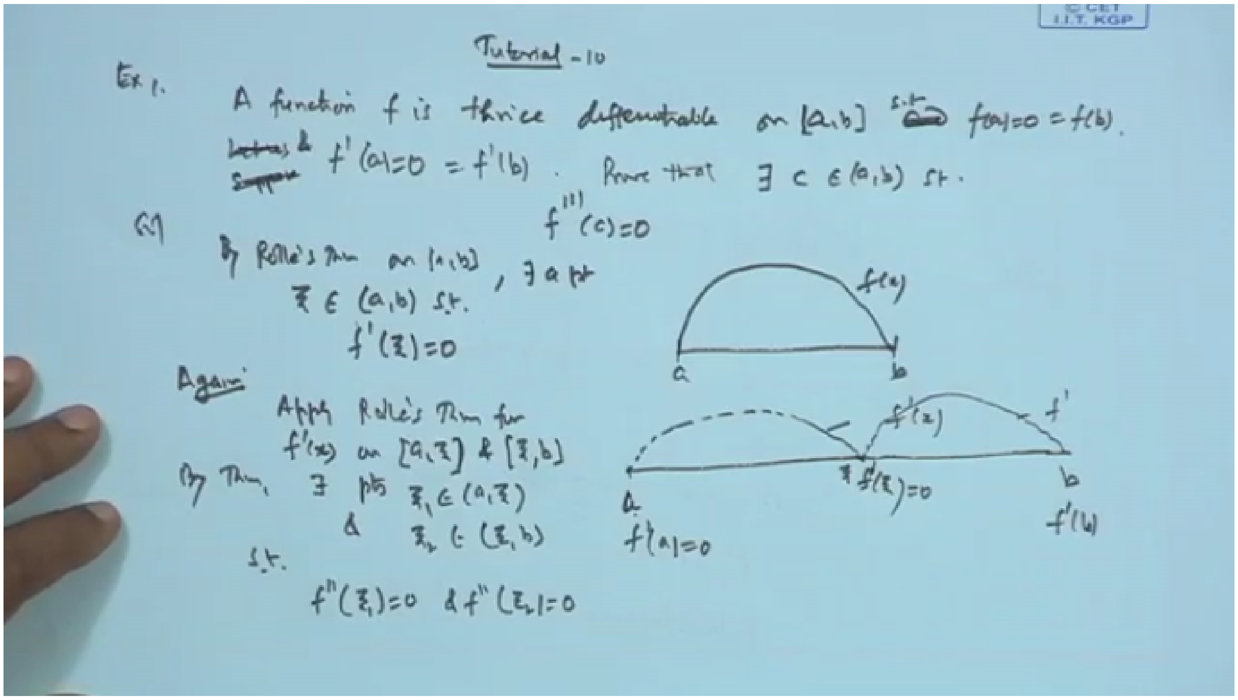
(Refer Slide Time: 03:27)



and it is given that F dash(a) is 0, F dash X xi is 0, F dash(b) is 0.

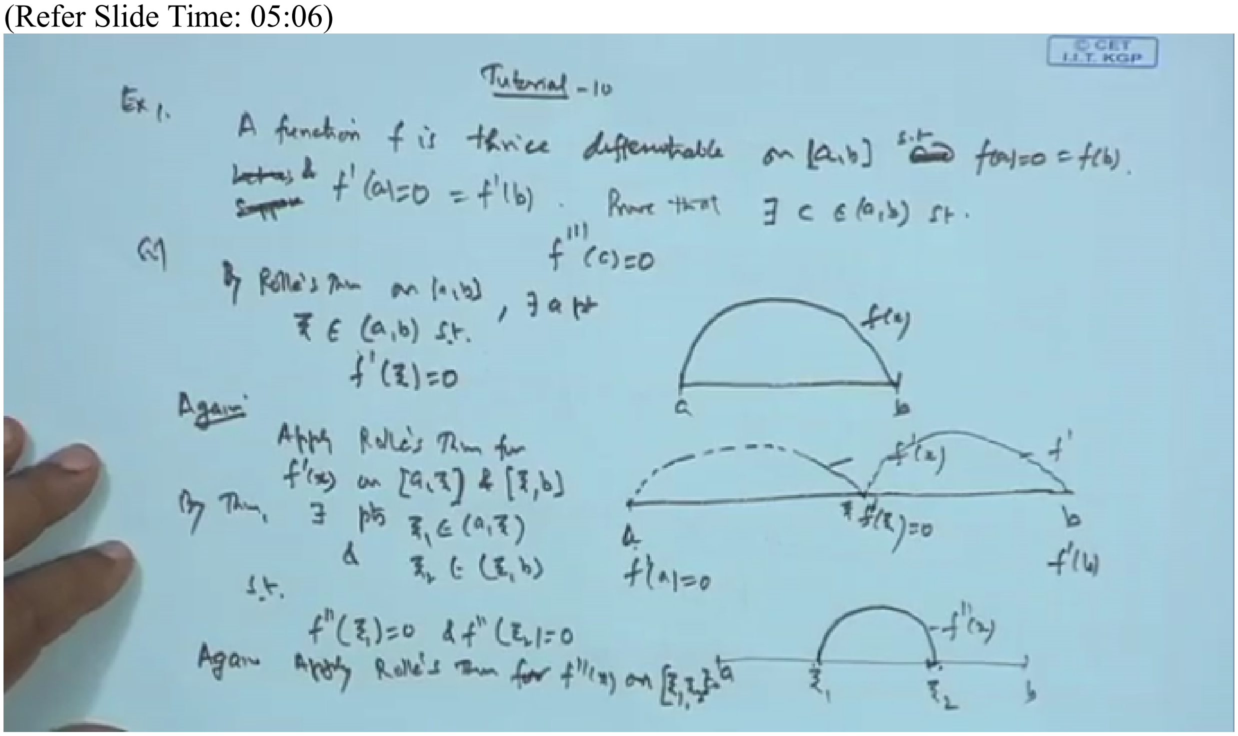
So now apply the Rolle's theorem, for the function F dash(x) on the interval A to xi, and xi to B, so by Rolle's theorem, by this theorem there exists points xi 1 in the interval A to xi, and xi to interval xi to B such that the derivative, second derivative of this function or derivative of prime vanishes at this point, okay.

(Refer Slide Time: 04:22)



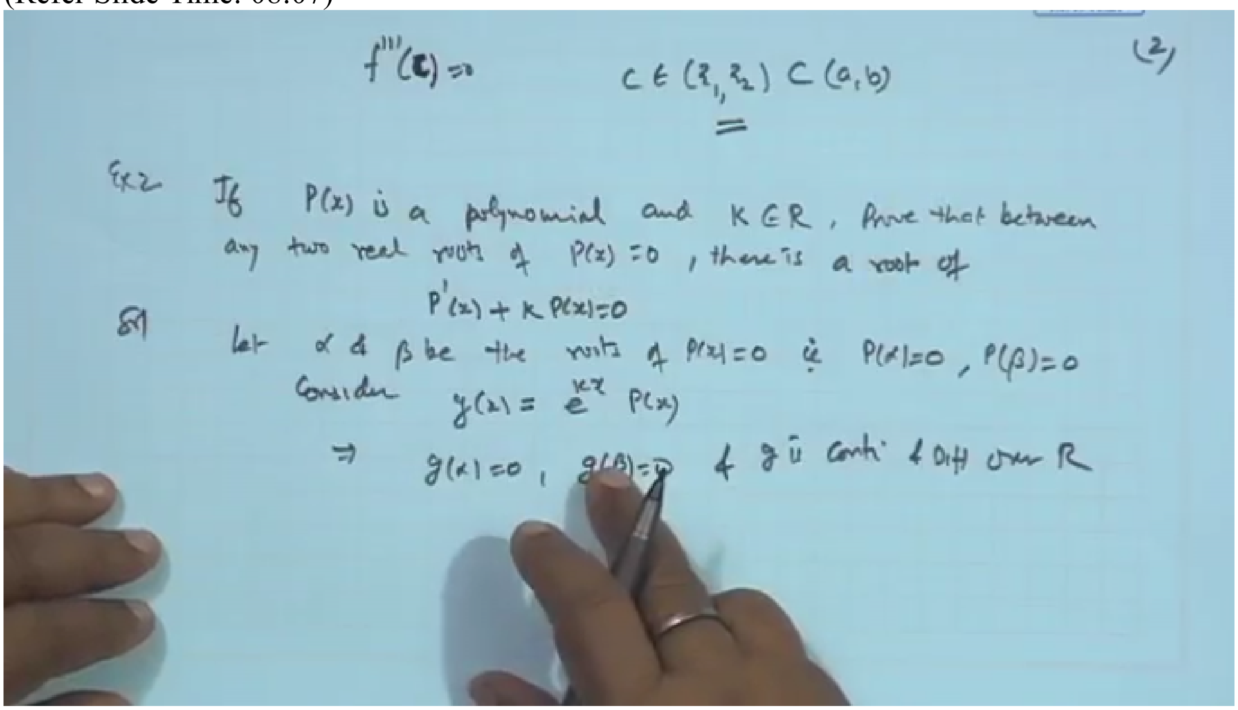
So now again the figure is like this, a function this is A, B, here is point xi 1, xi 2, the second derivative of the function, this is the F double dash X vanishes at the point xi 1, xi 2 function F is thrice differentiable, so this function F double dash is continuous and differentiable, so again apply Rolle's theorem for the function F prime X on the interval xi 1, xi 2,

(Refer Slide Time: 05:06)



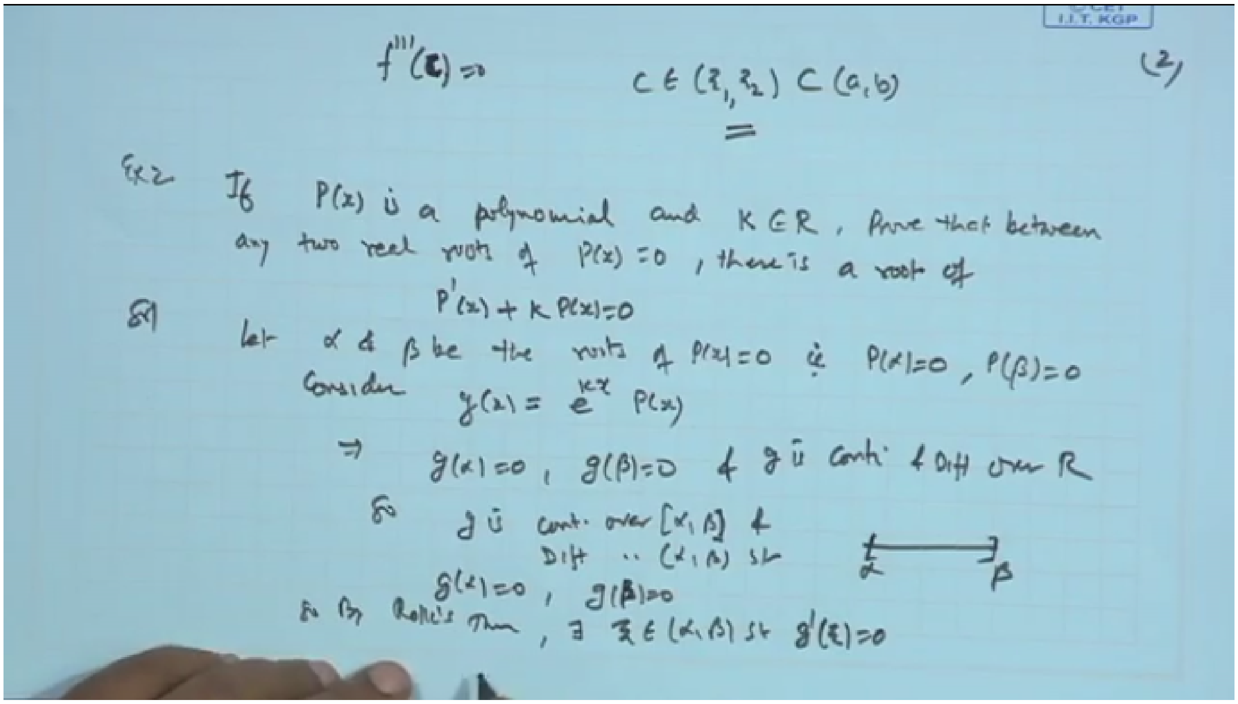
so from here we get there exists a point say C in the interval A, B such that in fact in the interval xi 1, xi 2 such that the value of the function, the derivative of the function third derivative at this point C is 0, so C lying between xi 1, xi 2 which is subset of A, B, so the result follows from there, okay. So basically we have using the Rolle's Theorem thrice and getting the results for this.

Now next, if P(x) is a polynomial and K belongs to R, any real number then prove that between two real roots of P(x) = 0 there is a root of the equation P dash(x) + K times P(x) = 0, so P(x) is a polynomial of this, proved that between any two real roots of P(x) = 0, there is a root for this, so let alpha and beta be the roots of equation P(x) = 0, it means that is P(alpha) is 0, P(beta) is 0.

Now consider a function G(x) as E to the power KX into P(x). Now since alpha beta all the roots of this therefore this implies the G(alpha) is 0, G(beta) is 0, because at the point alpha P(alpha) is 0, at the point beta the P(beta) is also 0, so G(alpha) G(beta) is 0, and G is continuous and differentiable over the interval R, because polynomial is always a continuous and differentiable function of R and E to the power X is also continuous and differentiable,
(Refer Slide Time: 08:07)


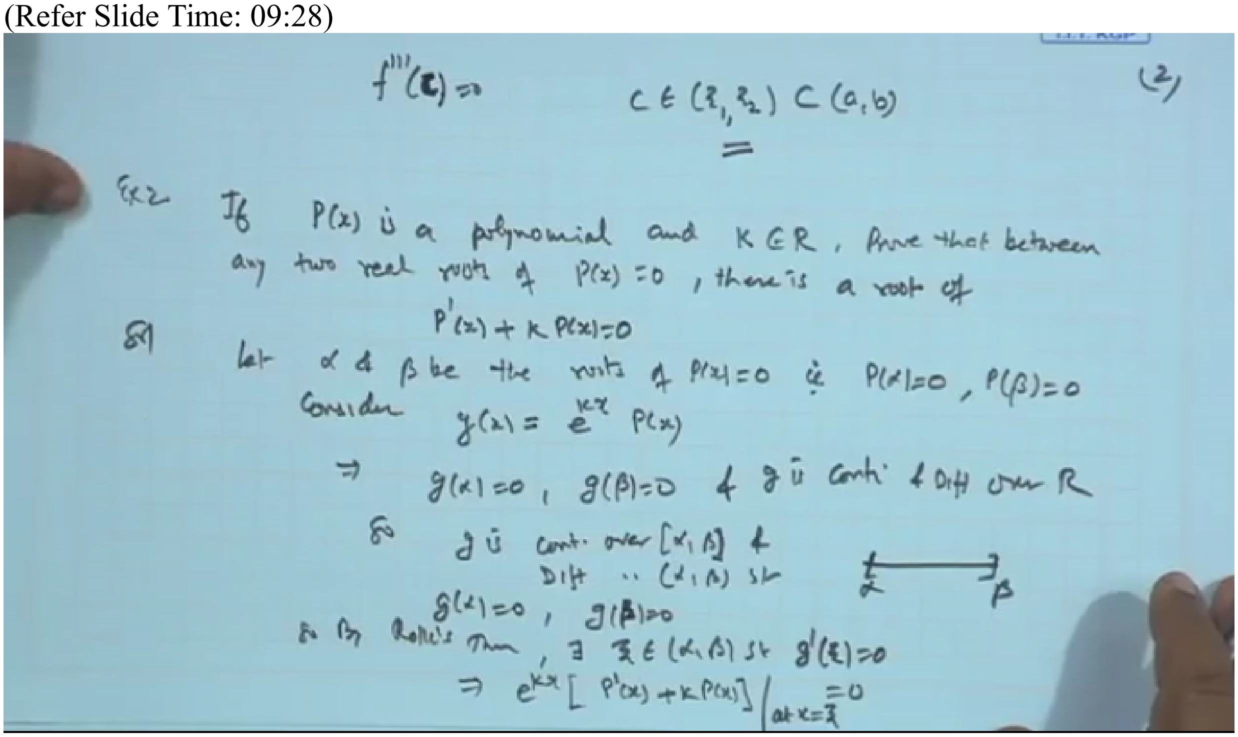
therefore the function G is continuous and differentiable over the entire R, so if we take any interval it will remains continuous and differentiable of this.

Now if we look the interval alpha to beta over this closed interval the function G is continuous, so G is continuous over the closed interval alpha beta, and differentiable over the open interval alpha beta, such that G(alpha) is 0, G(beta) is also 0, so by Rolle's theorem again, so by Rolle's theorem there exists a point xi in between alpha beta such that the derivative of G dash at xi is 0, but what is G dash(x)?
(Refer Slide Time: 09:07)



Is the same as P dash E to the Power KX, P dash(x) + K times P(x) at the point X = xi this vanishes, so this shows that this implies the result,

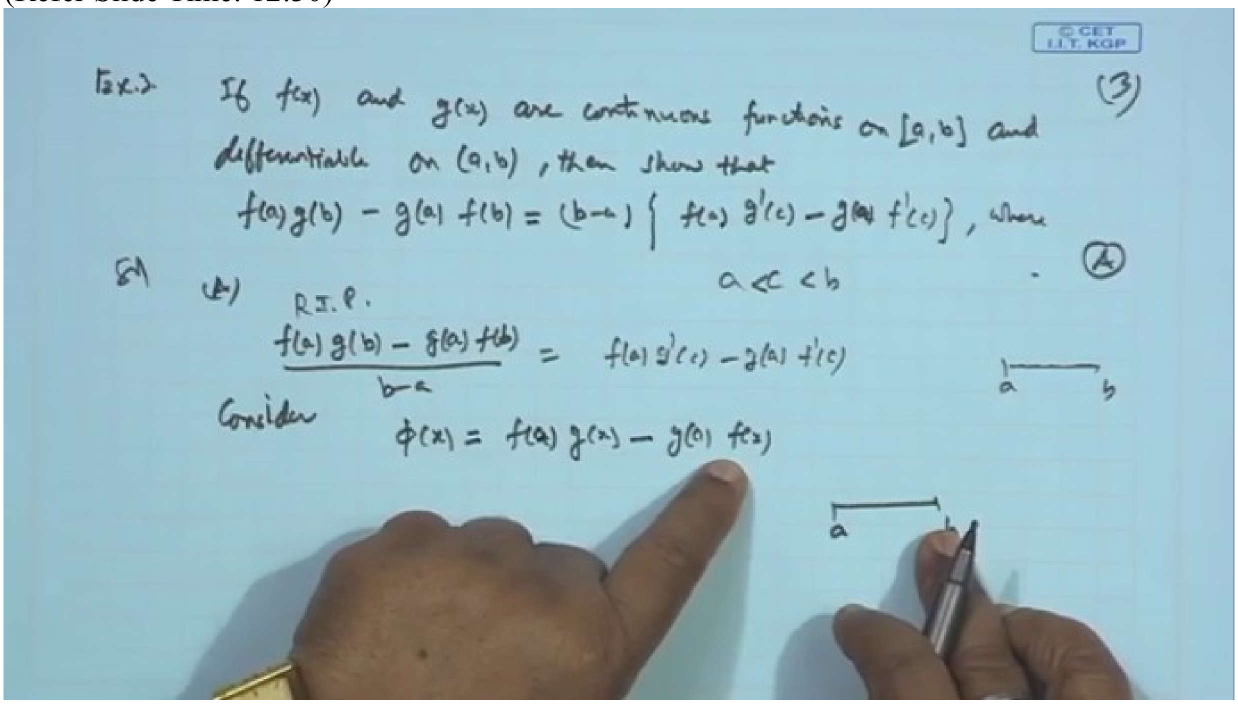
(Refer Slide Time: 09:28)



because E to the power alpha xi is not 0 therefore this has to be 0 and we get the result, okay, so this completes this existence, okay.

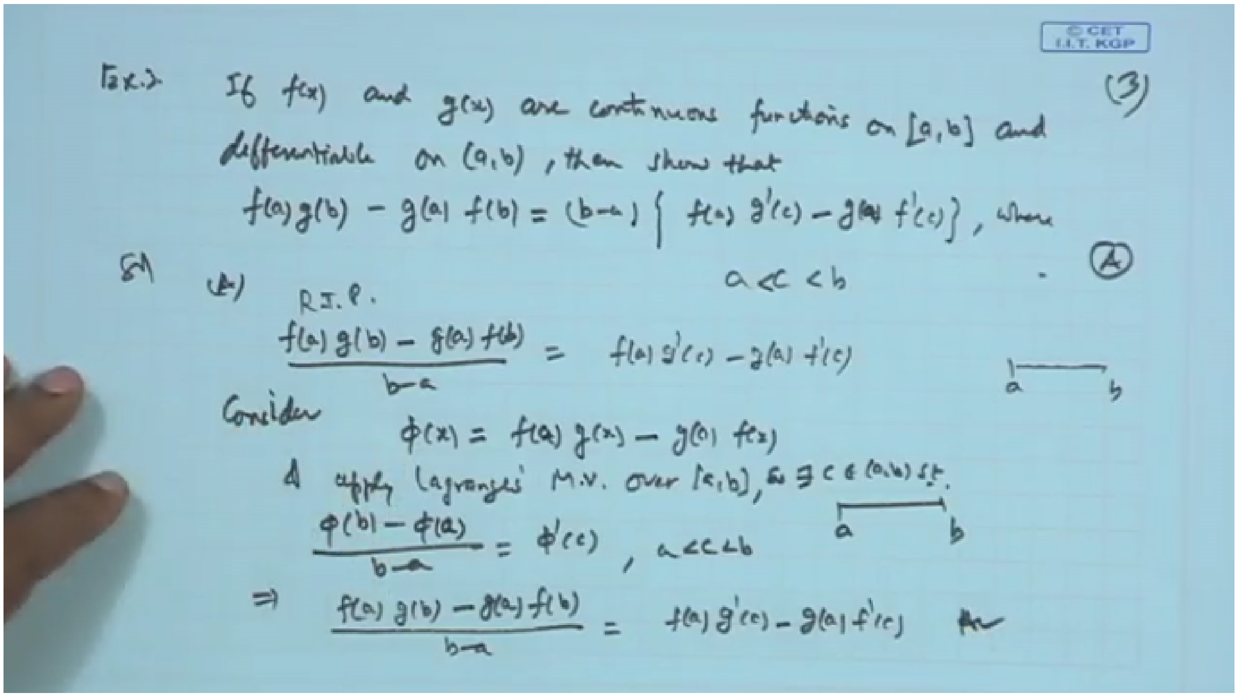
Third one, if F(x) and G(x) are continuous functions on the closed interval A, B and differentiable on the open interval A, B, then so that F(a) G(b) - G(a) F(b) = B - A into F(a) G prime(c) – G(a) F prime(c), where C lies between A and B. So now function F and G are giving to be continuous on the closed interval A, B, it is also differentiable then we have to prove this thing, now if we look this one, then basically from the expression A, if we look this expression A this can be expressed as, it can be rewritten as F(a) G(b) – G(a) F(b)/B-A is this function, is it not? This we wanted to prove, required to prove it with this one.

So this suggest that if we consider the function phi(x) as F(a) G(x) – G(a) F(x) then this function is continuous and differentiable over the interval A, B, because this function G and F both are given to be continual differentiable, and then at the point B it has this value, and at the point A it has the value 0,
(Refer Slide Time: 12:30)



so it suggests that this function phi X if we apply the Lagrange's Mean Value theorem for the function phi X over the interval A, B then we may get this result, okay, so consider this function and apply Lagrange's Mean Value theorem over the interval A, B, so what we get phi(b) – phi(a) over B-A, so there we get this thing, where C is a point in between A, B, so there exists C in the interval A, B such that this is at hold.

Now substitute the value and get the result, so phi(b) means F(a) G(b) – G(a) F(b) divided by B-A, and what is the derivative of this function? F(a) and G prime(x), so G prime(c) – G(a) F prime(c) and that is the answer which we wanted, okay,
(Refer Slide Time: 13:51)

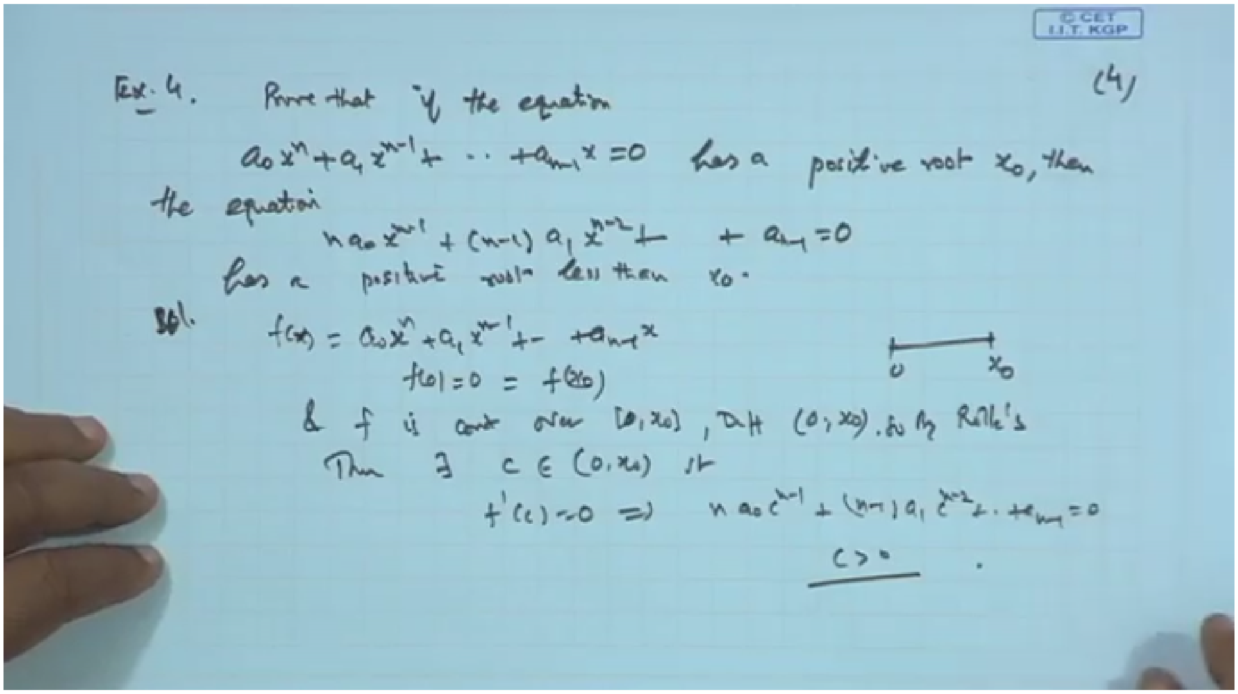


so sometimes we have to see the answer and according to the expression one can decide that can we have a suitable function so that after applying either Lagrange's mean value theorem or Rolle's theorem we can get immediately the result, okay so that is what, okay.

Now prove that if the equation A naught X to the power N, A1 XN-1 + AN-1 X = 0 has a positive root say X naught, then the equation N AN XN-1 + N-1 A1 XN-2 + AN-1 = 0 has a positive root less than X naught, I think that's very easy question. Solution, this is the interval 0 and here is the point X naught, the function has given a positive root this.

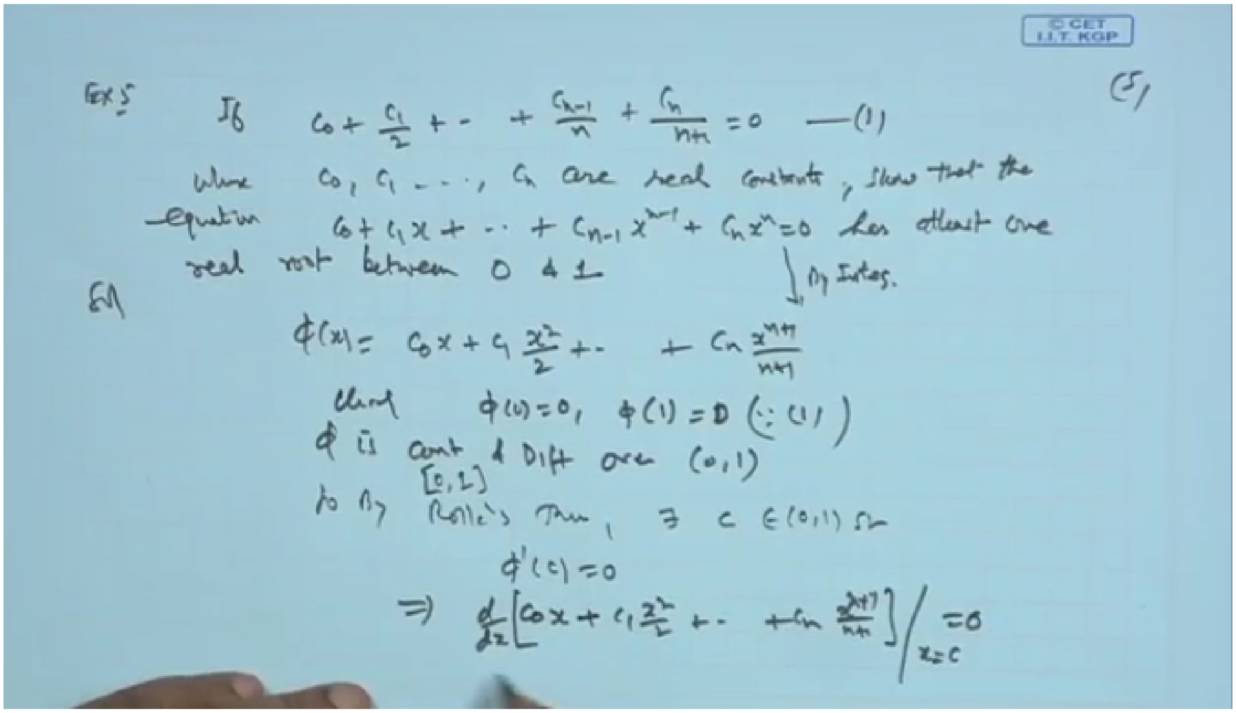
What is the function F(x)? F(x) if I take X naught X to the power N + A1 XN-1 + AN-1X, then at the point 0 it is 0, at the point X naught is also 0, because it has a root, X naught has a root of this equation, so function is continuous over the interval 0 to X naught differentiable on the open interval 0 to X naught, so by the Rolle's theorem there exists a point C in the interval 0 to X naught such that the derivative of this function vanishing, but derivative means N A naught C N-1 + N-1 A1, CN-2 + N-1 is 0, where C is positive, and that's the answer which we are looking, okay, so this will be,

(Refer Slide Time: 17:04)



and the exercise if C naught + C1/2 + CN-1/N + CN/N+1 is 0 where C naught, C1, C2, CN are real constants, then so that the equation C naught + C1X + C2 + CN-1 XN-1, CN XN is 0 has at least one real root between 0 and 1 solution, so this is given B1 this function has it is one real root in 0, 1, so if we look the function phi X, what is this function is? Construct the function phi(x) C naught(x) + C1 X square/2 + CN X to the power N+1/N+1, this basically I have taken from here by integrating, because when you consider this function and apply the Rolle's theorem then finally result comes as a derivative of the function, so if I prove that this function phi satisfy all the conditions of the Rolle's theorem over the interval 0 to 1, then there will be a point C X some point where the derivative of this function vanish, it means this will come, okay, so clearly phi(0) is 0, phi(1) is 0 because of the condition 1, because of one, is it not? Because of this condition this vanish, therefore this phi is also continuous and differentiable over the interval 0, 1, continuity has the closed interval 0, 1, differentiable at the open interval 0, 1, so by Rolle's theorem.

There exists a point C in the interval 0, 1 where the derivative of this function vanishes, but this derivative means C naught X, C naught derivative of this, so derivative of this C naught X + C1 X square/2 CN XN+1/N+1 at the point X = C vanishing and that is the result,
(Refer Slide Time: 20:11)

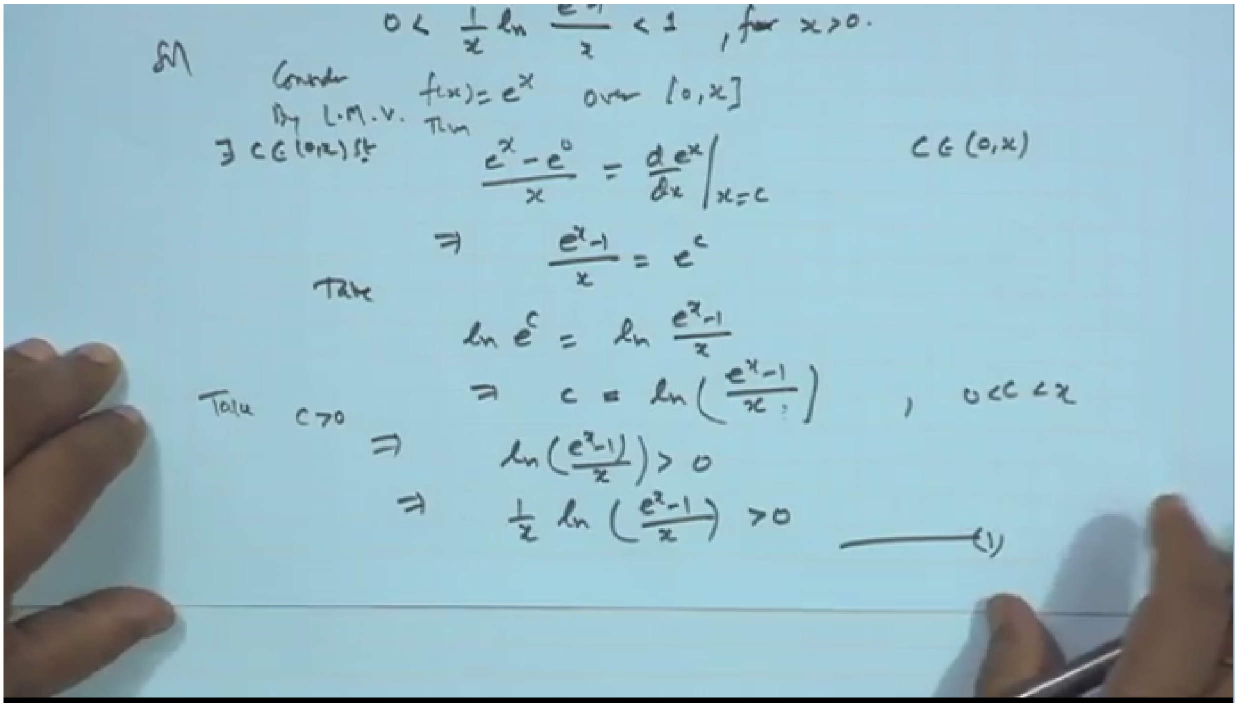


so answer okay, please do that part, answer is this.

Now using the Mean Value theorem we can also obtain or we can also drive certain inequality, say using mean value theorem proved that 0 less than 1/X LN E to the power X-1 divided/X is less than 1 for X greater than 0, so let's see the solution for this, this inequality we wanted to drive with the help of Mean Value theorem in fact the Lagrange's mean value theorem will help you.

So consider the function F(x) = E to the power X and over the interval 0 to X, then by Lagrange's Mean Value theorem we get the value of the function at a point X minus value of the function at the point 0 divided by X is equal to the derivative of the function E to the power X at the point X = C where the C lies between 0, X, okay, C lies between 0 and X, so by Mean Value theorem there exists a C in the interval 0, X, such that this is at hold, and this is equal to E to the power X-1/X = E to the power C. Take the log so we get log LN E to the power C = LN E to the power X-1/X and this implies C = LN E to the power X-1/X, where the C lies between 0 and X.

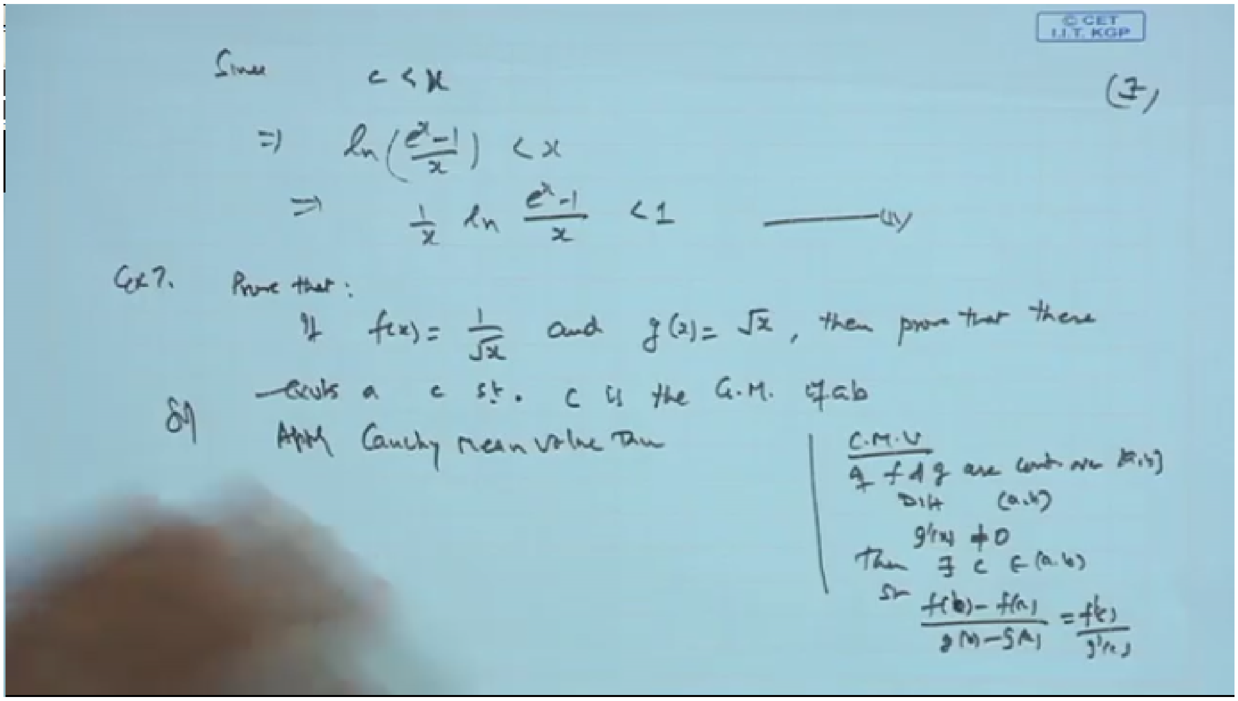
Now if we take C to be greater than 0, then obviously this inequality LN-1/X this will be greater than 0, but X is also positive so this implies that 1/X LN E to the power X-1/X will also be positive, so that is the first part of this result.
(Refer Slide Time: 23:07)



The second side is obtained like this, since C is less than 1, less than sorry X, less than X therefore C means what? This one, E to the power X-1/X, log of this, this is log, so this implies the log of LN E to the power X-1/X is less than X, therefore 1/X LN E to the power X-1/X is still less than 1, so this proves the second part of it, and this completes the results for it, okay.

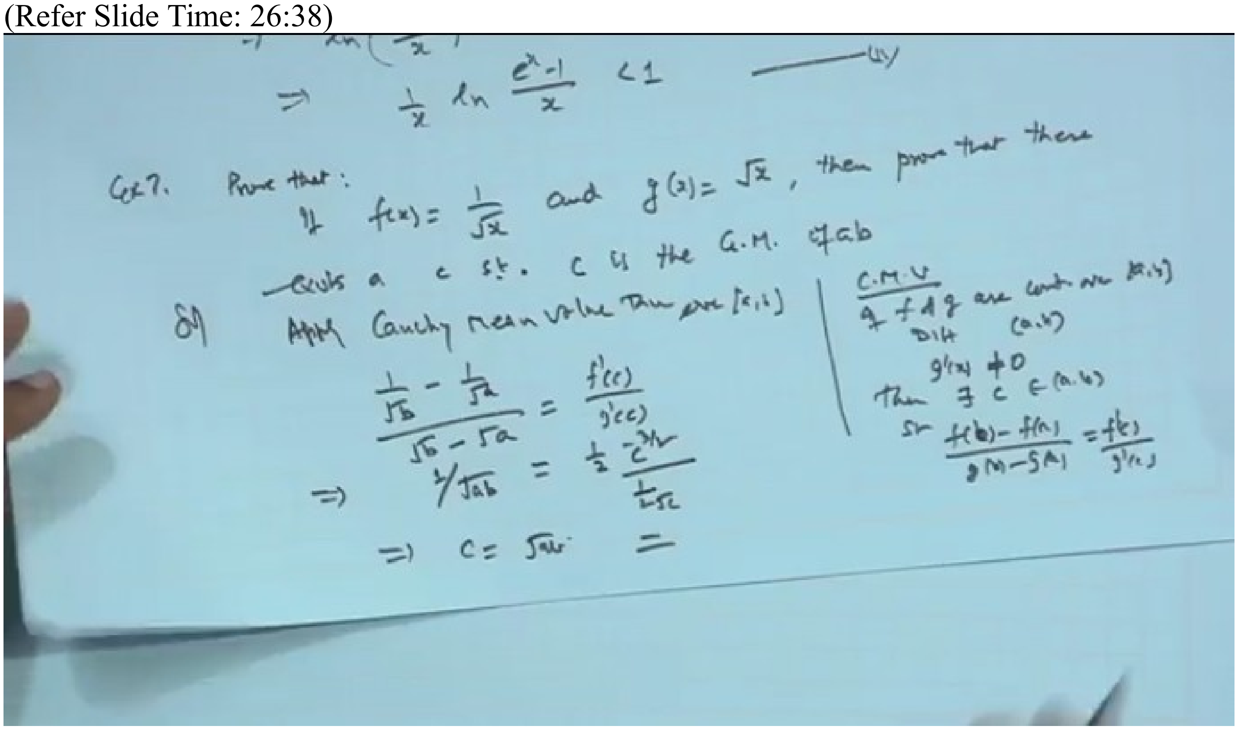
And next is say F(x) is proved that, if F(x) is 1/root X and G(x) is equal to root X, then prove that there exists a C such that C is the geometric mean of A, B, so this, so let us see apply the Cauchy Mean Value theorem, what the Cauchy Mean Value theorem says? The Cauchy Mean Value theorem is if F and G are continuous and different over the interval say A, B, differentiable on the open interval A, B, G prime(x) is not equal to 0 then there exists a point C in the interval A, B such that F(x) F(b) – F(a)/G(b) – G(a) is the derivative F dash(c)/G dash(c), this is the Mean Value theorem.

(Refer Slide Time: 25:39)



So apply this here F(x) and G(x) are such, so find out the F prime(c), so by Cauchy Mean Value theorem over the interval say A, B we get from here is 1/root B - 1/root A divided by root B minus root A is the derivative F prime(c)/G prime(c), and this is equal to nothing but 1 by this, this is root A, B, this root A, B 1/root A, B okay equal to what is the F prime(c)? F prime(c) will be 1/2 C to the power -3/2 over this will be 1/2 by root 2 by root C, so from here we get C = root A, B and that's the answer for it.

(Refer Slide Time: 26:38)



Thank you very much. Thanks.